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NOVIKOV-JORDAN ALGEBRAS

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ABSTRACT

Algebras with the identity $(a \star b) \star (c \star d) - (a \star d) \star (c \star b) = (a, b, c) \star d - (a, d, c) \star b$ are studied. Novikov algebras under Jordan multiplication and Leibniz dual algebras satisfy this identity. If algebra with such identity has a unit, then it is associative and commutative.

Key Words: Novikov algebras; Jordan algebras; Leibniz dual algebras; Polynomial identities

1. INTRODUCTION

Let $(A, \circ)$ be an algebra with multiplication $A \times A \to A$, $(a, b) \mapsto a \circ b$ and algebra $A^+ = (A, \{\ ,\ \})$ be its Jordan algebra under a product $\{a, b\} = a \circ b + b \circ a$. The algebra A^+ satisfies the commutativity identity

$$\{a, b\} = \{b, a\}.$$

Let $\mathcal{V}$ be a variety of algebras. Denote by $\mathcal{V}^+$ a variety generated by algebras A^+, where $A \in \mathcal{V}$. One can ask about a minimal identity of minimal

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degree for $\mathcal{V}^+$ that does not follow from the commutativity identity. If such identity exists call it *Jordan identity*.

Let *Ass* be a class of associative algebras. Well known, that Jordan identity for Ass^+ is the following identity of degree 4

$$\{\{a,b\},\{c,d\}\} + \{\{a,c\},\{d,b\}\} + \{\{a,d\},\{b,c\}\} \\ - \{\{b,c\},a\},d\}\} - \{\{c,d\},a\},b\}\} - \{\{d,b\},a\},c\} = 0.$$

An algebra $(A,\circ)$ with the identity

$$a\circ(b\circ c) - (a\circ b)\circ c = a\circ(c\circ b) - (a\circ c)\circ b$$

is called *right-symmetric*. This identity can be written in terms of associators $(a,b,c) = a\circ(b\circ c) - (a\circ b)\circ c$, as follows

$$(a,b,c) = (a,c,b).$$

That is the reason why this identity is called right-symmetric. Right-symmetric algebras were introduced around 1960's in[1–3].

A right-symmetric algebra A is called (right) *Novikov*,[4,5] if the following *left-commutativity* identity is true

$$a\circ(b\circ c) = b\circ(a\circ c).$$

In fact right-symmetric algebras were considered about 100 years before in[6] and Novikov algebras appeared first in[7]. Right-symmetric algebras are Lie-admissible. Any right-symmetric algebra under the commutator $[a,b] = a\circ b - b\circ a$ is a Lie algebra.

An algebra with a multiplication $A\times A\to A$, $(a,b)\mapsto a\times b$ and the identity

$$(a\times b)\times c = a\times(b\times c + c\times b) \tag{1}$$

is called (left) Leibniz dual.[8] Notice that any Leibniz dual algebra is right-commutative

$$(a\times b)\times c = (a\times c)\times b. \tag{2}$$

For these algebras one can define their right and left versions under the opposite multiplication $(a,b)\mapsto b\circ a$. In our paper we consider mainly right symmetric and right Novikov algebras. If otherwise is not stated the names Leibniz dual and Novikov mean left Leibniz dual and right Novikov correspondingly.

We consider algebras with the identity of degree 4

$$(a \star b) \star (c \star d) - (a \star d) \star (c \star b) = (a, b, c) \star d - (a, d, c) \star b. \qquad (3)$$

Here $(a, b) \mapsto a \star b$ is a multiplication of algebra and $(a, b, c) = a \star (b \star c) - (a \star b) \star c$ is an associator for the multiplication $\star$. Call this identity a (right) *Tortken* identity. Call any algebra with identity (5) (right) *Tortken* algebra. In kazak "tort" or "tirt" means four.

Examples of Tortken algebras.

1. Polynomial algebra $K[x]$ under the multiplication

   $$(a, b) \mapsto \partial(ab).$$

2. Polynomial algebra $K[x]$ under the multiplication

   $$(a, b) \mapsto a \int_0^x b\, dx.$$

3. Divided power algebra $O_1(m)$ under the multiplication

   $$(a, b) \mapsto \partial(\partial^{p-1}(a)\partial^{p-1}(b))$$

   if $\operatorname{char} K = p > 0$.
4. Divided power algebra $O_1(m)$ under the multiplication

   $$(a, b) \mapsto \partial^{2^k+1}(\partial^{2^k-1}(a)\partial^{2^k-1}(b))$$

   if $p = 2$ and $k > 0$.

Let $\mathcal{N}$ be a class of Novikov algebras. Call A^+ *Novikov-Jordan* algebra if A is Novikov. We show that Jordan identity for $\mathcal{N}^+$ is Tortken. We prove that some Osborn algebras under Jordan product are simple. We establish that Leibniz dual algebras are also Tortken. These algebras are not commutative, but they are right-commutative. So, in general the class of Tortken algebras does not coincide with the class of Novikov-Jordan algebras.

One can consider the left Tortken identity

$$(a \star b) \star (c \star d) - (c \star b) \star (a \star d) = -a \star (b, c, d) + c \star (b, a, d).$$

If $(A, \star)$ is a right Tortken, then $(A, \bar{\star})$, where $a \,\bar{\star}\, b = b \star a$ is a left Tortken. For commutative Tortken algebras, like Novikov-Jordan algebras, left and right Tortken identities are coincide. In general they are different identities.

We prove that Tortken algebras with unit are associative and commutative. In[9–13]algebras with identity of degree 3 and 4 are classified. Their classification was done under the condition that algebras have a unit. Therefore, the class of algebras with identity (5) is not in their list.

Let $[a,b,c]=(a,b,c)-(a,c,b)$ be a deviation of associators. In[14] is proved that many known classes of algebras (Lie, Jordan, right-symmetric algebras, LT-algebras) are satisfy the identity

$$[a*b,c,d]=a*[b,c,d]+[a,c,d]*b.$$

Novikov-Jordan algebras $Os^{+}(\alpha,\beta)$ do not satisfy this identity. Therefore, Tortken algebras are not in this class.

Identities of right-symmetric algebras were also studied in[15].

2. TORTKEN IDENTITY AND SOME CONSEQUENCES OF DEGREE 4

Let A be an algebra over a field K of characteristic $p\geq 0$ and $f=f(t_1,\ldots,t_k)$ be some non-associative polynomial in variables $t_1,\ldots,t_k$. We say that $f=0$ or $f(a_1,\ldots,a_k)=0$ is identity on A if $f(a_1,\ldots,a_k)=0$ for any substitution $t_i:=a_i\in A$.

Let A be an algebra A with multiplication $A\times A\rightarrow A,(a,b)\mapsto a\star b$. Recall that an element $e\in A$ is called *left unit* if $e\star a=a$, for any $a\in A$. Similarly, e is called *right unit* if $a\star e=a$, for any $a\in A$. Left and right unit is called *unit*.

Theorem 2.1. *Let A be a Tortken algebra.*

i) *If A has a right unit, then A satisfies the following identity*

$$a\star(b\star c+c\star b)=2(a\star b)\star c. \tag{4}$$

ii) *If A has a left unit, then A is associative and commutative.*

Proof. i) Let e be the right unit. Then

$$(a,b,e)=(a\star b)\star e-a\star(b\star e)=0, \tag{5}$$

for any $a,b,c\in A$. Take $c:=e$ in (3). By (5) we have

$$(a\star b)\star(e\star d)-(a\star d)\star(e\star b)=0, \tag{6}$$

for any $a, b, d \in A$. In (6) take $d := e$. We obtain that

$$a \star b - a \star (e \star b) = (a \star b) \star (e \star e) - (a \star e) \star (e \star b) = 0,$$

for any $a, b \in A$. In other words,

$$(a \star e) \star b - a \star (e \star b) = (a, e, b) = 0. \tag{7}$$

Now take $d := e$ in (3). By (7) we have

$$(a \star b) \star (c \star e) - (a \star e) \star (c \star b) = (a, b, c) \star e.$$

So,

$$(a \star b) \star c - a \star (c \star b) = (a, b, c).$$

In other words, (4) is true.

ii) Let e be the left unit. Then

$$(e, a, b) = e \star (a \star b) - (e \star a) \star b = 0,$$

for any $a, b \in A$. Therefore, for $a := e$ it follows from (3) that

$$b \star (c \star d) - d \star (c \star b) = 0. \tag{8}$$

In (8) take $c := e$. We have

$$b \star d - d \star b = 0,$$

for any $b, d \in A$. So, the algebra A is commutative. Thus, according to (8),

$$b \star (c \star d) = d \star (c \star b) = (c \star b) \star d = (b \star c) \star d.$$

Corollary 2.2. *Any Tortken algebra with a unit is associative and commutative.*

Remark. For $p \neq 2$ the right-alternative identity $(a, b, b) = 0$ and the right-commutativity identity $(a \star b) \star c = (a \star c) \star b$ follow from identity (4), and vice versa, from these two identities one can obtain (4). Notice that, the right-alternative identity itself is not enough to obtain (4).

For an algebra A with a multiplication $A \times A \rightarrow A, (a, b) \mapsto a \star b$, denote by $r_a : A \rightarrow A, b \mapsto a \star b$ a right multiplication operator.

Proposition 2.3. *Let A be a commutative Tortken algebra. Then*

$$\sum_{\sigma \in \mathrm{Sym}_3} \operatorname{sign} \sigma \, r_{a_{\sigma(1)}} r_{a_{\sigma(2)}} r_{a_{\sigma(3)}} = 0,$$

for any $a_1, a_2, a_3 \in A$.

