Z. Phys. C 72, 509–517 (1996)

ZEITSCHRIFT
FÜR PHYSIK C

Virasoro type Lie algebras and deformations

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Received: 6 December 1995

Abstract. The first and second cohomologies of Cartan Type Lie algebras with coefficients in irreducible tensor modules are calculated. The space $H^1(L, U)$ is interpreted as a space of deformations of (L, U)-modules. $H^2(L, L) \neq 0$ if $L = S_2, S_2^+$ or $L = H_n, H_n^+$. Lie algebra of divergenceless vector fields S_2^+ has only one nontrivial local deformation. The two-sided simple hamiltonian algebra H_n has $2n^2 + n$ new local deformations in addition to Moyal cocycle. The Lie algebras $L = W_n (n > 3)$, $S_{n-1}(n > 2)$, $H_n(n > 1)$, $K_{n+1}(n > 1)$ have 3, 1, 1, 3 nonisomorphic tensor modules with irreducible bases and nonzero 1-cohomologies; respectively, the corresponding numbers for 2-cohomologies are 9, 6, 7 and 9.

1 Introduction

The main applications of cohomologies of Lie algebras $H^k(L, M)$ concern small $k = 0, 1, 2, 3$. The gauge group of cohomology introduced in physics by Faddeev [7] is related to the cohomology of the gauge Lie algebras [24]. Wess-Zumino functional is interpreted as a 1-cocycle of the space-time gauge group [23] and Schwinger term as a 2-cocycle of gauge group [25]. Abelian extensions appear in gauge theory and in gravitational theory. A 3-cocycles concern to the failure of the Jacobi identity in the presence of a Dirac monopole. Such fundamental results of the theory of finite-dimensional Lie algebras as Levi-Mal'cev theorem (any Lie algebra is a sum of a semisimple subalgebra and the radical) and Weyl theorem (any representation of a classical Lie algebra is a direct sum of irreducible representations) mean that, for a semisimple Lie algebra L, the first and second cohomologies with coefficients in irreducible modules are trivial.

In the nonclassical case both of these results are not true, but counterexamples to these facts also have important applications. Well known example of nonsplit extensions gives Virasoro algebra, i.e. a central extension of the Lie algebra of vector fields on the circle [9] :

$$Vir = \{e_i, z : [e_i, e_j] = (j - i)\, e_{i+j} + \delta(i + j = 0)(i^3 - i)\, z,$$
$$[e_i, z] = 0, i, j \in \mathbb{Z}\}$$

Indecomposable modules, outer derivations and generators of nilpotent algebras can be described in terms of first cohomologies. Second cohomologies are responsible for (non) split extensions and deformations [10]. Nonsplit extensions of Cartan Type Lie algebras called as Virasoro type Lie algebras. We interested in the following problem:

For given Lie algebra L find irreducible modules M such that $H^k(L, M) \neq 0$ and find basic cocycles

As mentioned in [1] this problem for $k = 2$ is originated from Cartan. We solve it for Cartan Type Lie algebras and their irreducible tensor modules in cases $k = 1, 2$.

Denote by $\kappa_k(L)$ the number of irreducible modules M such that $H^k(L, M) \neq 0$. For any Lie algebra L of prime characteristic p the following inequality is true: $0 < \kappa_k(L) < \infty$, $0 < k < \dim L$ [3, 4]. For example, $\kappa_2(sl_2) = 1, p > 3$. Some infinite-dimensional analoges of this result are also true for Lie algebras of Cartan type over complex numbers. From our results it follows that in the class of tensor modules,

$$\kappa_2(W_n) = 9 - \delta(n = 2, 3) - 4\delta(n = 1),$$
$$\kappa_2(S_{n-1}) = 6 - \delta(n = 3), n > 2,$$
$$\kappa_2(H_n) = 7 - 2\delta(n = 1),$$
$$\kappa_2(K_{n+1}) = 9,$$
$$\kappa_1(W_n) = \kappa_1(K_{n+1}) = 3,$$
$$\kappa_1(S_{n-1}) = \kappa_1(H_n) = 1$$

For example, W_1 has five irreducible tensor modules with nonsplit extensions. One of them is trivial module and the corresponding extension is Virasoro algebra.

In our paper we consider two-sided Cartan type Lie algebras L (Lie algebras of vector fields in Laurent power series) and one-sided Cartan type Lie algebras L^+ (Lie algebras of vector fields in formal power series). In Sect. 2

we give a constructive description of tensor modules for Cartan type Lie algebras. In Sect. 4 we describe all irreducible L_0^+-modules M_0, such that $H^2(\mathscr{L}_0^+, M_0) \neq 0$. As it turns out calculation of $H^2(\mathscr{L}_0^+, M_0)$, more precisely $H^2(\mathscr{L}_1, \mathbb{C})$, is equivalent to calculation of the following things:

i) a space of defining relations of maximal graded nilpotent subalgebra $\mathscr{L}_1^+$;
ii) invariant bilinear forms $L_1^+ \wedge L_1^+ \to M_0$ and sometimes $L_k^+ \wedge L_q^+ \to M_0, k + q = 3, 4$;
iii) all nonsplit extensions of L by irreducible (L, U)-modules.

Recall that L_0^+ is isomorphic to one of the classical Lie algebras of types A_n, C_n or their split central extensions. We prove that all irreducible L_0^+-modules M_0 with the property $H^2(L, U \otimes M_0) \neq 0$ can be realised as subspaces of tensor modules of type (2, 4) (1, 3),(0, 2), (0, 1) (1, 2), very like to the spaces of tensors of connection, curvature and etc. Here (s, k) denotes type of tensor, s is the number of contravariant components and k is the number of covariant components. For example, $H^2(\mathscr{L}(S)_0^+, M_0) \neq 0$ exactly in the following five cases:

$$M_0 \cong \langle R_{ab} : R_{ab} = -R_{ba} \rangle \qquad (0, 2),$$

$$M_0 \cong \langle R^a_{bcd} : R^a_{bcd} = R^a_{cbd}, R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0 \rangle \qquad (1, 3),$$

$$M_0 \subseteq \langle R^{ab}_{cdef} : R^{ab}_{cdef} = -R^{ab}_{efcd} = R^{ab}_{dcef} = R^{ab}_{cdfe} = R^{ba}_{cdef} \rangle \qquad (2, 4),$$

$$M_0 \subseteq \langle R^{ab}_{cdef} : R^{ab}_{cdef} = R^{ab}_{dcef} = R^{ab}_{cedf} = R^{ab}_{cdfe} = -R^{ba}_{cdef} \rangle \qquad (2, 4),$$

$$M_0 \subseteq \langle R^{ab}_{cdef} : R^{ab}_{cdef} = R^{ab}_{efcd} = -R^{ab}_{dcef} = -R^{ab}_{cdfe} = -R^{ba}_{cdef} \rangle \qquad (2, 4)$$

