COMMUNICATIONS IN ALGEBRA®
Vol. 32, No. 9, pp. 3315–3326, 2004

Representations of Vector Product n-Lie Algebras#

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ABSTRACT

Let $V_n = \langle e_1, \ldots, e_{n+1} \rangle$ be the vector product n-Lie algebra with n-Lie commutator $[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = (-1)^i e_i$ over the field of complex numbers. Any finite-dimensional n-Lie V_n-module is completely reducible. Any finite-dimensional irreducible n-Lie V_n-module is isomorphic to an n-Lie extension of so_{n+1}-module with highest weight $t\pi_1$ for some nonnegative integer t.

Key Words: Vector products algebra; Lie algebras; n-Lie algebras; Nambu algebras; Representations; N-Commutators.

1991 Mathematics Subject Classification: Primary 17C50; 17B66; 22E65.

#Communicated by I. Shestakov.

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DOI: 10.1081/AGB-120038644
0092-7872 (Print); 1532-4125 (Online)
www.dekker.com

1. INTRODUCTION

An n-algebra $A = (A, [,\ldots,])$ with a skew-symmetric n-multiplication $[,\ldots,] : \wedge^n A \to A, (a_1,\ldots,a_n) \mapsto [a_1,\ldots,a_n]$ is called *n-Lie*, if

$$[a_1,\ldots,a_{n-1},[a_n,\ldots,a_{2n-1}]] = \sum_{i=n}^{2n-1} (-1)^{i+n}[[a_1,\ldots,a_{n-1},a_i],a_n,\ldots,\hat{a}_i,\ldots,a_{2n-1}],$$

for any $a_1,\ldots,a_{2n-1} \in A$. Here $\hat{a}_i$ means that the element a_i is omitted. n-Lie algebras was firstly defined in Filippov (1985). Sometimes they are called as Filippov, Nambu or Takhtajan algebras.

To any n-Lie algebra one can associate a Lie algebra $L(A) = \wedge^{n-1}A$, called *basic* Lie algebra, with a multiplication given by

$$[a_1 \wedge \cdots \wedge a_{n-1}, b_1 \wedge \cdots \wedge b_{n-1}] = \sum_{i=1}^{n-1} (-1)^{i+n}[[a_1,\ldots,a_{n-1},b_i],b_1,\ldots,\hat{b}_i,\ldots,b_{n-1}],$$

or by

$$[a_1 \wedge \cdots \wedge a_{n-1}, b_1 \wedge \cdots \wedge b_{n-1}] = \sum_{i=1}^{n-1} (-1)^{i+1}[a_1,\ldots,\hat{a}_i,\ldots,a_n,[a_i,b_1,\ldots,b_{n-1}]],$$

where $\hat{b}_i$ means that the element b_i is omitted.

Example 1. Let $A = K[x_1,\ldots,x_n]$ under Jacobian map

$$(a_1,\ldots,a_n) \mapsto \det(\partial_i(a_j)).$$

Then A is n-Lie (Filippov, 1985, 1998) and its basic algebra is isomorphic to divergenceless vector fields algebra S_{n-1} (Dzhumadil'daev, 2002).

Example 2. Let V_n be $(n+1)$-dimensional vector space with a basis $\{e_1,\ldots,e_{n+1}\}$. Then V_n under a n-Lie multiplication

$$[e_1,\ldots,\hat{e}_i,\ldots,e_{n+1}] = (-1)^i e_i$$

can be endowed by a structure of n-Lie algebra. This algebra is called *vector product n-Lie algebra*. For $n = 2$ we obtain well known vector product algebra on K^3. From results of Filippov (1985) it follows that $L(V_n) \cong so_{n+1}$.

One can expect that the n-Lie algebra V_n plays in a theory of n-Lie algebras a role like sl_2 in theory of Lie algebras. The aim of our paper is to describe all finite-dimensional representations of vector products n-Lie algebra over the field of complex numbers.

Let $\pi_1, \ldots, \pi_{[n+1/2]}$ be the fundamental weights for so_{n+1}. Recall that $so_4 \cong sl_2 \oplus sl_2$ and any irreducible so_4-module can be realized as $sl_2 \oplus sl_2$-module $M_{t,r} = M_t \otimes M_r$, where M_t denotes $(t+1)$-dimensional irreducible sl_2-module with highest weight t. The main result of our paper is the following:

Theorem 1.1. $K = \mathbf{C}$.

(i) *Any finite-dimensional n-Lie representation of $V_n, n \geq 2$, is completely reducible.*
(ii) *Let $M_{t,r}$ be an irreducible so_4-module with highest weight (t, r). Then $M_{t,r}$ can be prolonged to 3-Lie module over V_3, if and only if $t = r$.*
(iii) *Let M be an irreducible module of Lie algebra so_{n+1}, $n > 3$, with highest weight α. Then M can be prolonged to n-Lie module of V_n, if and only if α has a form $t\pi_1$, for some nonnegative integer t.*

So, we obtain a complete description of finite-dimensional n-Lie V_n-modules over $\mathbf{C}$. Our result shows that any irreducible n-Lie representation of V_n is ruled by some nonnegative integer t as in Lie case $V_2 \cong sl_2$. Call t mentioned in Theorem 1.1 *n-Lie highest weight.*

Corollary 1.2. $(K = \mathbf{C}, n > 2)$ *The dimension of any irreducible n-Lie V_n-module with highest weight t is equal to* $\frac{n+2t-1}{n+t-1}\binom{n+t-1}{t}$.