Proof.

$$\begin{aligned}
&((x\star a)\star b)\star c + ((x\star b)\star c)\star a + ((x\star c)\star a)\star b - ((x\star a)\star c)\star b \\
&\quad - ((x\star b)\star a)\star c - ((x\star c)\star b)\star a \\
&\quad = ((x\star a)\star b)\star c - ((x\star b)\star a)\star c + ((c\star \underline{b)\star a})\star x - ((c\star \underset{\sim\sim\sim}{a)\star b})\star x \\
&\qquad + ((x\star b)\star c)\star a - ((x\star c)\star b)\star a - ((a\star \underline{\underline{b)\star c}})\star x + ((a\star \underset{\sim\sim\sim}{c)\star b})\star x \\
&\qquad + ((x\star c)\star a)\star b) - ((x\star a)\star c)\star b + ((b\star \underline{\underline{a)\star c}})\star x - ((b\star \underline{c)\star a})\star x \\
&\quad = \text{(by identity (3))}\ ((b\star x)\star(a\star c)) - ((b\star c)\star(a\star x)) \\
&\qquad + ((c\star x)\star(b\star a)) - ((c\star a)\star(b\star x)) + ((a\star x)\star(c\star b)) \\
&\qquad - ((a\star b)\star(c\star x))\ \text{(by the commutativity condition)} = 0.
\end{aligned}$$

Corollary 2.4. *Let A be a commutative Tortken algebra. Then for any* $a, b, c, x \in A$,

$$(a, b \star x, c) + (b, c \star x, a) + (c, a \star x, b) = 0.$$

Proof. By proposition 2.3,

$$\begin{aligned}
&(c, a \star x, b) + (a, b \star x, c) + (b, c \star x, a) \\
&\quad = ((x \star a) \star b) \star c) - ((x \star a) \star c) \star b) + ((x \star b) \star c) \star a) \\
&\qquad - ((x \star b) \star a) \star c) + ((x \star c) \star a) \star b) - ((x \star c) \star b) \star a) \\
&\quad = 0.
\end{aligned}$$

Corollary 2.5. *Let A be a commutative Tortken algebra. Then for any* $a, b, c, x \in A$,

$$(a, x, b) \star c + (b, x, c) \star a + (c, x, a) \star b = 0.$$

Proof. By proposition 2.3,

$$
\begin{aligned}
&(a, x, b) \star c + (b, x, c) \star a + (c, x, a) \star b \\
&\quad = ((x \star b) \star a) \star c) - ((x \star a) \star b) \star c) + ((x \star c) \star b) \star a) \\
&\qquad - ((x \star b) \star c) \star a) + ((x \star a) \star c) \star b) - ((x \star c) \star a) \star b) = 0.
\end{aligned}
$$

3. NOVIKOV ALGEBRAS UNDER JORDAN PRODUCT

Proposition 3.1. *Let A be a Novikov algebra. Then for any $a_1, a_2, a_3 \in A$,*

$$\sum_{\sigma \in \mathrm{Sym}_3} \operatorname{sign} \sigma \, r_{a_{\sigma(1)}} r_{a_{\sigma(2)}} r_{a_{\sigma(3)}} = 0.$$

Here r_a denotes a right-multiplication operator: $(b)\, r_a = b \circ a$.

Proof. For any $a, b, c, d \in A$ we have

$$
\begin{aligned}
&((a \circ b) \circ c) \circ d + ((a \circ c) \circ d) \circ b + ((a \circ d) \circ b) \circ c - ((a \circ b) \circ d) \circ c \\
&\quad - ((a \circ c) \circ b) \circ d - ((a \circ d) \circ c) \circ b \\
&\quad = \text{(by right-symmetric rule) } (a \circ b) \circ (c \circ d - d \circ c) \\
&\qquad + (a \circ c) \circ (d \circ b - b \circ d) + (a \circ d) \circ (b \circ c - c \circ b) \\
&\quad = \text{(by left-commutativity rule)} c \circ ((a \circ b) \circ d) - d \circ ((a \circ b) \circ c) \\
&\qquad + d \circ ((a \circ c) \circ b) - b \circ ((a \circ c) \circ d) + b \circ ((a \circ d) \circ c) - c \circ ((a \circ d) \circ b) \\
&\qquad \text{(by right-symmetric rule)} b \circ (a \circ (d \circ c - c \circ d)) \\
&\qquad + c \circ (a \circ (b \circ d - d \circ b)) + d \circ (a \circ (c \circ b - b \circ c)) \\
&\quad = \text{(by left-commutativity rule)} b \circ (d \circ (a \circ c)) - b \circ (c \circ (a \circ d)) \\
&\qquad + c \circ (b \circ (a \circ d)) - c \circ (d \circ (a \circ b)) + d \circ (c \circ (a \circ b)) \\
&\qquad - d \circ (b \circ (a \circ c)) \text{(by left-commutativity rule)} = 0.
\end{aligned}
$$

Corollary 3.2. *Let $a \circ b = a \circ f(b)$ be a new multiplication in a Novikov algebra $(A, \circ)$, where $f \in \mathrm{End}\, A$. Then the algebra $(A, \underline{\circ})$ satisfies the following identity of degree 4*

$$
\begin{aligned}
&((a \underline{\circ} b) \underline{\circ} c) \underline{\circ} d + ((a \underline{\circ} c) \underline{\circ} d) \underline{\circ} b + ((a \underline{\circ} d) \underline{\circ} b) \underline{\circ} c \\
&\quad - ((a \underline{\circ} c) \underline{\circ} b) \underline{\circ} d - ((a \underline{\circ} d) \underline{\circ} c) \underline{\circ} b - ((a \underline{\circ} b) \underline{\circ} d) \underline{\circ} c = 0,
\end{aligned}
$$

for any $a, b, c \in A$.

Proof. Notice that $a \underset{-}{\circ} b = ar_{f(b)}$. Therefore our identity is equivalent to the relation

$$\sum_{\sigma\in \mathrm{Sym}_3} \mathrm{sign}\,\sigma\, r_{f(a_{\sigma(1)})} r_{f(a_{\sigma(2)})} r_{f(a_{\sigma(3)})} = 0.$$

Example 1. Any associative algebra is right-symmetric.

Example 2. Let U be the algebra of Laurent polynomials $K[x^{\pm 1}]$ (if $p = 0$) or divided power algebra $O_1(m)$ (if $p > 0$) and W_1 or $W_1(m)$ be the Lie algebra of special derivations $u\partial$, where $\partial(x^i) = ix^{i-1}$ (if $p = 0$) or $\partial(x^{(i)}) = x^{(i-1)}$. Define a multiplication on W_1 or $W_1(m)$ by

$$u\partial \circ v\partial = v\partial(u)\partial.$$

We obtain a right-symmetric algebra. Call it right-symmetric Witt algebra. This construction can be easily generalized for the case of any associative commutative algebra U with derivation ∂.

Example 3. An algebra $A = \langle e_1, \ldots, e_n\rangle$ with multiplication $e_i \circ e_j = e_j$ is left-commutative and associative. In particular, A is Novikov. Its Jordan algebra arises in genetics and is called gametic algebra[7]. The algebra A^+ satisfies (associative)-Jordan identity $\{\{\{x,x\},y\},x\} = \{\{x,x\},\{y,x\}\}$ and identity (3).

Any associative algebra is Lie-admissible: under commutator $[a,b] = a \circ b - b \circ a$ it can be endowed by a structure of Lie algebra. It is also known that any associative algebra with multiplication $(a,b) \mapsto a \circ b$ is Jordan under anti-commutator $\{a,b\} = a \circ b + b \circ a$. Right-symmetric algebras are also Lie-admissible. Below we describe analog of Jordan algebras for Novikov algebras.

Commutative Tortken algebra is called *special Tortken*, if it is isomorphic to a subalgebra of some Tortken algebra of the form A^+, where A is Novikov.

Theorem 3.3. *Let A be a Novikov algebra. Then the algebra A^+ satisfies the Tortken identity; moreover, this identity is a Jordan identity for A^+.*

Proof of theorem 3.3 needs several lemmas. Below we assume that A is a Novikov algebra and $a,b,c,d \in A$.

Lemma 3.4.

$$(a,b,c)^+ = \{\{b,c\},a\} - \{\{b,a\},c\} = \{b,[c,a]\}.$$

Proof. It is easy to see that

$$(a,b,c)^+ = \{\{b,c\},a\} - \{\{b,a\},c\}.$$

Therefore,

$$\begin{aligned}(a,b,c)^+ &= (b\circ c)\circ a + (c\circ b)\circ a + a\circ(b\circ c) + a\circ\underline{(c\circ b)}\\ &\quad - (b\circ a)\circ c - (a\circ b)\circ c - c\circ(b\circ a) - c\circ\underline{(a\circ b)}\\ &= \text{(left-commutativity identity)}\\ &= (b\circ c)\circ a - (b\circ a)\circ c - (a\circ c)\circ b + (c\circ a)\circ b\\ &\quad + \underline{\underline{(a\circ c)}}\circ b - \utilde{(c\circ a)}\circ b + \utilde{(c\circ b)}\circ a - \underline{\underline{(a\circ b)}}\circ c\\ &\quad + a\circ(b\circ c) - c\circ(b\circ a).\end{aligned}$$

By left-commutativity,

$$a\circ(b\circ c) - c\circ(b\circ a) = b\circ(a\circ c) - b\circ(c\circ a) = b\circ[a,c].$$

Therefore,

$$\begin{aligned}(a,b,c)^+ &= b\circ[c,a] + [c,a]\circ b + a\circ[c,b] + c\circ[b,a] + b\circ[a,c]\\ &= \{b,[c,a]\}.\end{aligned}$$

Lemma 3.5.

$$\{(a,b,c)^+,d\} - \{(a,d,c)^+,b\} = \{[c,a],[b,d]\}.$$

Proof. By Lemma 3.4,

$$\begin{aligned}\{(a,b,c)^+,d\} &= \{\{b,[c,a]\},d\},\\ \{(a,d,c)^+,b\} &= \{\{d,[c,a]\},b\}.\end{aligned}$$

Thus,

$$\begin{aligned}\{(a,b,c)^+,d\} - \{(a,d,c)^+,b\} &= \{\{b,[c,a]\},d\} - \{\{d,[c,a]\},b\}\\ &= (d,[c,a],b)^+ = \{[c,a],[b,d]\}.\end{aligned}$$

Lemma 3.6.