(here $a, b, c, d, e, f = 0, 1, 2, \ldots, n$)

We would like to draw the attention to some corollaries of our result. The one-sided hamiltonian algebra H_n^+ has only one nontrivial cocycle in the adjoint module, this cocycle is called the Moyal cocycle [17]. Corresponding local deformation can be prolongated to a global deformation and obtained deformation of Poisson bracket, called the Moyal bracket, is important in quantum mechanics. We prove that two-sided algebra $[H_n, H_n]$ has, in addition to the Moyal cocycle, $2n^2 + n$ more 2-cocycles in adjoint module. It is also interesting to note that Lie algebra of divergenceless vector fields on the 3-dimensional sphere S_2^+ has exactly one nontrivial 2-cocycle in the adjoint module:

$$\psi(u\partial_i, v\partial_j) = \varepsilon_{abc}\partial_a\partial_j(u)\,\partial_b\partial_i(v)\,\partial_c,$$

where ε_{abc} is Levi-Chivita tensor. It seems to be that prolongation question of local deformations to global deformations is not so simple. In hamiltonian case H_n, for example, some of local deformations have obstructions.

The calculation of $H^*(\mathscr{L}_0^+, M_0)$ may be useful for calculation of cohomologies or homologies of the nilpotent subalgebra $\mathscr{L}_1^+$ with coefficients in trivial module:

$$\bigoplus_{M_0} M_0 \otimes H^k(\mathscr{L}_0^+, L_0^+, M_0) \cong H^k(\mathscr{L}_1^+, \mathbb{C})$$

(here M_0 runs through all the irreducible L_0^+-modules). Note that for Cartan Type Lie algebras L the cohomologies $H^*(\mathscr{L}_1^+, \mathbb{C})$ was completely calculated only in case of $L = W_1$. [11]. At least eight proofs of this result are known, but it still remains one of the difficult results in cohomology theory of infinite-dimensional Lie algebras [8].

Another direction arising from our approach concerns defining relations of simple Lie algebras. As we mentioned above our problem is "almost" equivalent to the problem of calculating the second homologies of nilpotent subalgebra $\mathscr{L}_1^+$. The space $H^2(\mathscr{L}_1^+, \mathbb{C})$ can be interpreted as a space of defining relations of $\mathscr{L}_1^+$. Defining relations of classical Lie algebras were found by Serre [22]. Nonclassical simple Lie algebras (Cartan Type Lie algebras) have big nilpotent subalgebras $\mathscr{L}_1^+$ and $H^2(\mathscr{L}_1^+, \mathbb{C})$ constitutes the main part of their defining relations. Calculations of $H_2(\mathscr{L}_1^+, \mathbb{C})$ made jointly with Kerimbaev will be published elsewhere. Recall that 1-homologies of $\mathscr{L}_1^+$ correspond to generators of $\mathscr{L}_1^+$. In characteristic zero generators were found by Gelfand and its collaborators [8], in characteristic $p > 0$ it was calculated by Kostrikin and Shafarevich [13].

Many mathematicians and physicists were interested in generalizations of Virasoro algebras. Central extensions of two-sided Cartan type Lie algebras were described in [2]. In this paper proved that in the class of two-sided Cartan type Lie algebras only the following algebras have nonsplit central extensions: W_1 (Virasoro algebra), S_n and H_n. From a physical viewpoint the problem of describing nonsplit extensions of Lie algebras of vector fields by modules of tensor fields was interested also in [15, 14, 16]. For $L = W_n$ Larsson found two modules (differential 1-forms and 2-forms) having nontrivial 2-cocycles. For $L = W_1, W_2$ and H_1, second cohomolgies in all irreducible tensor modules was calculated in [4, 5] (actually in this paper the case of characteristic $p > 0$ was considered, but the results contain the charactersitic 0 case as $p \to \infty$). Nonsplit extensions of $L = W_1$ independently described also in [21]. Some cocycles for $L = W_n$ and $L = H_n$ was also found in [20, 12, 18]

2 Preliminaries

All vector spaces are considered over $\mathbb{C}$. For a set of vectors $\{u, v, \ldots\}$ we will denote by $\langle u, v, \ldots \rangle$ its linear span. If $\mathscr{A}$ is some statement, we will denote by $\delta(\mathscr{A})$ its Kroneker symbol: $\delta(\mathscr{A}) = 1$ if $\mathscr{A}$ is true, and $= 0$ if $\mathscr{A}$ is false. Usually $\delta(x = y)$ is denoted by $\delta_{x,y}$.

Let Γ_n, or more precisely Γ_I be a set of n-tuples $\{\alpha = (\ldots, \alpha_i, \ldots), \alpha_i \in \mathbb{Z}, i \in I\}$, where I is a set of indices and $n = |I|$ is the number of its elements. Let Γ_n^+ be the subset consisting of all α, such that $\alpha_i \geq 0$. Let $\mathbb{C}_I$ be the algebra of Laurent power series $\mathbb{C}[x_i^{\pm} : i \in I] = \langle x^\alpha = \prod_{i \in I} x_i^{\alpha_i} : \alpha \in \Gamma_I \rangle$. Instead of $\mathbb{C}_I$ we shall write $\mathbb{C}_{|I|}$ or simply U (I will be clear from the context). The Lie algebra of General Type W_n is defined as the algebra of derivations of U with the usual commutator:

$$[u\partial_i, v\partial_j] = u\partial_i(v)\,\partial_j - v\partial_j(u)\,\partial_i, I = \{1, \dots, n\}.$$

Let

$$\omega_S = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n, I = \{0, 1, \dots, n\}, \text{(volume form)}$$

$$\omega_H = \sum_{i=1}^{n} dx_{-i} \wedge dx_i, I = \{\pm 1, \dots, \pm n\}, \text{ (hamiltonian form)}$$

$$\omega_K = dx_0 + \sum_{i=1}^{n} dx_{-i} \wedge dx_i, I = \{0, \pm 1, \dots, \pm n\}, \text{(contact form)}$$