$$
\begin{aligned}
&\{\{a,b\},\{c,d\}\} - \{\{a,d\},\{c,b\}\}\\
&\quad = \{d\circ c, a\circ b\} + \{c\circ d, b\circ a\} - \{b\circ c, a\circ d\} - \{c\circ b, d\circ a\}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\{\{a,b\},\{c,d\}\} - \{\{a,d\},\{c,b\}\}\\
&\quad = (a\circ b)\circ(c\circ d) + (b\circ a)\circ(c\circ d) + (a\circ b)\circ(d\circ c) + (b\circ a)\circ(d\circ c)\\
&\qquad + (c\circ d)\circ(a\circ b) + (c\circ d)\circ(b\circ a) + (d\circ c)\circ(a\circ b) + (d\circ c)\circ(b\circ a)\\
&\qquad - (a\circ d)\circ(c\circ b) - (a\circ d)\circ(b\circ c) - (d\circ a)\circ(c\circ b) - (d\circ a)\circ(b\circ c)\\
&\qquad - (c\circ b)\circ(a\circ d) - (b\circ c)\circ(a\circ d) - (c\circ b)\circ(d\circ a) - (b\circ c)\circ(d\circ a)\\
&\quad = \text{(left-commutativity rule)}\\
&\quad = c\circ((a\circ b)\circ d) + c\circ((b\circ a)\circ d) + d\circ((a\circ b)\circ c) + d\circ((b\circ a)\circ c)\\
&\qquad + a\circ((c\circ d)\circ b) + b\circ((c\circ d)\circ a) + a\circ((d\circ c)\circ b) + b\circ((d\circ c)\circ a)\\
&\qquad - c\circ((a\circ d)\circ b) - b\circ((a\circ d)\circ c) - c\circ((d\circ a)\circ b) - b\circ((d\circ a)\circ c)\\
&\qquad - a\circ((c\circ b)\circ d) - a\circ((b\circ c)\circ d) - d\circ((c\circ b)\circ a) - d\circ((b\circ c)\circ a)\\
&\qquad + a\circ((c\circ d)\circ b + (d\circ c)\circ b - (c\circ b)\circ d - (b\circ c)\circ d) + b\circ((c\circ d)\circ a\\
&\qquad + (d\circ c)\circ a - (a\circ d)\circ c - (d\circ a)\circ c) + c\circ((a\circ b)\circ d + (b\circ a)\circ d\\
&\qquad - (a\circ d)\circ b - (d\circ a)\circ b) + d\circ((a\circ b)\circ c) + (b\circ a)\circ c\\
&\qquad - (c\circ b)\circ a - (b\circ c)\circ a)\\
&\quad = \text{(right-symmetric and left-commutativity rules)}\\
&\quad = a\circ\underline{(c\circ[d,b])} + (d\circ c)\circ(a\circ b) - (b\circ c)\circ(a\circ d) + b\circ\underline{\underline{(d\circ[c,a])}}\\
&\qquad + (c\circ d)\circ(b\circ a) - (a\circ d)\circ(b\circ c) + c\circ\underline{(a\circ[b,d])} + (b\circ a)\circ(c\circ d)\\
&\qquad - (d\circ a)\circ(c\circ b) + d\circ\underline{\underline{(b\circ[a,c])}} + (a\circ b)\circ(d\circ c) - (c\circ b)\circ(d\circ a)\\
&\quad = \text{(left-commutativity rule)}(d\circ c)\circ(a\circ b) - (b\circ c)\circ(a\circ d)\\
&\qquad + (c\circ d)\circ(b\circ a) - (a\circ d)\circ(b\circ c) + (b\circ a)\circ(c\circ d) - (d\circ a)\circ(c\circ b)\\
&\qquad + (a\circ b)\circ(d\circ c) - (c\circ b)\circ(d\circ a)\\
&\quad = \{d\circ c, a\circ b\} + \{c\circ d, b\circ a\} - \{b\circ c, a\circ d\} - \{d\circ a, c\circ b\}.
\end{aligned}
$$

Lemma 3.7.

$$
\begin{aligned}
\{[c,a],[b,d]\} = {} & \{d\circ c, a\circ b\} + \{c\circ d, b\circ a\} - \{b\circ c, a\circ d\}\\
& - \{c\circ b, d\circ a\}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\{[c,a],[b,d]\} - \{d\circ c, a\circ b\} - \{c\circ d, b\circ a\} + \{b\circ c, a\circ d\} + \{c\circ b, d\circ a\}\\
&\quad = (c\circ a)\circ(b\circ d) - (c\circ a)\circ(d\circ b) - (a\circ c)\circ(b\circ d) + (a\circ c)\circ(d\circ b)\\
&\qquad + (b\circ d)\circ(c\circ a) - (d\circ b)\circ(c\circ a) - (b\circ d)\circ(a\circ c) + (d\circ b)\circ(a\circ c)\\
&\qquad - (d\circ c)\circ(a\circ b) - (a\circ b)\circ(d\circ c) - (c\circ d)\circ(b\circ a) - (b\circ a)\circ(c\circ d)\\
&\qquad + (b\circ c)\circ(a\circ d) + (a\circ d)\circ(b\circ c) + (c\circ b)\circ(d\circ a) + (d\circ a)\circ(c\circ b)\\
&\quad = (\text{left-commutativity rule})\\
&\quad = b\circ((c\circ a)\circ d) - d\circ((c\circ a)\circ b) - b\circ((a\circ c)\circ d) + d\circ((a\circ c)\circ b)\\
&\qquad + c\circ((b\circ d)\circ a) - c\circ((d\circ b)\circ a) - a\circ((b\circ d)\circ c) + a\circ((d\circ b)\circ c)\\
&\qquad - a\circ((d\circ c)\circ b) - d\circ((a\circ b)\circ c) - b\circ((c\circ d)\circ a) - c\circ((b\circ a)\circ d)\\
&\qquad + a\circ((b\circ c)\circ d) + b\circ((a\circ d)\circ c) + d\circ((c\circ b)\circ a) + c\circ((d\circ a)\circ b)\\
&\quad = a\circ(-(b\circ d)\circ c + (d\circ b)\circ c - (d\circ c)\circ b + (b\circ c)\circ d)\\
&\qquad + b\circ((c\circ a)\circ d - (a\circ c)\circ d - (c\circ d)\circ a + (a\circ d)\circ c)\\
&\qquad + c\circ((b\circ d)\circ a - (d\circ b)\circ a - (b\circ a)\circ d + (d\circ a)\circ b)\\
&\qquad + d\circ(-(c\circ a)\circ b) + (a\circ c)\circ b - (a\circ b)\circ c + (c\circ b)\circ a)\\
&\quad = a\circ(b\circ[c,d] + d\circ[b,c]) + b\circ(c\circ[a,d] - a\circ[c,d])\\
&\qquad + c\circ(b\circ[d,a] - d\circ[b,a]) + d\circ(-c\circ[a,b] + a\circ[c,b])\\
&\quad = (\text{left-commutativity rule})\\
&\qquad\ a\circ(b\circ[c,d]) + a\circ(d\circ[b,c]) + b\circ(c\circ[a,d]) - a\circ(b\circ[c,d])\\
&\qquad + b\circ(c\circ[d,a]) - c\circ(d\circ[b,a]) + c\circ(-d\circ[a,b]) + a\circ(d\circ[c,b]) = 0.
\end{aligned}
$$

Lemma 3.7. is proved.

Denote by $O_1(\infty)$ an infinite-dimensional algebra $\{x^{(i)} : 0 \le i\}$ with the multiplication

$$x^{(i)}x^{(j)} = \binom{i+j}{i}x^{(i+j)}.$$

If $p > 0$, then it has p^m-dimensional subalgebra

$$O_1(m) = \{x^{(i)} : 0 \le i < p^m\},$$

for any nonnegative integer m. The algebra $O_1(m)$ is called *divided power algebra*. If $p = 0$, then $O_1(\infty)$ is isomorphic to the polynomial algebra $K[x]$. The isomorphism can be given by $x^{(i)} \mapsto x^i/i!$. Take now new basis of $O_1(\infty)$

which consists of elements $e_i = x^{(i+1)}$, $-1 \le i$. Endow the vector space $O_1(m)$, $p > 0$ by multiplication

$$e_i \star e_j = \binom{i+j+2}{i+1} e_{i+j}.$$

This algebra is commutative Tortken. In notation of Sec. 4 this algebra is isomorphic to $Os^+(0,0,m)$.

Lemma 3.8. $(p \ne 2)$. *Let* $Os^+(0,0) = \{e_i : -1 \le i, i \in \mathbf{Z}\}$ *be a Novikov-Jordan algebra with the multiplication* $e_i \star e_j = (i+j+2)e_{i+j}$ *(we consider this algebra more detailed in section* (4)). *Any multilinear polynomial identity of degree* 3 *of the algebra* $(Os^+(0,0), \star)$ *follows from commutativity identity.*

Proof. According to commutativity rule, we can assume that f has the form

$$f(t_1, t_2, t_3) = \lambda_1(t_1 t_2)t_3 + \lambda_2(t_2 t_3)t_1 + \lambda_3(t_3 t_1)t_2.$$

Then

$$f(e_i, e_j, e_s) = ((i+j+s+2)(i+j+2)\lambda_1 + (j+s+2)\lambda_2 + (s+i+2)\lambda_3)e_{i+j+1}).$$

Therefore, the condition $f(e_i, e_j, e_s) = 0$ gives us that

$$(i+j+2)\lambda_1 + (j+s+2)\lambda_2 + (s+i+2)\lambda_3 = 0,$$

for any $i,j,s \in \mathbf{Z}, i,j,s \ge -1$, such that $i+j+s+2 \ne 0$. So, for $(i,j,s) = (1,2,3),(2,3,1),(3,1,2)$, we obtain a system of linear equations with determinant 54. Thus, this system is non-degenerate if $p \ne 2,3$. Let $p = 3$. Then for $(i,\ j,\ s) = (1,1,0),(1,0,1),(0,1,1)$ we obtain a non-degenerate system of 3×3 equations. Hence, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, if $p \ne 2$.

Remark. If $p = 2$, lemma 3.8 is not true. In this case Jordan product and Lie products are coincide and any special Tortken algebra satisfies Jacobi identity.

Lemma 3.9. $(p \ne 2)$. *For the algebra* $(O_1(m), \star)$ *a space of multilinear polynomial identities of degree* 4 *is generated by the following polynomials*

$$\begin{aligned}\mathrm{Tortken}(t_1,t_2,t_3,t_4) := {} & (t_1t_2)(t_3t_4) - (t_1t_4)(t_3t_2) - (t_1(t_2t_3))t_4 \\ & + ((t_1t_2)t_3)t_4 + (t_1(t_4t_3))t_2 - ((t_1t_4)t_3)t_2,\end{aligned}$$

$$\begin{aligned}\mathrm{Tortken}'(t_1,t_2,t_3,t_4) := {} & (t_1t_3)(t_2t_4) + (t_1t_4)(t_2t_3) + ((t_1t_3)t_4)t_2 \\ & + ((t_1t_4)t_2)t_3 + ((t_2t_3)t_1)t_4 + ((t_2t_4)t_3)t_1, \\ \text{if } (p,m) = {} & (3,1), \\ \mathrm{Com}(t_1,t_2) := {} & t_1t_2 - t_2t_1.\end{aligned}$$

Proof. According to commutativity condition, we can assume that f has the following form

$$\begin{aligned}f(t_1,t_2,t_3,t_4) = {} & \mu_1(t_1t_2)(t_3t_4) + \mu_2(t_1t_3)(t_2t_4) + \mu_3(t_1t_4)(t_2t_3)\mu_4((t_1t_2)t_3)t_4 \\ & + \mu_5((t_1t_2)t_4)t_3 + \mu_6((t_1t_3)t_2)t_4 + \mu_7((t_1t_3)t_4)t_2\mu_8((t_1t_4)t_2)t_3 \\ & + \mu_9((t_1t_4)t_3)t_2 + \mu_{10}((t_2t_3)t_1)t_4 + \mu_{11}((t_2t_3)t_4)t_1 \\ & + \mu_{12}((t_2t_4)t_1)t_3 + \mu_{13}((t_2t_4)t_3)t_1 + \mu_{14}((t_3t_4)t_1)t_2 \\ & + \mu_{15}((t_3t_4)t_2)t_1.\end{aligned}$$