The Lie algebras of Special Type S_{n-1}, Hamiltonian Type H_n and Contact Type K_{n+1} are defined as the subalgebras of W_I saving the corresponding differential forms:

$$S_n = \{D \in W_{n+1}\colon D(\omega_H) = 0\},$$

$$H_n = \{D \in W_{2n}\colon D(\omega_H) = 0\},$$

$$K_{n+1} = \{D \in W_{2n+1}\colon D(\omega_K) = \omega_K\},$$

The Hamiltonian algebra can be defined as the vector space U with an even number variables with the Poisson bracket

$$\{u, v\} = \sum_{i}^{n} sgn\, i\, \partial_{-i}(u)\,\partial_i(v).$$

The Contact algebra is defined on vector space U with odd number varuables as follows:

$$[u, v] = \partial_0(u)\,\Delta(v) - \partial_0(v)\,\Delta(u) + \sum_{i}^{n} sgn\, i\, \partial_{-i}(u)\,\partial_i(v),$$

where, $\Delta(u) = (2 - \sum_{i \neq 0} x_i\partial_i)\,(u)$. In general these algebras are not simple. To obtain simple ones we take second commutators and factorize by center (in hamiltonian case). Lower indices denote dimensions of standard tori. We endow U and its derivation algebras with gradings:

$$U = \bigoplus_k U_k,\; L = \bigoplus_k L_k,\; U_k U_q \subseteq U_{k+q},\; [L_k, L_q] \subseteq L_{k+q}$$

$$U_k = \{x^\alpha\colon |\alpha| = \sum_{i \in I}^{n} \alpha_i = k\},$$

$$L = W_n, L_k = \{x^\alpha \partial_i\colon |\alpha| = k + 1, \},$$

$$L = H_n, L_k = \{x\alpha\colon |\alpha| = k + 2\},$$

$$L = K_{n+1}, L_k = \left\{x^\alpha\colon 2\alpha_0 + \sum_{i=1}^{n} \alpha_i = k + 2\right\}$$

The filtrations corresponding to these gradins are:

$$L = \bigcup_k \mathscr{L}_k, \quad \mathscr{L}_k = \bigoplus_{j \geqq k} L_j, \quad \mathscr{L}_k \supseteq \mathscr{L}_{k+1}, k \in Z$$

Here gradations and filtrations run through all positive as well as neagtive integers. This is why we call these algebras two-sided Cartan type Lie algebras. They have subalgebras that we call one-sided Cartan Type Lie algebras, because their gradations and filtrations run integers from -2 or -1 only to positive parts. Namely, for the algebra $U^+ = \mathbb{C}[[x_i : i \in I]]$ we introduce one-sided Cartan type Lie algebras as follows:

$$W_n^+ = Der\, U^+, \quad S_n^+ = W_n^+ \cap S_n,$$

H_n^+, K_{n+1}^+ are subalgebras of H_n, K_{n+1} spanned on U^+

Then for $L = W_n, S_n, H_n, K_{n+1}$ algebra L^+ has gradation:

$$L^+ = \bigoplus_{k \geqq -2} L_k^+, \quad L_k^+ = L^+ \cap L_k$$

and filtration

$$\mathscr{L}_k^+ = \bigoplus_{j \geqq k} L_j^+, \quad L = \mathscr{L}_{-2}^+ \subseteq \mathscr{L}_{-1}^+ \mathscr{L}_0^+ \mathscr{L}_1^+ \cdots$$

The subalgebra L_0^+ is isomorphic to $gl_n, sl_n, sp_n, sp_n \oplus \mathbb{C}$ for $L = W_n, S_n, H_n, K_{n+1}$ respectively.

Let $J(L)$ be a subalgebra L_{-1}^+, if $L = W_n, S_{n-1}$ and $L_{-1}^+ + L_{-2}^+$, if $L = H_n, K_{n+1}$.

3 Tensor modules

Let Q be a Lie algebra of derivations of an associative commutative algebra V. We say that M is a *(Q, V)-module* if M has structures of module over the Lie algebra Q and over the associative algebra V, such that

$$D(vm) = D(v)m + vD(m)$$

for any $D \in Q, v \in V$ and $m \in M$. For $L = W_n, S_n, H_n, K_n$, $Q = L, L^+$, $V = U, U^+$ denote by $\mathfrak{I}(V)$ the category of (Q, V)-modules. Let $\mathfrak{I}_0$ be the category of L_0^+-modules. Make any $M_0 \in \mathfrak{I}_0$ a module over $\mathscr{L}_0^+$ by $\mathscr{L}_1^+ M_0 = 0$. Define a Functor

$$\mathfrak{I}_0 \to \mathfrak{I}(V), \quad M_0 \mapsto V \otimes M_0$$

by the rule

$$D(v \otimes m) = D(v)\; m + \sum_a v E_a(D)\; \otimes a(m),$$

$$u(v \otimes m) = uv \otimes m, \forall u, v \in U, \forall D \in L, \forall m \in M_0,$$

where $E_a : Q \to V$ are the linear maps invariant under $L_1^+ + L_2^+$ constructed in [6] for any basic element $a \in \mathscr{L}_0^+$. We call the module $V \otimes M_0$ the *tensor Q-module with base* M_0. So, actions of L and its subalgebra L^+ on tensor modules are given by the formulas

$$L = W_n, (u\partial_i)\;(v \otimes m) = u\partial_i(v) \otimes m + \sum_s \partial_s(u) \otimes x_s \partial_i(m),$$

$$\begin{aligned} L = S_n, &(\partial_i(u)\partial_j - \partial_j(u)\partial_i)(v \otimes m) \\ &= (\partial_i(u)\,\partial_j(v) - \partial_j(u)\,\partial_i(v)) \otimes m \\ &\quad + \sum_s v\partial_s\partial_i(u) \otimes x_s\partial_j(m) - v\partial_s\partial_j(u) \otimes x_s\partial_i(m), \end{aligned}$$

$L = H_n$, $u(v \otimes m) = \{u, v\} \otimes m + (1/2) \sum_{i,j} u \partial_i \partial_j (v) \otimes x_i x_j (m)$,

$$L = K_{n+1},\ u(v \otimes m) = ([u, v] - 2\partial_0(u)v) \otimes m + v\partial_0(u) \otimes x_0(m) + (1/2) \sum_{i,j \neq 0} v(\partial_i + sgn\, i x_{-i}\partial_0)(\partial_j + sgn\, j x_{-j}\partial_0)(u) \otimes x_i x_j(m)$$

Notice that (L^+, U^+)-modules can be described in the language of induced or coinduced modules. For example, $U^+ \otimes M_0 \cong Ind(\mathscr{L}_0, M_0)$. But an (L, U)-module in general is not isomorphic either to an induced or a coinduced module.

For Cartan Type Lie algebra of toroidal dimension n,(L, U)-modul $M = U \otimes M_0$ has weight decomposition $M = \bigoplus_{\alpha \in \Gamma_n} M_\alpha$ such that for any weight space $M_\alpha = \{x^\alpha \otimes M_0\}$ we have $\dim M_\alpha = \dim M_0$. This property was used in [19] for constructing a W_n-module with weight subspaces of dimension n (take M_0 as Λ^1), n^2 (take M_0 as the adjoint module of gl_n), n^3, etc.

4 Cohomologies and interpretations

For a Lie algebra Q and a Q-module M denote by $C^k(Q, M)$ the space of skewsymmetric polylinear maps with k arguments in Q and coefficients in M, if $k > 0$, $C^0(Q, M) = M$, and $C^k(Q, M) = 0$, if $k < 0$. Let ϱ: $Q \to \mathrm{End} C^k(Q, M)$ be the standard representation of Q:

$$\varrho(D)\psi(D_1, \ldots, D_k) = D(\psi(D_1, \ldots, D_k)) + \sum_{i=1}^{k} (-1)^i \psi([D, D_i], D_1, \ldots, \hat{D}_i, \ldots, D_k)$$

and let $\iota: C^k(Q, M) \to C^{k-1}(Q, M)$ be the inner product:

$$\iota(D)\psi(D_1, \ldots, D_{k-1}) = \psi(D, D_1, \ldots, D_k)$$

(here $\hat{D}$ means that the element D is omitted). For a subalgebra R of Q denote by $C^k(Q, R, M)$ the subspace of $C^k(Q, M)$ consisting of the maps ψ such that $\iota(D)\psi = 0$ and $\varrho(D)\psi = 0$ for any $D \in R$. Let $C^*(Q, M) = \bigoplus_k C^k(Q, M)$. In the cochain complex $C^*(Q, M)$ the coboundary operator

$$d: C^k(Q, M) \to C^{k+1}(Q, M)$$

is defined by

$$d\psi(D_1, \ldots, D_{k+1}) = \sum_{i<j} \psi([D_i, D_j], D_1, \ldots, \hat{D}_i, \ldots, \hat{D}_j, \ldots, D_{k+1} + \sum_{i=1}^{k+1} (-1)^i D_i \psi(D_1, \ldots, \hat{D}_i, \ldots, D_{k+1})$$

Let

$Z^*(Q, M) = Ker\, d$ (space of cycles),

$B^*(Q, M) = Im\, d$ (space of coboundaries),

$H^*(Q, M) = Z^*(Q, M)/B^*(Q, M)$ (space of cohomologies).

For example,

$H^0(Q, M) = M^L = \{m \in M : D(m) = 0, \forall D \in Q\}$

(submodule of invariants),

$H^1(Q, \mathbb{C}) = Q/[Q, Q]$ (subspace of generators),

$H^1(Q, Q)$ (space of derivations),

$H^2(L, L)$ is the space of local deformations (see [10]).

For Cartan Type Lie algebra L and its nilpotent subalgebra $\mathscr{L}_1^+$ the space $H^2(\mathscr{L}_1^+, \mathbb{C})$ can be interpretered as a space of defining relations of $\mathscr{L}_1^+$ (see [8]). In next section we give an interpretation of $H^1(L, U)$, $H^1(L^+, U^+)$ as a spaces of deformations of representations.

For subalgebra R of Q relative cohomologies are defined as the cohomologies of subcomplex $C^*(Q, R, M) = \bigoplus_k C^k(Q, R, M)$.

Let L be a graded Lie algebra of Cartan Type: $L = \bigoplus_k L_k$ and $\mathscr{L}_0^+ = \bigoplus_{k>0} L_k^+$, M_0-$\mathscr{L}_0^+$-module such that $\mathscr{L}_1^+ M_0 = 0$. The cochain complex $C^*(\mathscr{L}_0^+, M_0)$ also has a gradation:

$$C^*(\mathscr{L}_0^+, M_0) = \bigoplus_{r \geq 0} C_r^*(\mathscr{L}_0^+, M_0),$$

$$C_r^*(\mathscr{L}_0^+, M_0) = \bigoplus_k C_r^k(\mathscr{L}_0^+, M_0),$$

$$C_r^k(\mathscr{L}_0^+, M_0) = \left\{\psi \in C^k(\mathscr{L}_0^+, M_0) : \psi(D_1, \ldots, D_k) = 0,\ D_i \in L_{k_i}, \sum_i k_i \neq k\right\}.$$

Denote by $H_r^*(\mathscr{L}_0^+, M_0)$ the cohomologies of the cochain subcomplex $C_r^*(\mathscr{L}_0^+, M_0)$.

For a Cartan Type Lie algebra L and $Q = L, L^+$, $V = U, U^+$ the following map induces isomorphism of cochain complexes:

$$E: C^k(\mathscr{L}_0^+, M_0) \to C^k(Q, J(L), V \otimes M_0),$$

$$E\psi(D_1, \ldots, D_k) = \sum_{a_1, \ldots, a_k} E_{a_1}(D_1) \cdots E_{a_k}(D_k) \otimes \psi(a_1, \ldots, a_k)$$

here $a_1, \ldots, a_k$ runs through the basic vectors of $\mathscr{L}_0^+$.

5 Deformations of (Q, V)-modules

Let M be an (Q, V)-module with action $Q \times M \to M$, $(D, m) \mapsto D(m)$. Let M be exact as a V-module: $u(m) = 0, \forall m \in M \Rightarrow u = 0$. Define on $M_\lambda = M \otimes \mathbb{C}\{\lambda\}$ following Gerstenhaber a new structure of (Q, V) -module with action of Q given by a power series

$$D_\lambda(m) = D(m) + \lambda f_1(D)m + \lambda^2 f_2(D)m + \cdots.$$

We do not change the action of V. We shall say that M_λ is a deformation of the (Q, V)-module M and denote it by $f = (f_1, f_2, \ldots)$, if these actions really give (Q, V) -module structures. Deformations **f** and **g** are *equivalent* if the following diagramm is commutative

$$\begin{array}{ccc} M_\lambda & \xrightarrow{f} & M_\lambda \\ \downarrow\Phi & & \downarrow\Phi \\ M_\lambda & \xrightarrow{g} & M_\lambda \end{array}$$

under with

$$\Phi: M_\lambda \to M_\lambda, m \mapsto m + \lambda\phi_1(m) + \lambda^2\phi_2(m) + \cdots,$$

where $\phi_1, \phi_2, \dots$ are linear operators on M. The condition that $D \to D_\lambda$ is a representation of a Lie algebra is equivalent to

$f_k \in Z^1(Q, V), \forall k > 0.$
In particular,

$$f_1 \in Z^1(Q, V)$$

and for equivalent deformations **f**,**g**,

$$f_1 - g_1 = d\phi \in B^1(Q, V).$$

Moreover any cocycle $f \in Z^1(Q, V)$ as a local deformation $f_1 = f$ can be prolongated to global deformation **f** in a trivial way: $f = (f_1, 0, \dots)$ is a deformation of M. So, we give an interpretation of the first space of cohomology $H^1(Q, V)$ as a space of deformations of (Q, V)-modules.