Suppose that $p > 3$ or $m > 1$, if $p = 3$. Let $1 = x^{(0)}, x = x^{(1)}$. Make the following 10 substitutions of (t_1,t_2,t_3,t_4) :

$$\begin{aligned}&(x,x,x,1),(x,x,1,x),(x,1,x,x),(1,x,x,x),(1,1,x,x^{(2)}),(1,1,x^{(2)},x),\\ &(1,x^{(2)},x,1),(x,1,1,x^{(2)}),(x^{(2)},1,1,x),(1,1,1,x^{(3)}).\end{aligned}$$

We obtain the system of linear equations $M\mu^t = 0$, where $\mu = (\mu_1,\dots,\mu_{15})$ is a row with 15 components, μ^t is the corresponding column and M is the 10×15-matrix

$$\begin{pmatrix} 2&2&2&4&2&4&2&1&1&4&2&1&1&1&1\\ 2&2&2&2&4&1&1&4&2&1&1&4&2&1&1\\ 2&2&2&1&1&2&4&2&4&1&1&1&1&4&2\\ 2&2&2&1&1&1&1&1&1&2&4&2&4&2&4\\ 0&1&1&0&0&0&1&1&2&0&1&1&2&3&3\\ 0&1&1&0&0&1&2&0&1&1&2&0&1&3&3\\ 1&1&0&2&1&1&0&0&0&3&3&1&2&0&1\\ 1&1&0&0&1&0&1&3&3&0&0&2&1&2&1\\ 1&1&0&1&2&1&2&3&3&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1&1&0&0&1&1&1&1 \end{pmatrix}$$

It has rank 10 and has the following fundamental system of solutions

$$\begin{array}{l}\{(1,\ 0,\ -1,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0,\ 1),\\ (1,\ 0,\ -1,\ 0,\ 1,\ 1,\ -1,\ -1,\ 0,\ -1,\ 0,\ 0,\ 0,\ 1,\ 0),\\ (0,\ 1,\ -1,\ -1,\ 1,\ 1,\ 0,\ -1,\ 0,\ 0,\ -1,\ 0,\ 1,\ 0,\ 0),\\ (0,\ 1,\ -1,\ 0,\ 0,\ 1,\ 0,\ -1,\ 0,\ -1,\ 0,\ 1,\ 0,\ 0,\ 0),\\ (0,\ 0,\ 0,\ -1,\ 1,\ 1,\ -1,\ -1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\}\end{array}$$

So, we can take $\mu_9, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{15}$ as a free parameters. Then

$$\begin{aligned}\mu_1 &= \mu_{14} + \mu_{15},\\ \mu_2 &= \mu_{12} + \mu_{13},\\ \mu_3 &= -\mu_{12} - \mu_{13} - \mu_{14} - \mu_{15},\\ \mu_4 &= -\mu_{13} - \mu_9,\\ \mu_5 &= \mu_{13} + \mu_{14} + \mu_{15} + \mu_9,\\ \mu_6 &= \mu_{12} + \mu_{13} + \mu_{14} + \mu_9,\\ \mu_7 &= -\mu_{14} - \mu_9,\\ \mu_8 &= -\mu_{12} - \mu_{13} - \mu_{14} - \mu_{15} - \mu_9,\\ \mu_{10} &= -\mu_{12} - \mu_{14},\\ \mu_{11} &= -\mu_{13} - \mu_{15}.\end{aligned}$$

Thus,

$$f = \mu_9 f_1 + \mu_{12} f_2 + \mu_{13} f_3 + \mu_{14} f_4 + \mu_{15} f_5,$$

where

$$\begin{aligned}f_1 &= -\sum_{\sigma\in \mathrm{Sym}_3} \mathrm{sign}\,\sigma\,((t_1 t_{\sigma(2)})t_{\sigma(2)})t_{\sigma(3)},\\ f_2 &= \mathrm{Tortken}(t_1, t_3, t_2, t_4),\\ f_3 &= (t_1t_3)(t_2t_4) - (t_1t_4)(t_2t_3) - ((t_1t_2)t_3)t_4 + ((t_1t_2)t_4)t_3\\ &\quad + ((t_1t_3)t_2)t_4 - ((t_1t_4)t_2)t_3 - ((t_2t_3)t_4)t_1 + ((t_2t_4)t_3)t_1,\\ f_4 &= (t_1t_2)(t_3t_4) - (t_1t_4)(t_2t_3) + ((t_1t_2)t_4)t_3 + ((t_1t_3)t_2)t_4\\ &\quad - ((t_1t_3)t_4)t_2 - ((t_1t_4)t_2)t_3 - ((t_2t_3)t_1)t_4 + ((t_3t_4)t_1)t_2,\\ f_5 &= (t_1t_2)(t_3t_4) - (t_1t_4)(t_2t_3) + ((t_1t_2)t_4)t_3)\\ &\quad - ((t_1t_4)t_2)t_3 - ((t_2t_3)t_4)t_1 + ((t_3t_4)t_2)t_1.\end{aligned}$$

We see that, f_2 is the identity (3) for t_1, t_3, t_2, t_4. By proposition 2.3, $f_1 = 0$ is the identity for any commutative Tortken algebra. It is easy to see that $f_3 = 0,\ f_4 = 0,\ f_5 = 0$ are follow from corollary 2.5 and (3).

We omit details of tedious calculations for $(p, m) = (3, 1)$. We just mention that

$$\begin{aligned}&(a\star c)\star(b\star d)+(a\star d)\star(b\star c)+((a\star c)\star d)\star b+((a\star d)\star b)\star c\\&\quad+((b\star c)\star a)\star d+((b\star d)\star c)\star a\\&=2\partial^3(a\cdot b\cdot c\cdot d)\end{aligned}$$

if $p = 3$ and $a\star b = \partial(a\cdot b)$. Thus *Tortken*$'$ is identity for $(p, m) = (3, 1)$.

Proof of theorem 3.3. Tortken identity for Jordan product follows from lemma 3.5, 3.6, 3.7. Second part of theorem 3.3 follows from lemma 3.8 and 3.9.

4. SIMPLE NOVIKOV-JORDAN ALGEBRAS

Let $O_1(m) = \{x^{(i)}: 0 \le i \le p^m - 1\}$ be the divided power algebra with the multiplication

$$x^{(i)}x^{(j)} = \binom{i+j}{j}x^{(i+j)}$$

if $p > 0$. If $p = 0$ we can consider an algebra $O_1(\infty) = \{x^{(i)} : i \in \mathbf{Z},\ i \ge 0\}$ with the same multiplication.

Let $U = K[[x^{\pm}]]$ be Laurent formal power series algebra if $p = 0$, and $U = O_1(m)$ if $p > 0$. So, any element of U for $p = 0$ has a form $\sum_{j\ge i}\lambda_j x^j$, where the number of non-zero terms in positive part $\lambda_j \in K,\ j \ge 0$, may be infinite, but the number of non-zero terms in negative part, $|\{\lambda_j : j < 0\}|$ is finite.

In[5,17,18] it is proved that the algebra defined on polynomials algebra $U = K[x]$ by the rule

$$(a, b) \mapsto \partial(a)b$$

and on Laurent formal power series algebra $U = K[[x^{\pm 1}]], p = 0$, by

$$(a, b) \mapsto \partial(a)b + \alpha x^{-1}ab + \beta x^{-2}ab, \quad \alpha, \beta \in K,$$

is Novikov and simple. In the case of $p > 0, U = O_1(m)$ this multiplication is changed by obvious way

$$(a, b) \mapsto \partial(a)b + \alpha x^{(p^m-1)}ab + \beta x^{(p^m-2)}ab, \quad \alpha, \beta \in K.$$

Denote such algebras in honor of M. Osborn by Os if $U = K[x], p = 0$, $Os(\alpha, \beta)$ if $U = K[[x^{\pm}]], p = 0$, and $Os(\alpha, \beta, m)$ if $U = O_1(m), p > 2$.

In this section we denote by $\star$ the Jordan bracket $\{\ ,\ \}$.

Let $p = 0$. Take $U = K[[x^{\pm}]]$. Let $\overline{Os}(\alpha, 0)$ be a subspace of $Os(\alpha, 0)$ generated by the set $\{x^i : i \neq -2\alpha - 1, i \in \mathbf{Z}\}$ if $\alpha \in \frac{1}{2}\mathbf{Z}$. Notice that

$$x^i \star x^j = (i + j + 2\alpha)x^{i+j-1} + 2\beta x^{i+j-2}.$$

Therefore, the coefficient at $\langle x^{-2\alpha-1}\rangle$ of any element of the form $X = x^i \star \sum_j \lambda_j x^j = \sum_j \lambda_j (i + j + 2\alpha)x^{i+j-1}$ is equal to $\lambda_j(i + j + 2\alpha)\delta_{i+j+2\alpha,0}$, that is 0. So, $\overline{Os}^+(\alpha, 0)$ is not only subalgebra, but also an ideal of $Os^+(\alpha, 0)$.

Let now $p > 0$. The multiplication on $Os^+(\alpha, \beta, m)$ can be given by

$$\begin{aligned} x^{(i)} \star x^{(j)} = \binom{i+j}{j} x^{(i+j-1)} + 2\beta\delta_{i,0}\delta_{j,0}x^{(p^m-2)} \\ + 2(\alpha\delta_{i,0}\delta_{j,0} - \beta(\delta_{i,0}\delta_{j,1} + \delta_{i,1}\delta_{j,0}))x^{(p^m-1)}. \end{aligned}$$

In particular,

$$\begin{aligned} 1 \star 1 &= 2\beta x^{(p^m-2)} + 2\alpha x^{(p^m-1)}, \\ 1 \star x^{(1)} &= 1 - 2\beta x^{(p^m-1)}, \\ 1 \star x^{(j)} &= x^{(j-1)}, \quad j > 1. \end{aligned}$$

If $\alpha = 0$, the algebra $Os^+(\alpha, \beta, m)$ has the ideal

$$\overline{Os}^+(0, \beta, m) = \{1 - 2\beta x^{(p^m-1)}, x^{(i)} : 0 < i < p^m - 1\}$$

of dimension $p^m - 1$.