It is easy to check that for a Cartan Type Lie algebra Q (one-sided or two-sided does not matter) and a two-sided module U the 1-cochains $Sq_i \in C^1(Q, U)$, $i \in I(L)$ defined by

$$Sq_i(D) = D(\ln x_i) : D \mapsto x_i^{-1} D(x_i),$$

are cocycles. The same is true for the following cochains in $C^1(Q, V)$:

$$Div: u\partial_i \mapsto \partial_i(u),$$

for $Q = W_n, W_n^+, V = U, U^+$,

$$\Delta: u \mapsto \left(2 - \sum_i x_i\partial_i\right)(u), D_\theta: x^\alpha \mapsto \delta_{\alpha=-\theta},$$

for $L = H_n, H_n^+, V = U, U^+$, (here $\theta = (1, \dots, 1) \in \Gamma_{2n}$)

$$\partial_0: u \mapsto \partial_0(u),$$

for $L = K_{n+1}$.

Theorem 5.1. *Let* $Q = L, L^+$, *where* $L = W_n, S_{n-1}, H_n, K_{n+1}$, *and* $V = U, U^+$, *where* $U = \mathbb{C}[[x_i^{\pm} : i \in I = I(L)]]$. Then

$$H^1(L, U) \cong H^1(L^+, U) \cong H^1(J(L), U) \oplus H^1(\mathscr{L}_0, \mathbb{C}),$$

$$H^1(L^+, U^+) \cong H^1(J(L), U^+) \oplus L_0^+/[L_0^+, L_0^+],$$

$$H^1(J(L), V) \cong H^1(\Omega(V)) \cong \mathbb{C}^{|I(L)|},$$

if $L \neq H_n$, *and*

$$H^1(J(L), V) \cong H^1(\Omega(V)) \oplus \langle\Delta\rangle \cong \mathbb{C}^{2n+1},$$

if $L = H_n$ *where*

$$H^1(\Omega(U)) \cong \langle d(\ln x_i) : i \in I(L)\rangle \cong \langle Sq_i : i \in I(L)\rangle, H^1(\Omega(U^+) = 0,$$

$H^1(\mathscr{L}_0^+, \mathbb{C}) \cong \langle Div\rangle \cong \mathbb{C}$, *if* $L = W_n$, $\cong \langle D_\theta\rangle \cong \mathbb{C}$, *if* $L = H_n$, *and* $\cong \langle\partial_0\rangle \cong \mathbb{C}$, *if* $L = K_{n+1}$.

Let $\Lambda^k = \langle dx_{i_1} \wedge \dots \wedge dx_{i_k} : i_1, \dots, i_k \in I\rangle$ be the L_0^+-module of k-differential forms, $\Lambda^* = \bigoplus_k \Lambda^k$, and $\Omega^k(V) = V \otimes \Lambda^k$, $\Omega^* = \oplus\,\Omega^k(V)$. Endow $\Omega^*(V)$ with a coboundary operator:

$$\Omega^k(V) \to \Omega^{k+1}(V),$$

$$d(v \otimes dx_{i_1} \wedge dx_{i_k}) = \sum_{i \in I} \partial_i(v) \otimes dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

In this way we obtain the de Rham complex $\Omega^*(V)$ with coefficients in V. Let, as usual,

$Z^*(\Omega(V)) = Ker d$ (space of closed forms)

$B^*(\Omega(V)) = Im\, d$ (space of exact forms)

and,

$H^*(\Omega(V)) = Z^*(\Omega(V))/B^*(\Omega(V))$ (de Rham cohomologies).

Proposition 5.2. $H^*(U) \cong \Lambda^* = \langle d(\ln x_{i_1}) \wedge \cdots \wedge d(\ln x_{i_k}) : i_1, \dots, i_k \in I\rangle$, $H^*(U^+) \cong 0$.

In particular,

$$\langle Sq_i : i \in I\rangle \cong H^1(\Omega(U)), Sq_i \mapsto d(\ln x_i)\rangle.$$

Recall that any irreducible L_0^+-module M_0 is uniquely determined by its highest weight π, and conformal weight λ (if L_0^+ is simple, then $\lambda = 0$). L_0^+ has 1-dimensional center in the following cases:

$$L = W_n, L_0^+ = \langle x_i\partial_j : i, j = 1, \dots, n\rangle \cong gl_n,$$

$$(x_i\partial j)_\lambda(m) = x_i\partial_j(m) + (1 - \lambda)\delta(i = j)m, m \in M_0$$

and

$$L = K_{n+1},$$

$$L_0^+ = \langle x_i x_j : i, j = \pm 1, \dots, \pm n\rangle \oplus \langle x_0\rangle \cong sp_n \oplus \mathbb{C},$$

$$u_\lambda(m) = u(m) + (\lambda - 2)\partial_0(u)m, u \in L_0^+, m \in M_0).$$

We will write $M = R(\pi)$ or $M = R(\pi, \lambda)$. Denote by π_i, $1 \le i \le \dim T$, the fundamental weights of L_0^+. Any highest weight can be represented as a linear combination of fundamental weights with nonnegative integer coefficients $\pi = \sum_i l_i\pi_i$. It is well known that any irreducible module $M_0 = R(\pi)$ is isomorphic to a submodule (or factor-module) of some module $\hat{M}_0$, obtained from tensor ($\otimes$), exterior ($\wedge$) and symmetric ($\circ$) products of fundamental representations $R(\pi_i)$:

$$\hat{M}_0 = \bigotimes_i S^{l_i}(\Lambda^i).$$

For tensors a and b the following notations are used: $a \wedge b = a \otimes b - b \otimes a$, $a \circ b = a \otimes b + b \otimes a$. Let $\Lambda^k = \langle dx_{i_1} \wedge \cdots \wedge dx_{i_k}\rangle$ be the k-th exterier power and $S^k = \langle dx_{i_1} \circ \cdots \circ dx_{i_k}\rangle$ be the k-th symmetric power, here