Theorem 4.1. *Let*

- $A = Os^+$, *if* $p = 0$,
- $A = Os^+(\alpha, \beta)$, *if* $p = 0$ *and* $\beta \neq 0$, *or* $\beta = 0, \alpha \notin \frac{1}{2}\mathbf{Z}$,
- $A = \overline{Os}^+(\alpha, 0)$, *if* $p = 0$ *and* $\beta = 0$, $\alpha \in \frac{1}{2}\mathbf{Z}$,

- $A = Os^+(\alpha, \beta, m)$, *if* $p > 2$ *and* $\alpha \neq 0$,
- $A = \overline{Os}^+(0, \beta, m)$, *if* $p > 2$ *and* $\alpha = 0$.

Then A is simple Novikov-Jordan algebra.

Proof. Let $U = K[x], p = 0$. Then $1 \star x^i = ix^{i-1}$. If J is an ideal of Os^+ and $0 \neq X = \sum_{i_0 \le i \le i_1} \lambda_i x^i \in J, \lambda_{i_1} \neq 0$, then multiplication of 1 to X i_1 times gives the element $\lambda_{i_1} i_1! x^0 \in J$. Therefore, $1 \in J$ and $x^i = (i+1)^{-1} x^{i+1} \star 1 \in J$, for any $i \geq 0$. This means that $J = Os^+$.

Below in the case of $p = 0$ we assume that U is Laurent formal power series. Let $X \in A$. Present X as a linear combination of basic elements: $X = \sum_{j \geq i} \lambda_j x^j, \lambda_j \in K$ if $p = 0$ or $X = \sum_{j \geq i} \lambda_j x^{(j)}, \lambda_j \in K$ if $p > 0$. Set $|X| = i$ if $\lambda_i \neq 0$.

Let J be a non-zero ideal of A. Take some $0 \neq X \in J$. Suppose that $|X| = i$.

Step 1. Let M be a set of elements $X \in J$, such that $|X| \geq 0$. Prove that $M \neq \emptyset$.

If $p > 2$ this statement is trivial.

Let $p = 0$. Suppose that $|X| = i < 0$. There exists some $j > |i| + 1 > 0$ such that

- $x^j \in A$
- $j \neq -i - 2\alpha$, and
- $X' = X \star x^j = \lambda_i(i + j + 2\alpha)x^{i+j-1} + 2\beta\lambda_i x^{i+j-2} + X'' \in J$, where $|X''| \geq i + j - 1$, and $|X''| > i + j - 1$, if $\beta = 0$.

If $\beta = 0$, then $\lambda_i(i + j + 2\alpha) \neq 0$ and $|X''| > i + j - 1$. Therefore, in this case $X' \neq 0$. If $\beta \neq 0$, then x^{i+j-2} enters with non-zero coefficient 2β in decomposition of X' by basic elements. In particular, $X' \neq 0$. So, in both cases, $X' \neq 0, X' \in J$ and $|X'| \geq 0$. Therefore, the set $M = \{X \in J : |X| \geq 0\}$ is not empty.

Step 2. Take $0 \neq X_0 \in M$ with a minimal degree $|X_0|$. Let $|X_0| = i_0 \geq 0$. Prove that $i_0 = 0$ if $(\alpha, \beta) \neq (-1/2, 0)$ and $i_0 = 1$ if $(\alpha, \beta) = (-1/2, 0)$.

Consider firstly the case $\alpha = -1/2, \beta = 0$. Suppose that $i_0 > 1$. Then $1 \notin \overline{Os}^+(-1/2, 0)$, but $x^{-1} \in \overline{Os}^+(-1/2, 0)$. We have

$$X_1 = x^{-1} \star X_0 = \lambda_{i_0}(i_0 - 2)x^{i_0-2} + X_1' \in J,$$

where $|X_1'| \geq i_0 - 1$. Thus, $i_0 = 2$. Then

$$X_2 = x^2 \star X_0 = 3\lambda_2 x^3 + X_2' \in J, |X_2'| > 3.$$

We have

$$X_3 = x^{-1} \star X_2 = 3\lambda_2 x + X_3' \in J, |X_3'| > 1.$$

Since $\lambda_{i_0} \neq 0, i_0 = 2$, we obtain a contradiction. So, $i_0 = 1$ if $(\alpha, \beta) = (-1/2, 0)$.

Now consider the case, when $(\alpha, \beta) \neq (-1/2, 0)$. In this case $1 \in A$ or $1 - 2\beta x^{(p^m-1)} \in A$, if $\alpha = 0, p > 0$. Suppose that $i_0 > 0$. Then $\bar{X}_0 := 1 \star X_0 \in J$ or $\bar{X}_0 := (1 - 2\beta x^{(p^m-1)}) \star X_0 \in J$ in case of $\alpha = 0, p > 0$, satisfies the condition

$$0 \le |\bar{X}_0| < |X_0|.$$

Therefore, $\bar{X}_0 = 0$, and $i_0 = 0$.

So, our statement in step 2 is proved.

Step 3. Now prove that $J = A$.

Let $p = 0$. If $\beta \neq 0$, then $x^{i+2} \in A = Os^+(\alpha, \beta)$, for any $i \in \mathbf{Z}$. Thus

$$X_0 \star x^{i+2} = 2\beta x^i + X_0' \in J, \quad |X_0'| > i,$$

for any $i \in \mathbf{Z}$. This means that in J one can get basis with elements of the form $2\beta x^i + X_0', |X_0'| > i, i \in \mathbf{Z}$. Recall that in case of $p = 0$ our algebras consist of Laurent formal power series, i.e., infinite sums in positive part are allowed. Therefore, $J = Os^+(\alpha, \beta)$, if $\beta \neq 0$.

If $p = 0, \alpha \neq -1/2, \beta = 0$, *then*

$$X_0 \star x^{i+1} = (i + 1 + 2\alpha)x^i + X_0' \in J, \quad |X_0'| > i.$$

So, $J = Os(\alpha, 0)$, if $\alpha \notin \frac{1}{2}\mathbf{Z}$. If $2\alpha + 1 \in \mathbf{Z}$, then $X_0 \in \overline{Os}(\alpha, 0)$ and $J = \{x^i : i \in \mathbf{Z}, i \neq -2\alpha - 1\} = \overline{Os}(\alpha, 0)$.

If $p = 0, \alpha = -1/2, \beta = 0$, *then*

$$X_0 \star x^i = ix^i + X_0' \in J, \quad |X_0'| > i,$$

for any $i \neq 0$. So, $J = \overline{Os}^+(-1/2, 0)$.

If $p > 2$ *and* $x^{(i)} \in A, i > 0$, *then*

$$X_0 \star x^{(i)} = x^{(i-1)} + X_0' \in J, \quad |X_0'| \geq i.$$

So, J has elements of the form $Y_i = x^{(i)} + \lambda_i x^{(p^m-1)}, 0 \le i < p^m - 1$. Notice that

$$Y_1 \star Y_1 = 2x^{(1)} \in J.$$

Therefore, $\lambda_1 x^{(p^m-1)} \in J$.

Suppose that J has the element $x^{(p^m-1)}$. The multiplication of this element $p^m - i - 1$ times by $x^{(0)} := 1$, gives us the element $x^{(i)} \in J, i > 0$. Finally, $1 \star x^{(1)} = 1 - 2\beta x^{(p^m-1)} \in J$. Therefore, $1 \in J$, and $J = Os(\alpha, \beta, m)$. So, if $\lambda_1 \neq 0$, then our theorem is proved.

Consider the case, $\lambda_1 = 0$. Then $x^{(1)} \in J$ and

$$Y_0 \star x^{(1)} = 1 - 2\beta x^{(p^m-1)} \in J.$$

So, $\lambda_0 = -2\beta$. Further,

$$Y_0 \star Y_0 = 1 \star 1 + 2\lambda_0 1 \star x^{(p^m-1)} = 2\alpha x^{(p^m-1)} + 2\lambda_0 x^{(p^m-2)} + 2\beta x^{(p^m-2)} \in J.$$

Thus, $2\alpha x^{(p^m-1)} \in J$. So, if $\alpha \neq 0$, then $J = Os(\alpha, \beta, m)$. If $\alpha = 0$, then $J = \overline{Os}^+(0, \beta, m)$. Our theorem is proved completely.

Remark. If in the case $p = 0$ we consider Laurent polynomials algebra instead of formal power series, then theorem 4.1 is not true. Below we give one counterexample.

Set $\overline{x^i} = x^i$, if $i < -1$ and $\overline{x^i} = x^i + 2(i+1)^{-1}\beta x^{i-1}$, if $i > -1$. Let $\overline{Os}(0, \beta)$ be a subspace of $Os(0, \beta)$ generated by elements $\overline{x^i}, i \neq -1$. Notice that, if $\alpha = 0, \beta = 0$ then these two definitions of subspace $\overline{Os}(0, 0)$ are coincide.

Lemma 4.2. $(p = 0)$. *Let* $U = K[x^{\pm 1}]$. *The subspace* $\overline{Os}(0, \beta) \subset Os(0, \beta)$ *is closed under multiplication* $(a, b) \mapsto a \star b = \partial(ab) + 2\beta x^{-2} ab$. *Moreover, it is an ideal of codimension* 1.

Proof. It is obvious.

Remark. The algebra $\overline{Os}(0, 0) = \{x^i : i \neq -1\}$ has a nontrivial central extension. The cocycle ψ corresponding to this central extension can be given by

$$\psi(x^i, x^j) = \delta_{i+j,0}.$$

It has the following property:

$$\psi(\{x^i, x^j\}, x^s) + \psi(\{x^j, x^s\}, x^i) + \psi(\{x^s, x^i\}, x^j) = 2,$$

if

$$\psi(\{x^i, x^j\}, x^s) + \psi(\{x^j, x^s\}, x^i) + \psi(\{x^s, x^i\}, x^j) \neq 0,$$

i.e., if $i+j+s=1$. In the case of characteristic $p>0$ the definition of ψ should be slightly changed:

$$\psi(x^{(i)},x^{(j)})=\frac{1}{p}\binom{p^m}{i}\delta_{i+j,p^m}.$$

Notice that $\binom{p^m}{i}\equiv 0(\mathrm{mod}\, p)$, if $0\le i<p^m$, and this definition is correct.

5. LEIBNIZ DUAL ALGEBRAS

An algebra A with multiplication $\circ$ is called (*left*) *Leibniz*, if it satisfies the identity

$$(a\circ b)\circ c-a\circ(b\circ c)+b\circ(a\circ c)=0.$$

Similarly, an algebra with the identity

$$a\circ(b\circ c)-(a\circ b)\circ c+(a\circ c)\circ b=0$$

is called right Leibniz[19]. An algebra $(A,\times)$ is called (left) Leibniz dual, if it satisfies the identity

$$(a\times b)\times c=a\times(b\times c)+a\times(c\times b).$$

Proposition 5.1. *Let A be a Leibniz dual algebra. Then it satisfies* (*right*) *Tortken identity*.

Proof. According to (1)

$$(a,b,c)=a\times(b\times c)-(a\times b)\times c=-a\times(c\times b).$$

According to (2)

$$(a,b,c)\times d=-(a\times d)\times(c\times b).$$

By analogous reasons

$$-(a,d,c)\times b=(a\times b)\times(c\times b).$$

Thus

$$(a \times b) \times (c \times d) - (a \times d) \times (c \times b) = (a, b, c) \times d - (a, d, c) \times c.$$

Example 1. In[8] it is proved that the Leibniz cohomology group of trivial module $H^*_{lei}(L, K)$ for Leibniz algebra is Leibniz dual under the cup product. Statements of this paper can be generalized. Let us give some of such generalizations.

Proposition 5.2. *Let* $\mathfrak{g}$ *be a right Leibniz algebra and* R *be a left Leibniz dual algebra. Then the tensor product* $\mathfrak{g} \otimes R$ *equipped with the multiplication*

$$(x \otimes r) \circ (y \otimes s) = [x, y] \otimes rs$$

is a right-symmetric algebra.

Proof. By definition

$$\begin{aligned}
&(x \otimes r) \circ ((y \otimes s) \circ (z \otimes t)) - ((x \otimes r) \circ (y \otimes s)) \circ (z \otimes t)) \\
&\quad - (x \otimes r) \circ ((z \otimes t) \circ (y \otimes s)) + ((x \otimes r) \circ (z \otimes t)) \circ (y \otimes s)) = \\
&[x,[y,z]] \otimes r(st) - [[x,\underline{y],z] \otimes (rs)}t - [x,[z,y]] \otimes r(ts) + [[x,z],\underset{\sim\sim\sim}{y] \otimes (rt)}s =
\end{aligned}$$

(by left Leibniz dual condition)

$$\begin{aligned}
&[x, \underline{[y,z]] \otimes} r(st) - [[x, \underline{y], z] \otimes} r(st) - [[x, \underset{\sim\sim\sim}{y], z] \otimes} r(ts) \\
&\quad - [x, \underset{\sim\sim\sim}{[z, y]] \otimes} r(ts) + [[x, \underset{\sim\sim\sim}{z], y] \otimes} r(ts) + [[x, z], \underline{y] \otimes} r(st) = 0
\end{aligned}$$

(by right Leibniz dual condition)

$$= 0.$$

Corollary 5.3 (Proposition 1.3 of[8]). *Let* $\mathfrak{g}$ *be a right Leibniz algebra and* R *be a left Leibniz dual algebra. Then the tensor product* $\mathfrak{g} \otimes R$ *equipped with the multiplication*

$$(x \otimes r) \circ (y \otimes s) = [x, y] \otimes rs - [y, x] \otimes sr$$

is a Lie algebra.

Second generalization concerns cup-products. Cup-products of Leibniz cohomologies can be defined not only for pairing of trivial modules as it done in.[8] Let $\mathfrak{g}$ be a left Leibniz algebra, M be a symmetric module, N be an anti-symmetric module and S be an anti-symmetric module. Suppose that there is given a bilinear map

$$M \times N \to S, \quad (m, n) \mapsto m \cup n,$$

such that

$$X(m \cup n) = X(m) \cup n + m \cup X(n)$$

for any $X \in \mathfrak{g}, m \in M, n \in N$. This cup product can be prolonged until the bilinear map

$$C^k_{lei}(\mathfrak{g}, M) \times C^l_{lei}(\mathfrak{g}, N) \to C^{k+l}_{lei}(\mathfrak{g}, S)$$

by

$$\psi \cup \phi(X_1, \ldots, X_{k+l}) \\ = \sum_{\sigma} \operatorname{sign} \sigma\, \psi(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \cup \phi(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}).$$

Here summation indexes run $\sigma \in Sym_{k+l}$, such that $\sigma(1) < \cdots \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l), \sigma(k+l) = k+l$. This bilinear map is compatible with the action of a coboundary operator:

$$d(\psi \cup \phi) = d\psi \cup \phi + (-1)^{|\psi|} \psi \cup d\phi.$$

If $M = K$ be the trivial module and N is an antisymmetric module and $(K, N) \to N, (\lambda, n) \mapsto \lambda n$ be a natural bilinear map, then the cup-product satisfies the Leibniz dual rule

$$\psi \cup (\phi \cup \chi) = (\psi \cup \phi + (-1)^{|\psi||\phi|} \phi \cup \psi) \cup \chi,$$

where $\psi \in C^*_{lei}(\mathfrak{g}, K), \phi \in C^*_{lei}(\mathfrak{g}, N), \chi \in C^*_{lei}(\mathfrak{g}, N)$. This means that $H^*_{lei}(\mathfrak{g}, K)$ is not only right Leibniz dual algebra, but also $H^*_{lei}(\mathfrak{g}, N)$ is a left module over right Leibniz dual algebra $H^*_{lei}(\mathfrak{g}, K)$ if N is antisymmetric.

Another generalization concerns cup-products of right-symmetric algebras.

Example 2. Let A be a right-symmetric algebra and M be a right-symmetric module with a cup product

$$M \times M \to M, (m, n) \mapsto m \cup n.$$

For instance we can take as M the trivial module with a usual multiplication $K \times K \to K,\ (\lambda, \mu) \mapsto \lambda\mu$. Let

$$C^*_{\text{rsym}}(A, M) \times C^*_{\text{rsym}}(A, M) \to C^*_{\text{rsym}}(A, M), \quad (\psi, \phi) \mapsto \psi \cup \phi$$

be the cup product of corresponding cochain complexes.[20] Then this cup product satisfies Leibniz dual law

$$(\psi \cup \phi) \cup \chi = \psi \cup (\phi \cup \chi + (-1)^{|\phi||\chi|} \chi \cup \phi),$$

where $\psi \in C^*_{\text{rsym}}(A, M), \phi \in C^*_{\text{rsym}}(A, M), \chi \in C^*_{\text{rsym}}(A, M)$. For instance, the right-symmetric cohomology of the trivial module $H^*_{\text{rsym}}(A, K)$ is a left Leibniz dual algebra under the cup product and $H^*_{\text{rsym}}(A, M)$ is a right module over left Leibniz dual algebra $H^*_{\text{rsym}}(A, K)$.

Proof of this statement repeats arguments of.[8]

Example 3. Let $A = K[x]$ and $a \star b = a \int_0^x b\, dx$. Then $(A, \star)$ is a left Leibniz dual. It is easy to see that identity (1) is equivalent to the condition of integration by parts.

6. TORTKEN ALGEBRAS IN CHARACTERISTIC $p > 0$

Theorem 6.1. *Let K be a field of characteristic* $\operatorname{char} K = p > 0$, $(A, \circ)$ *an associative commutative algebra and D a derivation of $(A, \circ)$. Let $\Box = \Box_{k,l}$ be a new multiplication on A depending on some integers $0 \le k \le l$, such that*

$$a \Box b = D(D^{p^k-1}(a) \circ D^{p^l-1}(b) + D^{p^l-1}(a) \circ D^{p^k-1}(b)),$$

if $k \neq l$ and

$$a \Box b = D(D^{p^k-1}(a) \circ D^{p^k-1}(b)),$$

if $k = l$. Then

i) $(A, \square)$ *is Tortken, if* $k = l,\ p > 0$ *or* $l = k + 1,\ p = 2$.
ii) *If* $k \neq l,\ p > 2$ *or* $l - k > 1,\ p = 2$, *then there exists some associative commutative algebra* $(A, \circ)$, *such that* $(A, \circ)$ *is not Tortken.*

Proof. Define multiplications $\star$ and $\star'$ by

$$a \star b = D(D^{p^k-1}(a) \circ D^{p^l-1}(b)),$$
$$a \star' b = D(D^{p^l-1}(a) \circ D^{p^k-1}(b)).$$

Then

$$a \square b = a \star b + a \star' b$$

if $k \neq l$ and

$$a \circ b = a \star b,$$

if $k = l$.

Let

$$f(a,b,c,d) = (a \square b) \square (c \square d) - (a \square d) \circ (c \square b) - (a,b,c)^{\square} \square d + (a,d,c)^{\square} \square b.$$

For the multiplication $\star$ we have

$$\begin{aligned}(a,b,c)^{\star} &= a \star (b \star c) - (a \star b) \star c \\ &= D(D^{p^k-1}(a) \circ D^{p^l-1}(D(D^{p^k-1}(b) \circ D^{p^l-1}(c)))) \\ &\quad - D(D^{p^k-1}(D(D^{p^k-1}(a) \circ D^{p^l-1}(b))) \circ D^{p^l-1}(c)) \\ &= D(D^{p^k-1}(a) \circ D^{p^l}(D^{p^k-1}(b) \circ D^{p^l-1}(c)) \\ &\quad - D^{p^k}(D^{p^k-1}(a) \circ D^{p^l-1}(b)) \circ D^{p^l-1}(c)) \\ &= D\big(D^{p^k-1}(a) \circ \underline{D^{p^k+p^l-1}}(b) \circ D^{p^l-1}(c) + D^{p^k-1}(a) \\ &\qquad \circ D^{p^k-1}(b) \circ D^{2p^l-1}(c) - D^{2p^k-1}(a) \circ D^{p^l-1}(b) \circ D^{p^l-1}(c) \\ &\quad - D^{p^k-1}(a) \circ \underline{D^{p^k+p^l-1}}(b) \circ D^{p^l-1}(c) \big) \\ &= D\big(D^{p^k-1}(a) \circ D^{p^k-1}(b) \circ D^{2p^l-1}(c) \\ &\quad - D^{2p^k-1}(a) \circ D^{p^l-1}(b) \circ D^{p^l-1}(c)).\end{aligned}$$

Thus

$$\begin{aligned}(a,b,c)^{\star}\star d = {} & D(D^{p^k}\big(D^{p^k-1}(a)\circ D^{p^k-1}(b)\circ D^{2p^l-1}(c)\\ & \quad - D^{2p^k-1}(a)\circ D^{p^l-1}(b)\circ D^{p^l-1}(c)\big)\circ D^{p^l-1}(d))\\ = {} & D\big(D^{2p^k-1}(a)\circ D^{p^k-1}(b)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(d)\\ & + D^{p^k-1}(a)\circ D^{2p^k-1}(b)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(d)\\ & + D^{p^k-1}(a)\circ D^{p^k-1}(b)\circ D^{p^k+2p^l-1}(c)\circ D^{p^l-1}(d)\\ & - D^{3p^k-1}(a)\circ D^{p^l-1}\underline{(b)\circ D}^{p^l-1}(c)\big)\circ D^{p^l-1}(d)\\ & - D^{2p^k-1}(a)\circ D^{p^k+p^l-1}(b)\circ D^{p^l-1}(c)\big)\circ D^{p^l-1}(d)\\ & - D^{2p^k-1}(a)\circ D^{p^l-1}\underline{\underline{(b)\circ D}}^{p^k+p^l-1}(c)\circ D^{p^l-1}(d)\big).\end{aligned}$$