Table 1. $H^2(\mathscr{L}_0^+, M_0), L = W_n$

highest weight of M_0 and $\hat{M}_0$	$\dim M_0, l = 1, r = 2$, if otherwise is not mentioned	cocycles $\psi: L_k^+ \wedge L_q^+ \to \hat{M}_0$, $\psi(u\partial_i, v\partial_j), 0 \le k \le q$, $k + q = r$
$\pi_1, \lambda = 1$, $M_0 \cong \Lambda^1$	$n, r = 1$	ψ_1^W: $\partial_i(u)\ d\partial_j(v) - \partial_j(v)\ d\partial_i(u)$
$2\pi_1 + \pi_{n-1}$, $\lambda = 2, n > 1$, $\hat{M}_0 \cong S^2 \otimes L_{-1}^+$	$(n+2)\binom{n}{2}$	ψ_2^W: $\partial_i(u)\ v\partial_j - \partial_j(v)\ u\partial_i$
$\pi_2, \lambda = 1$, $n > 1, M_0 \cong \Lambda^2$	$\binom{n}{2}$ $l = 2$	ψ_3^W: $d\partial_i(u) \wedge d\partial_j(v)$; ψ_4^W: $d\partial_j(u) \wedge d\partial_i(v)$
$\pi_1 + \pi_2 + \pi_{n-1}$, $\lambda = 2, n > 2$ $\hat{M}_0 \cong \Lambda^1 \otimes \Lambda^2 \otimes L_{-1}^+$	$(n+2)\ n^2(n-2)\ /3$ $l = 2$	$\psi_5^W = \hat{\psi}_2^S$: $du \wedge d\partial_i(v)\ \partial_j - dv \wedge d\partial_j(u)\ \partial_i$ ψ_6^W: $d\partial_i(u) \wedge dv\partial_j - d\partial_j(v) \wedge du\partial_i$
$3\pi_1 + \pi_{n-1}$, $\lambda = 2, n > 1$, $\hat{M}_0 \cong S^2 \otimes \Lambda^1 \otimes L_{-1}^+$	$(n+3)\binom{n+1}{3}$	ψ_7^W: $d(\partial_i(u)\ v)\ \partial_j - d(\partial_j(v)\ u)\ \partial_i$
$2\pi_1 + \pi_2 + 2\pi_{n-1}$, $\lambda = 3, n > 2$, $\hat{M}_0 \cong S^2 \otimes \Lambda^2 \otimes S^2(L_{-1}^+)$	$\frac{3}{2}(n+4)\ (n+1)\binom{n+1}{4}$	ψ_8^W: $du \wedge dv\partial_i\partial_j$
$4\pi_1 + \pi_{n-2}$, $\lambda = 2, n > 2$, $\hat{M}_0 \cong S^3 \otimes \Lambda^1 \otimes \Lambda^2(L_{-1}^+)$	$\binom{n+1}{4}\binom{n+1}{4}$	$\psi_9^W = \hat{\psi}_4^S$: $d(uv)\ \partial_i \wedge \partial_j$
$2\pi_2 + \pi_{n-2}$, $\lambda = 2, n > 3$, $\hat{M}_0 \cong S^2(\Lambda^2) \otimes \Lambda^2(L_{-1}^+)$	$\frac{n+1}{2}\binom{n-2}{2}\binom{n+2}{3}$	$\psi_{10}^W = \hat{\psi}_5^S$: $\sum_{s,t}(d\partial_s(u) \wedge d\partial_t(v))$ $\circ(dx_s \wedge dx_t) \otimes (\partial_i \wedge \partial_j)$
π_1, $\lambda = 0, n = 2$	2 $l = 2, r = 3$	$\psi_{11}^W = \hat{\psi}_6^H$: $d\partial_j(u) \wedge d\partial_i(v)$ ψ_{12}^W: $d\{\partial_i(u), \partial_j(v)\}$
$5\pi_1$, $\lambda = 2, n = 2$	$6, r = 3$	ψ_{13}^W: $3\{\overline{pr_1}, \overline{pr_2}\} - 7\overline{pr_1} \wedge \widetilde{\overline{pr_2}}$
$7\pi_1$, $\lambda = 3, n = 2$	$8, r = 3$	$\psi_{14}^W = \hat{\psi}_7^H$: $\overline{pr_1} \wedge \overline{pr_2}$
$0, \lambda = -1, n = 1$	$1, r = 2$	$e_0 \wedge e_2 \mapsto 1$
$0, \lambda = -4, n = 1$	$1, r = 5$	$e_2 \wedge e_3 \mapsto 1$
$0, \lambda = -6, n = 1$	$1, r = 7$	$e_2 \wedge e_5 \mapsto 1, e_3 \wedge e_4 \mapsto -3$

Table 2. $H^2(\mathscr{L}_0^+, M_0), L = S_{n-1}, n > 2$

heighst weight of M_0 and $\hat{M}_0$	$\dim M_0, l = 1, r = 2$	cocycles ψ: $L_1^+ \wedge L_1^+ \to \hat{M}_0$, its values $\psi(u\partial_i, v\partial_j)$
$\pi_2, M_0 \cong \Lambda^2$	$\binom{n}{2}$	ψ_1^S: $d\partial_j(u) \wedge d\partial_i(v)$
$\pi_1 + \pi_2 + \pi_{n-1}$, $\hat{M}_0 \cong \Lambda^1 \otimes \Lambda^2 \otimes L_{-1}^+$	$\frac{(n+2)n^2(n-2)}{3}$	ψ_2^S: $du \wedge d\partial_i(v)\partial_j - dv \wedge d\partial_j(u)\partial_i$
$2\pi_1 + \pi_2 + 2\pi_{n-1}$, $\hat{M}_0 \cong S^2 \otimes \Lambda^2 \otimes S^2(L_{-1}^+)$	$\frac{3}{2}(n+4)(n+1)\binom{n+1}{4}$	ψ_3^S: $du \wedge dv\partial_i\partial_j$
$4\pi_1 + \pi_{n-2}$, $\hat{M}_0 \cong S^4 \otimes \Lambda^2(L_{-1}^+)$	$\binom{n+4}{2}\binom{n+1}{4}$	ψ_4^S: $d(uv)\ \partial_i \wedge \partial_j$
$2\pi_2 + \pi_{n-2}, n > 3$, $\hat{M}_0 \cong S^2(\Lambda^2) \otimes S^2(L_{-1}^+)$	$\frac{n+1}{2}\binom{n-2}{2}\binom{n+2}{3}$	ψ_5^S: $\sum_{s,t}(d\partial_s(u) \wedge d\partial_t(v))$ $\circ(dx_s \wedge dx_t) \otimes (\partial_i \wedge \partial_j)$

Table 3. $H^1(\mathscr{L}_0^+, M_0)$

L	weight	$\dim M_0$	r	l	cocycles
W_n	$0, \lambda = 1$	1	0	1	$Div: L_0^+ \to R(0) \cong \mathbb{C}$
$n > 1$	$\pi_1, \lambda = 1$	n	1	1	$Div: L_1^+ \to R(\pi_1) \cong (\mathbb{C}_n)_1^+$
	$2\pi_1 + \pi_{n-1}$ $\lambda = 2$	$\binom{n+2}{3}$	1	1	$\overline{pr_1}: L_1^+ \to R(2\pi_1 + \pi_{n-1}) \cong (S_{n-1})_1^+$
W_1	$0, \lambda = 1$	1	0	1	$e_0 \mapsto 1$
	$0, \lambda = 0$	1	1	1	$e_1 \mapsto 1$
	$0, \lambda = -1$	1	2	1	$e_2 \mapsto 1$
S_{n-1} $n > 2$	$2\pi_1 + \pi_{n-1}$	$\binom{n+2}{3}$	1	1	$pr_1: L_1^+ \to R(2\pi_1 + \pi_{n-1}) \cong L_1^+$
H_n	$3\pi_1$	$\binom{2n+2}{3}$	1	1	$pr_1: L_1^+ \to R(3\pi_1) \cong (\mathbb{C}_{2n})_3^+$
K_{n+1}	$3\pi_1, \lambda = 2$	$\binom{n+2}{3}$	1	1	$pr_1: L_1^+ \to R(3\pi_1) \cong (\mathbb{C}_{2n})_3^+$
	$\pi_1, \lambda = 0$	$2n + 1$	1	1	$\partial_0: L_1^+ \to R(\pi_1) \cong (\mathbb{C}_{2n})_1^+$
	$0, \lambda = 0$	1	0	1	$\partial_0: L_0^+ \to \mathbb{C}$

Table 4. $H^2(\mathscr{L}_0^+, M_0), L = H_n$

highest weight of M_0 and $\hat{M}_0$	$\dim M_0, l = 1, r = 2,$ if otherwise is not mentioned	cocycles $\psi: L_k^+ \wedge L_q^+ \to \hat{M}_0$ and $\psi(u \wedge v), 0 < k \leqq q, k + q = r$, nonwritten components are zero
$0, M_0 \cong \mathbb{C}$,	1	$\psi_1^H = \mu$
π_2, $\hat{M}_0 \cong \Lambda^2$, $n > 1$	$(n-1)(2n+1)$	ψ_2^H: $\sum_{i,j} - d\partial_{-i}\partial_{-j}(u) \wedge d\partial_i\partial_j(v) + 2d\partial_{-j}\partial_i(u) \wedge d\partial_{-i}\partial_j(v) - d\partial_i\partial_j(u) \wedge d\partial_{-i}\partial_{-j}(v)$
$2\pi_2, n > 1$, $\hat{M}_0 \cong S^2(\Lambda^2)$	$\frac{2(2n+3)(2n-1)}{3}\binom{n}{2}$	ψ_3^H: $\sum_{i,j}\sum_{s=1}^n (d\partial_i\partial_s(u) \wedge d\partial_j\partial_{-s}(v) - d\partial_i\partial_{-s}(u) \wedge d\partial_j\partial_s(v)) \circ dx_i \wedge dx_j$,
$3\pi_2, n > 1$, $\hat{M}_0 \cong S^3(\Lambda^2)$	$\frac{(2n+5)(2n+1)(2n-1)n}{9}\binom{n}{2}$	ψ_4^H: $\sum_{i,j,s,t}(d\partial_s\partial_i(u) \wedge d\partial_t\partial_j(v)) \circ (dx_s \wedge dx_t) \circ (dx_i \wedge dx_j)$,
$4\pi_1 + \pi_2$, $\hat{M}_0 \cong S^4 \otimes \Lambda^2$, $n > 1$	$\frac{(2n+5)(2n+3)(2n+1)}{3}\binom{n+1}{3}$	$\psi_5^H: du \wedge dv$
π_1, $M_0 \cong \Lambda^1$	$2n, r = 3$	ψ_6^H: $pr_1\mu(u \wedge v)$,
$7\pi_1, n = 1$, $M_0 \cong (\mathbb{C}_2)_7^+$	$8, r = 3$	ψ_7^H: $uv, u \in L_1^+, v \in L_2^+$
$2\pi_1, n = 1$, $M_0 \cong (\mathbb{C}_2)_2^+$	$3, r = 4$	$\psi_8^H: pr_2(6\mu + \mu_1)(u \wedge v)$

$i_1, \ldots, i_k \in \{1, \ldots, n\}$, if $L = W_n, S_{n-1}$ and $i_1, \ldots, i_k \in \{\pm 1, \ldots, \pm n\}$, if $L = H_n, K_{n+1}$.

It is easy to see that

$$H_2^2(\mathscr{L}_0^+, L_0^+, M_0) \cong C^2(L_1^+, M_0)^{L_0^+}$$

Since the commutators of 0-components of W_n^+, S_{n-1}^+ and H_n^+, K_{n+1}^+ coincide, and the natural imbeddings of 1-components

$$L_1^+(S) = (S_{n-1}^+)_1 \hookrightarrow L_1^+(W) = (W_n^+)_1,$$

$$L_1^+(H) = (H_n)_1^+ \hookrightarrow L_1^+(K) = (K_{n+1}^+)_1$$

are $[L_0^+, L_0^+]$-module monomorphisms, we can give

Table 5. $H^2(\mathscr{L}_0^+, L_0^+, M_0), L = K_{n+1}$

highest weight of M_0 and $\hat{M}_0$	$\dim M_0$, $l = 1, r = 2$, if otherwise is not mentioned	cocycles ψ and its values $\psi(u \wedge v)$, $u \in L_k^+$, $v \in L_q^+$ $0 < k \le q$, $k + q = r$
$0, \lambda = -2$, $M_0 \cong \mathbb{C}$	1	$\psi_1^K = \hat{\psi}_1^H$
$\pi_2, \lambda = -2$, $\hat{M}_0 \cong \Lambda^2, n > 1$	$(n-1)(2n+1)$ $l = 2$	$\psi_2^K = \hat{\psi}_2^H$; ψ_3^K: $d\partial_0(u) \wedge d\partial_0(v)$
$2\pi_2, \lambda = -4$, $\hat{M}_0 \cong S^2(\Lambda^2), n > 1$	$\frac{2}{3}(2n+3)(2n-1)\binom{n}{2}$	$\psi_4^K = \hat{\psi}_3^H$;
$3\pi_2, \lambda = -6$, $\hat{M}_0 \cong S^3(\Lambda^2), n > 1$	$\frac{(2n+5)(2n+1)(2n-1)n}{9}\binom{n}{2}$	$\psi_5^K = \hat{\psi}_4^H$
$4\pi_1 + \pi_2, \lambda = 0$, $\hat{M}_0 \cong S^4(\Lambda^1) \otimes \Lambda^2, n > 1$	$\frac{(2n+5)(2n+3)(2n+1)}{3}\binom{n+1}{3}$	$\psi_6^K = \hat{\psi}_5^H$
$2\pi_1 + \pi_2$, $\hat{M}_0 \cong S^2 \otimes \Lambda^2$, $\lambda = -2, n > 1$	$(2n+3)(2n+1)\binom{n}{2}$	$\psi_7^K : \partial_0(du \wedge dv)$
$4\pi_1, \lambda = 2$, $\hat{M}_0 \cong S^4$	$\frac{(2n+3)(2n+1)}{3}\binom{n+1}{2}$	ψ_8^K: $d(u\partial_0(v) - v\partial_0(u))$
$\pi_1, \lambda = -2$, $M_0 \cong (\mathbb{C}_2)_1^+$, $n = 1$	2 $l = 2$, $r = 3$	$\psi_9^K = \hat{\psi}_6^H$(prolongation see below) ψ_{10}^K: $x_0^2 \wedge x_0 x_i \mapsto x_i$,
$3\pi_1, \lambda = 0$, $\hat{M}_0 \cong (\mathbb{C}_2)_3^+$ $n = 1$	$4, r = 3$	$\psi_{11}^K : \partial_0(u)\partial_0(v)$ $+ 3u\partial_0^2(v), u \in L_1^+, v \in L_2^+$
$5\pi_1, \lambda = 2$, $M_0 \cong (\mathbb{C}_2)_5^+$ $n = 1$	$6, r = 3$	ψ_{12}^K: $3u\partial_0(v) + \partial_0(u)v$ $u \in L_1^+, v \in L_2^+$
$7\pi_1, \lambda = 0$, $M_0 \cong (\mathbb{C}_2)_7^+$ n = 1	$8, r = 3$	$\psi_{13}^K = \hat{\psi}_7^H$