Similarly,

$$\begin{aligned}(a,d,c)^{\star}\star b = {} & D\big(D^{2p^k-1}(a)\circ D^{p^k-1}(d)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(b)\\ & + D^{p^k-1}(a)\circ D^{2p^k-1}(d)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(b)\\ & + D^{p^k-1}(a)\circ D^{p^k-1}(d)\circ D^{p^k+2p^l-1}(c)\circ D^{p^l-1}(b)\\ & - D^{3p^k-1}(a)\circ D^{p^l-1}\underline{(d)\circ D}^{p^l-1}(c)\big)\circ D^{p^l-1}(b)\\ & - D^{2p^k-1}(a)\circ D^{p^k+p^l-1}(d)\circ D^{p^l-1}(c)\circ D^{p^l-1}(d)\\ & - D^{2p^k-1}(a)\circ D^{p^l-1}\underline{\underline{(d)\circ D}}^{p^k+p^l-1}(c)\circ D^{p^l-1}(b)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}& 3(a,b,c)^{\star}\star d - (a,d,c)^{\star}\star b\\ & \quad = D\big(D^{2p^k-1}(a)\circ D^{p^k-1}(b)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(d).\\ & \qquad + D^{p^k-1}(a)\circ D^{2p^k-1}(b)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(d)\\ & \qquad + D^{p^k-1}(a)\circ D^{p^k-1}(b)\circ D^{p^k+2p^l-1}(c)\circ D^{p^l-1}(d)\\ & \qquad - D^{2p^k-1}(a)\circ D^{p^k+p^l-1}(b)\circ D^{p^l-1}(c)\circ D^{p^l-1}(d)\\ & \qquad - D^{2p^k-1}(a)\circ D^{p^k-1}(d)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(b)\\ & \qquad - D^{p^k-1}(a)\circ D^{2p^k-1}(d)\circ D^{2p^l-1}(c)\circ D^{p^l-1}(b)\\ & \qquad - D^{p^k-1}(a)\circ D^{p^k-1}(d)\circ D^{p^k+2p^l-1}(c)\circ D^{p^l-1}(b)\\ & \qquad + D^{2p^k-1}(a)\circ D^{p^k+p^l-1}(d)\circ D^{p^l-1}(c)\circ D^{p^l-1}(b)\big).\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&(a \star b) \star (c \star d) \\
&\quad = D(D^{p^k}(D^{p^k-1}(a) \circ D^{p^l-1}(b)) \circ D^{p^l}(D^{p^k-1}(c) \circ D^{p^l-1}(d))) \\
&\quad = D(D^{2p^k-1}(a) \circ D^{p^l-1}(\underline{b) \circ D}^{p^k+p^l-1}(c) \circ D^{p^l-1}(d) \\
&\qquad + D^{p^k-1}(a) \circ D^{p^k+p^l-1}(b) \circ D^{p^k+p^l-1}(c) \circ D^{p^l-1}(d) \\
&\qquad + D^{2p^k-1}(a) \circ D^{p^l-1}(b) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(d) \\
&\qquad + D^{p^k-1}(a) \circ D^{p^k+p^l-1}(b) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(d)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(a \star d) \star (c \star b) \\
&\quad = D(D^{2p^k-1}(a) \circ D^{p^l-1}(\underline{d) \circ D}^{p^k+p^l-1}(c) \circ D^{p^l-1}(b) \\
&\qquad + D^{p^k-1}(a) \circ D^{p^k+p^l-1}(d) \circ D^{p^k+p^l-1}(c) \circ D^{p^l-1}(b) \\
&\qquad + D^{2p^k-1}(a) \circ D^{p^l-1}(d) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(b) \\
&\qquad + D^{p^k-1}(a) \circ D^{p^k+p^l-1}(d) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(b)).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(a \star b) \star (c \star d) - (a \star d) \star (c \star b) \\
&\quad = D(D^{p^k-1}(a) \circ D^{p^k+p^l-1}(b) \circ D^{p^k+p^l-1}(c) \circ D^{p^l-1}(d) \\
&\qquad + D^{2p^k-1}(a) \circ D^{p^l-1}(b) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(d) \\
&\qquad + D^{p^k-1}(a) \circ D^{p^k+p^l-1}(b) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(d) \\
&\qquad - D^{p^k-1}(a) \circ D^{p^k+p^l-1}(d) \circ D^{p^k+p^l-1}(c) \circ D^{p^l-1}(b) \\
&\qquad - D^{2p^k-1}(a) \circ D^{p^l-1}(d) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(b) \\
&\qquad - D^{p^k-1}(a) \circ D^{p^k+p^l-1}(d) \circ D^{p^k-1}(c) \circ D^{2p^l-1}(b)).
\end{aligned}
$$

Now we are ready to establish i). We see that, if $k = l$, then

$$
\begin{aligned}
f(a, b, c, d) = D(&D^{p^k-1}(a) \circ D^{2p^k-1}(\underline{b) \circ D}^{2p^k-1}(c) \circ D^{p^k-1}(d) \\
&- D^{2p^k-1}(a) \circ D^{2p^k-1}\underline{\underline{(b) \circ D}}^{p^k-1}(c) \circ D^{p^k-1}(d) \\
&- D^{p^k-1}(a) \circ D^{2p^k-1}\utilde{(d) \circ D}^{2p^k-1}(c) \circ D^{p^k-1}(b)
\end{aligned}
$$

$$+ D^{2p^k-1}(a) \circ D^{2p^k-1}\underset{\sim\sim\sim}{(d)} \circ D^{p^k-1}(c) \circ D^{p^k-1}(b)$$
$$- D^{p^k-1}(a) \circ D^{2p^k-1}\underline{(b)} \circ D^{2p^k-1}(c) \circ D^{p^k-1}(d)$$
$$- D^{2p^k-1}(a) \circ D^{p^k-1}\underset{\sim\sim\sim}{(b)} \circ D^{p^k-1}(c) \circ D^{2p^k-1}(d)$$
$$+ D^{p^k-1}(a) \circ D^{2p^k-1}\underset{\sim\sim\sim}{(d)} \circ D^{2p^k-1}(c) \circ D^{p^k-1}(b)$$
$$+ D^{2p^k-1}(a) \circ D^{p^k-1}\underline{\underline{(d)}} \circ D^{p^k-1}(c) \circ D^{2p^k-1}(b)) = 0.$$

The case $p = 2,\ l = k + 1$ can be established by analogous way. One just need to use the following relations $2p^k = p^l, p^k + p^l - 1 = 3p^k - 1$.

To prove ii) take the algebra of divided powers $O_1(m)$ as A. Then, substitutions $a = x^{(p^k-1)},\ b = x^{(2p^k-1)},\ c = x^{(p^l-1)},\ d = x^{(2p^l)}$ in formulas for $f(a, b, c, d)$ show that

$$f(x^{(p^k-1)},\ x^{(2p^k-1)},\ x^{(p^l-1)},\ x^{(2p^l)}) = -1,$$

if $l > k,\ p > 2$ or $l - k > 1,\ p = 2$.

Remark. We was not able to construct in $O_1(m)$ some Novikov multiplication $*$, such that $a \square b = a * b + b * a$. We do not know whether the divided power algebra $O_1(m)$ under Tortken multiplication $\square$ constructed in Theorem 6.1 is special.

7. CONSEQUENCES OF TORTKEN IDENTITY OF DEGREE 5

Theorem 7.1. *Let A be a commutative Tortken algebra and a, b, c, x, y are any elements of A. Then*

i)

$$(a, y, b) \star (x \star c) + (b, y, c) \star (x \star a) + (c, y, a) \star (x \star b)$$
$$- (a \star y) \star (b, x, c) - (b \star y) \star (c, x, a) - (c \star y) \star (a, x, b) = 0,$$

ii) $(p \neq 3)$

$$((x \star a) \star (y \star b) - (x \star b) \star (y \star a)) \star c$$
$$+ ((x \star b) \star (y \star c) - (x \star c) \star (y \star b)) \star a$$
$$+ ((x \star c) \star (y \star a) - (x \star a) \star (y \star c)) \star b = 0,$$

iii)

$$(x, y \star a, b) \star c + (x, y \star b, c) \star a + (x, y \star c, a) \star b \\ - (x, y \star b, a) \star c - (x, y \star c, b) \star a - (x, y \star a, c) \star b = 0,$$

iv)

$$(((x \star a) \star b) \star c) \star y + (((x \star b) \star c) \star a) \star y + (((x \star c) \star a) \star b) \star y \\ - (((x \star b) \star a) \star c) \star y - (((x \star c) \star b) \star a) \star y \\ - (((x \star a) \star c) \star b) \star y = 0.$$

Proof. i) According (3) for $a_1 = a,\ a_2 = b,\ a_3 = c$,

$$(a_{\sigma(1)}, x \star a_{\sigma(3)}, a_{\sigma(2)}) \star y \\ = (a_{\sigma(1)}, y, a_{\sigma(2)}) \star (x \star a_{\sigma(3)}) - (a_{\sigma(1)} \star y) \star (a_{\sigma(2)} \star (x \star a_{\sigma(3)})) \\ + (a_{\sigma(1)} \star (x \star a_{\sigma(3)})) \star (a_{\sigma(2)} \star y).$$

Let Sym_3^0 be the cyclic subgroup of Sym_3 of order 3. Therefore,

$$\sum_{\sigma \in \mathrm{Sym}_3^0} (a_{\sigma(1)}, y, a_{\sigma(2)}) \star (x \star a_{\sigma(3)}) - (a_{\sigma(1)} \star y) \star (a_{\sigma(2)} \star (x \star a_{\sigma(3)})) \\ + (a_{\sigma(1)} \star (x \star a_{\sigma(3)})) \star (a_{\sigma(2)} \star y) \\ = y \star \sum_{\sigma \in \mathrm{Sym}_3^0} (a_{\sigma(1)}, x \star a_{\sigma(3)}, a_{\sigma(2)}) \\ =$$