Prolongation of $\psi_6^H : \psi_9^K(u \wedge x_0 a) = 5(\sum_{i,j=\pm 1} sgn(ij)\,\partial_i \partial_j(a)\,\partial_{-i}\partial_{-j}(u))$, where $u \in L_1^+$, $v = x_0 a \in L_2^+$.

imbeddings

$$H_2^2(\mathscr{L}(S)_0^+, M_0) \hookrightarrow H_2^2(\mathscr{L}(W)_0^+, M_0),$$

$$H_2^2(\mathscr{L}(H)_0^+, M_0) \hookrightarrow H_2^2(\mathscr{L}_0^+, M_0).$$

So, if ψ is a cocycle from $Z_2^2(\mathscr{L}(S)_0^+, M_0)$ or $Z_2^2(\mathscr{L}(H)_0^+, M_0)$, then it has a trivial prolongation to $Z_2^2(\mathscr{L}(W)_0, M_0)$ or $Z_2^2(\mathscr{L}(K)_0, M_0)$, correspondingly. Denote it by $\hat{\psi}$. This denotion we will reserve also for cocycles from $Z_r^2(\mathscr{L}(S)_0^+, M_0)$, $Z_r^2(\mathscr{L}(H)_0^+, M_0)$, $r > 2$, that have prolongations to cocycles from $Z_r^2(\mathscr{L}(W)_0^+, M_0)$, $Z_r^2(\mathscr{L}(K)_0^+, M_0)$. Notice that $S_1 \cong H_1$ and $sp_1 \cong sl_2$, but we consider S_{n-1} from $n > 2$, thus in constructing cocycles for W_2 we use cocycles for H_1.

6 Main results

Theorem 6.1. *Let* $L = W_n, S_{n-1}, H_n, K_{n+1}$, $Q = L, L^+$, $(Q, V) = (L, U), (L^+, U), (L^+, U^+)$ *and* M_0 *be irreducible* $\mathscr{L}_0^+$*-module, such that* $\mathscr{L}_1^+ M_0 = 0$. *Then for* $k = 0, 1, 2$,

$$H^k(Q, V \otimes M_0) \cong \bigoplus_{s=0}^{k} H^s(J(L), V) \otimes H^{k-s}(\mathscr{L}_0^+, M_0),$$

$$H^k(\mathscr{L}_0^+, M_0) \cong L_0^+/[L_0^+, L_0^+] \otimes (H^{k-1}(\mathscr{L}_1^+, \mathbb{C}) \otimes M_0)^{L_0^+}$$
$$\oplus\, (H^k(\mathscr{L}_1^+, \mathbb{C}) \otimes M_0)^{L_0^+},$$

$$H^k(J(L), U) \cong \Lambda^k(J(L)),\ H^k(J(L), U^+) \cong \mathbb{C}^{\delta(L = H_n)}.$$

$H^1(Q, U \otimes M_0)$ has a basis consisting of classes of cocycles $\delta(M_0 \cong \mathbb{C}) Sq_i$, $i \in I$, $\delta(L = H_n)\delta(M_0 \cong \mathbb{C})\Delta$, and $E\psi(1)$. In $H^2(Q, U \otimes M_0)$ one can choose a basis consisting of classes of cocycles $E(\psi(2))$, $Sq_i \wedge E(\psi(1))$, $\delta(L = H_n)\Delta \wedge E(\psi(1))$, $\delta(M_0 = \mathbb{C}) Sq_i \wedge Sq_j$. Here $\psi(s)$ denotes the basic cocycles of $H^s(\mathscr{L}_0^+, M_0)$, $s = 0, 1, 2$. The space $H^2(L^+, U^+ \otimes M_0)$ has a basis consisting of classes of cocycles $E(\psi(2))$, $\delta(L = H_n)\Delta \wedge E(\psi(1))$. Nonzero $H^k(\mathscr{L}_0^+, M_0)$, $k = 1, 2$ see Tables 1–5.

In the tables we use some special denotions:

$L = W_2$. $\{u, v\}$ denote the Poisson bracket in subalgebra H_1 of L, $\overline{L_i}$ is the L_0-submodule of L_i generated by derivations without divergence and $\widetilde{L_i}$ is its additional submodule, $\overline{pr_i}: L_i \to \overline{L_i}$, $\widetilde{pr}_i$: $L_i \to \widetilde{L_i}$ are natural projections;

$L = W_1 . e_i = x^{i+1}\partial_1$;

$L = H_n . pr_i$: $U \to U_i$ is natural projection, $\mu = (\sum_{i=1}^{n} \partial_{-i} \wedge \partial_i)^3$ is the Moyal cocyle, μ_1 is the restriction of μ to the subalgebra $\mathscr{L}_2^+$, prolongated trivially to $\mathscr{L}_1^+$, i.e. $\mu_1(L_1^+, \mathscr{L}_1^+) = 0$;

Corollary 6.2. *Let* $L = W_n, S_{n-1}, n > 2, H_n, K_{n+1}$. *Then* $H^2(L, L) = H^2(L^+, L^+) = 0$, *except the following cases*: $L = S_n, H_n$. *Moreover,*

$$L = S_2, H^2(L^+, L^+)$$
$$\cong \mathbb{C} \cong \langle u\partial_i \wedge v\partial_j \mapsto \sum_{\{abc\}=\{0,1,2\}} sgn\binom{abc}{012} \partial_a \partial_j(u) \partial_b \partial_i(v) \partial_c \rangle,$$

$L = H_n$, $Q = L, L^+$, $H^2(Q, Q) \cong H^1(Q, Q) \wedge H^1(Q, Q) \oplus \langle \mu \rangle$, $H^1(L, L) \cong H^1(L, U) \cong \langle Sq_i, \Delta, D_\theta \rangle$, $H^1(L^+, L^+) \cong H^1(L^+, U^+) \cong \langle \Delta, D_\theta \rangle$.

Acknowledgements. The Author is deeply grateful to the Alexander von Humboldt and INTAS foundations for support and to prof. B. Pareigis for hospitality and to prof. A. I. Kostrikin, P. Schauenburg, M. Gerstenhaber, J. Wess, Yu. I. Manin for interest to results.

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