(by corollary 2.4)

$$= 0.$$

Since

$$(a, y, b) \star (x \star c) + (b, y, c) \star (x \star a) + (c, y, a) \star (x \star b) \\ - (a \star y) \underline{\star (b \star} (x \star c)) + (b \star y) \underline{\underline{\star (a \star}} (x \star c)) \\ - (b \star y) \underline{\underline{\star (c \star}} (x \star a)) + (c \star y) \utilde{\star (b \star} (x \star a)) \\ - (c \star y) \utilde{\star (a \star} (x \star b)) + (a \star y) \underline{\star (c \star} (x \star b))$$

$$= (a, y, b) \star (x \star c) + (b, y, c) \star (x \star a) + (c, y, a) \star (x \star b)$$
$$- (a \star y) \star (b, x, c) - (b \star y) \star (c, x, a) - (c \star y) \star (a, x, b),$$

our statement is proved.

ii) Set

$$g = ((x \star a) \star (y \star b) - (x \star b) \star (y \star a)) \star c$$
$$+ ((x \star b) \star (y \star c) - (x \star c) \star (y \star b)) \star a$$
$$+ ((x \star c) \star (y \star a) - (x \star a) \star (y \star c)) \star b.$$

According (3),

$$((x, a, y) \star b - (x, b, y) \star a) \star c$$
$$+ ((x, b, y) \star c - (x, c, y) \star b) \star a$$
$$+ ((x, c, y) \star a - (x, a, y) \star c) \star b$$
$$= ((x, a, y) \star b) \star c - ((x, a, y) \star c) \star b$$
$$+ ((x, b, y) \star c) \star a - ((x, b, y) \star a) \star c$$
$$- ((x, c, y) \star b) \star a + ((x, c, y) \star a) \star b$$
$$= (a, (x, b, y), c) + (b, (x, c, y), a) + (c, (x, a, y), b)$$
$$= (a, x \star (y \star b), c) - (a, y \star (x \star b), c)$$
$$(b, x \star (y \star c), a) - (b, y \star (x \star c), a)$$
$$(c, x \star (y \star a), b) - (c, y \star (x \star a), b).$$

We have

$$(a, x \star (y \star b), c) =$$

(by Corollary 2.4)

$$= (a, c \star x, y \star b) - (c, a \star x, y \star b).$$

Similarly,

$$(b, x \star (y \star c), a) = (b, a \star x, y \star c) - (a, b \star x, y \star c)$$
$$(c, x \star (y \star a), b) = (c, b \star x, y \star a) - (b, c \star x, y \star a)$$

$$
\begin{aligned}
-(a, y\star(x\star b), c) &= -(a, c\star y, x\star b) + (c, a\star y, x\star b),\\
-(b, y\star(x\star c), a) &= -(b, a\star y, x\star c) + (a, b\star y, x\star c),\\
-(c, y\star(x\star a), b) &= -(c, b\star y, x\star a) + (b, c\star y, x\star a).
\end{aligned}
$$

Hence

$$
\begin{aligned}
g = {} & (a, c\star x, y\star b) - (c, a\star x, y\star b)\\
& + (b, a\star x, y\star c) - (a, b\star x, y\star c)\\
& + (c, b\star x, y\star a) - (b, c\star x, y\star a)\\
& - (a, c\star y, x\star b) + (c, a\star y, x\star b)\\
& - (b, a\star y, x\star c) + (a, b\star y, x\star c)\\
& - (c, b\star y, x\star a) + (b, c\star y, x\star a).
\end{aligned}
$$

Collect together all associators with the first element a:

$$
\begin{aligned}
& (a, c\star x, y\star b) - (a, b\star x, y\star c) - (a, c\star y, x\star b) + (a, b\star y, x\star c)\\
&\quad = a\star((c\star x)\star(y\star b)) - (a\star(c\star x))\star(y\star b)\\
&\qquad - a\star((b\star x)\star(y\star c)) + (a\star(b\star x))\star(y\star c)\\
&\qquad - a\star((c\star y)\star(x\star b)) + (a\star(c\star y))\star(x\star b)\\
&\qquad + a\star((b\star y)\star(x\star c)) - (a\star(b\star y))\star(x\star c)\\
&\quad = 2a\star((c\star x)\star(y\star b)) - (a\star(c\star x))\star(y\star b)\\
&\qquad - 2a\star((b\star x)\star(y\star c)) + (a\star(b\star x))\star(y\star c)\\
&\qquad + (a\star(c\star y))\star(x\star b) - (a\star(b\star y))\star(x\star c).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& (b, a\star x, y\star c) - (b, c\star x, y\star a) - (b, a\star y, x\star c) + (b, c\star y, x\star a)\\
&\quad = 2b\star((a\star x)\star(y\star c)) - (b\star(a\star x))\star(y\star c)\\
&\qquad - 2b\star((c\star x)\star(y\star a)) + (b\star(c\star x))\star(y\star a)\\
&\qquad + (b\star(a\star y))\star(x\star c) - (b\star(c\star y))\star(x\star a)
\end{aligned}
$$

and

$$
\begin{aligned}
& (c, b\star x, y\star a) - (c, b\star x, y\star b) - (c, b\star y, x\star a) + (c, a\star y, x\star b)\\
&\quad = 2c\star((b\star x)\star(y\star a)) - (c\star(b\star x))\star(y\star a)\\
&\qquad - 2c\star((a\star x)\star(y\star b)) + (c\star(a\star x))\star(y\star b)\\
&\qquad + (c\star(b\star y))\star(x\star a) - (c\star(a\star y))\star(x\star b).
\end{aligned}
$$

Hence

$$\begin{aligned}
g &= 2a\star((c\star x)\star(y\star b)) - (a\star(c\star x))\star(y\star b)\\
&\quad - 2a\star((b\star x)\star(y\star c)) + (a\star(b\star x))\star(y\star c)\\
&\quad + (a\star(c\star y))\star(x\star b) - (a\star(b\star y))\star(x\star c)\\
&\quad + 2b\star((a\star x)\star(y\star c)) - (b\star(a\star x))\star(y\star c)\\
&\quad - 2b\star((c\star x)\star(y\star a)) + (b\star(c\star x))\star(y\star a)\\
&\quad + (b\star(a\star y))\star(x\star c) - (b\star(c\star y))\star(x\star a)\\
&\quad + 2c\star((b\star x)\star(y\star a)) - (c\star(b\star x))\star(y\star a)\\
&\quad - 2c\star((a\star x)\star(y\star b)) + (c\star(a\star x))\star(y\star b)\\
&\quad + (c\star(b\star y))\star(x\star a) - (c\star(a\star y))\star(x\star b)\\
&= -2g - (a\star(c\star x))\star(y\star b) + (a\star(b\star x))\star(y\star c)\\
&\quad + (a\star(c\star y))\star(x\star b) - (a\star(b\star y))\star(x\star c)\\
&\quad - (b\star(a\star x))\star(y\star c) + (b\star(c\star x))\star(y\star a)\\
&\quad + (b\star(a\star y))\star(x\star c) - (b\star(c\star y))\star(x\star a)\\
&\quad - (c\star(b\star x))\star(y\star a) + (c\star(a\star x))\star(y\star b)\\
&\quad + (c\star(b\star y))\star(x\star a) - (c\star(a\star y))\star(x\star b).
\end{aligned}$$

Collect in the last expression all terms of the form $(\cdots)\star(y\star b)$. We obtain

$$-(a\star(c\star x))\star(y\star b) + (c\star(a\star x))\star(y\star b) = (c,x,a)\star(y\star b).$$

Collect similarly, all terms of the form $(\cdots)\star(u\star v)$, where $u = x, y,\ v = a, b, c$. We have

$$\begin{aligned}
g &= -2g + (a,x,b)\star(y\star c) + (b,x,c)\star(y\star a) + (c,x,a)\star(y\star b)\\
&\quad - (a,y,b)\star(x\star c) - (b,y,c)\star(x\star a) - (c,y,a)\star(x\star b)\\
&=
\end{aligned}$$

(by (i)) $= -2g$.

Thus, $3g = 0$, and $g = 0$, if $p \neq 3$.

iii) Follows from Corollaries 2.4 and 2.5.
iv) Follows from Proposition 2.3.

Corollary 7.2. *If A is a commutative Tortken algebra over a field of characteristic $p \neq 3$, then*

$$(a,(b,x,c),y)+(b,(c,x,a),y)+(c,(a,x,b),y)\\ -(a,(b,y,c),x)-(b,(c,y,a),x)-(c,(a,y,b),x)=0,$$

for any $a,b,c,x,y \in A$.

Proof. By commutativity rule,

$$\begin{aligned}&((x\star a)\star(y\star b)-(x\star b)\star(y\star a))\star c+((x\star b)\star(y\star c)\\ &\quad-(x\star c)\star(y\star b))\star a+((x\star c)\star(y\star a)\\ &\quad-(x\star a)\star(y\star c))\star b\\ &=((a\star x)\star(b\star y)-(a\star y)\star(b\star x))\star c\\ &\quad+((b\star x)\star(c\star y)-(b\star y)\star(c\star x))\star a\\ &\quad+((c\star x)\star(a\star y)-(c\star y)\star(a\star x))\star b\end{aligned}$$

(identity (3))

$$\begin{aligned}&((a,x,b)\star y-(a,y,b)\star x)\star c+((b,x,c)\star y-(b,y,c)\star x)\star a\\ &\quad+((c,x,a)\star y-(c,y,a)\star x)\star b\\ &\qquad=((a,x,b)\star y)\star c-((a,y,b)\star x)\star c+((b,x,c)\star y)\star a\\ &\quad-((b,y,c)\star x)\star a+((c,x,a)\star y)\star b-((c,y,a)\star x)\star b=\end{aligned}$$

(by corollary 2.5.)

$$\begin{aligned}&((a,x,b)\star y)\star c-((a,y,b)\star x)\star c\\ &\quad+((b,x,c)\star y)\star a-((b,y,c)\star x)\star a\\ &\quad+((c,x,a)\star y)\star b-((c,y,a)\star x)\star b\\ &\quad-((a,x,b)\star c)\star y+((a,y,b)\star c)\star x\\ &\quad-((b,x,c)\star a)\star y+((b,y,c)\star a)\star x\\ &\quad-((c,x,a)\star b)\star y+((c,y,a)\star b)\star x\\ &\qquad=+(a,(b,x,c),y)+(b,(c,x,a),y)+(c,(a,x,b),y)\\ &\quad-(a,(b,y,c),x)-(b,(c,y,a),x)-(c,(a,y,b),x).\end{aligned}$$

It remains to use Theorem 7.1.

Remark. Very likely that all identities of degree 5 of free special Tortken algebra follow from four types of identities mentioned in Theorem 7.1 if

$p > 3$. It will be interesting to find some s-identity (if such identity exists), i.e., an identity that is true for any special Tortken algebra, but is not true for some commutative Tortken algebra.

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