For example, the dimension of any irreducible V_3-module with highest weight t is equal to $(t+1)^2$.

Remark. If $n = 3$ and if we consider infinite-dimensional modules, then studying of V_3-representations can be reduced to the problem on describing of gl_λ-modules. A definition of complex size matrices algebra gl_λ (see Dixmier, 1973; Feigin, 1988). One can prove that $U(V_3)$ has a subalgebra isomorphic to $gl_\lambda \otimes gl_\lambda$.

2. n-LIE MODULES

Let A be an n-Lie algebra. Let $\operatorname{End} A$ be a space of linear maps $A \to A$. Recall that an operator $D \in \operatorname{End} A$ is called *derivation*, if

$$D([a_1, \ldots, a_n]) = \sum_{i=1}^{n} [a_1, \ldots, a_{i-1}, D(a_i), a_{i+1}, \ldots, a_n],$$

for any $a_1, \ldots, a_n \in A$. Let $\operatorname{Der} A$ be a space of derivations of A. According n-Lie identity for any $n-1$ elements $a_1, \ldots, a_{n-1} \in A$ one can correspond adjoint derivation $\operatorname{ad}\{a_1, \ldots, a_{n-1}\} \in \operatorname{Der} A$ by the rule $\operatorname{ad}\{a_1, \ldots, a_{n-1}\}a_n = [a_1, \ldots, a_n]$. Denote by $\operatorname{Int} A$ a space generated by adjoint derivations of A. Call a derivation $D \in \operatorname{Der} A$ *inner*, if $D \in \operatorname{Int} A$. Then $\operatorname{Der} A$ is a Lie algebra under commutator $(D_1, D_2) \mapsto [D_1, D_2] := D_1 D_2 - D_2 D_1$ and $\operatorname{Int} A$ is its Lie ideal. If A has no center, $\operatorname{Int} A$ is isomorphic to $L(A)$.

An n-Lie module over n-Lie algebra A is defined as a vector space M such that a semi-direct sum $A+M$ is once again n-Lie. These mean that the n-Lie multiplication $[,\dots,]$ on A is continued to $A+M$ such that $[a_1,\dots,a_n]=0$, if at least two arguments among $a_1,\dots,a_n$ belong to M and the n-Lie identity is true for any $a_1,\dots,a_n\in A+M$. In other words,

$$[a_1,\dots,a_i,m,a_{i+1},\dots,a_n]=-[a_1,\dots,a_i,a_{i+1},m,\dots,a_n],\quad 1\le i<n,$$

$$[a_1,\dots,a_{n-1},[a_n,\dots,a_{2n-2},m]]-[a_n,\dots,a_{2n-2},[a_1,\dots,a_{n-1},m]]$$
$$=\sum_{i=n}^{2n-2}[a_n,\dots,a_{i-1},[a_1,\dots,a_{n-1},a_i],a_{i+1},\dots,a_{2n-2},m],$$

$$[a_1,\dots,a_{n-2},[a_n,\dots,a_{2n-1}],m]$$
$$=\sum_{i=n}^{2n-1}[a_n,\dots,a_{i-1},[a_1,\dots,a_{n-2},a_i,m],a_{i+1},\dots,a_{2n-1}],$$

for any $a_1,\dots,a_{n-2},a_n,\dots,a_{2n-1}\in A$ and $m\in M$. So, any module of n-Lie algebra is an usual module of Lie algebra, if $n=2$. If $n>2$, then any module of n-Lie algebra A is a Lie module of the basic Lie algebra $L(A)$ under representation $\rho:\wedge^{n-1}A\to \operatorname{End}M$ defined by $\rho(a_1\wedge\cdots\wedge a_{n-1})(m)=[a_1,\dots,a_{n-1},m]$, such that

$$\rho([a_1,\dots,a_n]\wedge a_{n+1}\wedge\cdots\wedge a_{2n-2})$$
$$=\sum_{i=1}^{n}(-1)^{i+n}\rho(a_1\wedge\ldots\wedge\hat{a}_i\wedge\ldots\wedge a_n)\rho(a_i\wedge a_{n+1}\wedge\cdots\wedge a_{2n-2}),\qquad(1)$$

for any $a_1,\dots,a_{2n-2}\in A$. If M is a Lie module over Lie algebra $L(A)$ that satisfies the condition (1) for any $a_1,\dots,a_{2n-2}\in A$, then we will say that Lie module structure on M over $L(A)$ can be *prolonged* to a n-Lie module structure over n-Lie algebra A, or shortly that Lie module M can be prolonged to n-Lie module.

Example. For any n-Lie algebra A its adjoint module, i.e., a module with vector space A and the action $(a_1\wedge\cdots\wedge a_{n-1})b=[a_1,\dots,a_{n-1},b]$ is n-Lie module.

Let A be an n-Lie algebra. Denote by $\widetilde{U}(A)$ the universal enveloping algebra of the Lie algebra $L(A)$. Let $Q(A)$ be an ideal of $\widetilde{U}(A)$ generated by elements

$$X_{a_1,\dots,a_{2n-2}}=[a_1,\dots,a_n]\wedge a_{n+1}\wedge\cdots\wedge a_{2n-2}$$
$$-\sum_{i=1}^{n}(-1)^{i+n}(a_1\wedge\cdots\wedge\hat{a}_i\cdots\wedge a_n)\,(a_i\wedge a_{n+1}\wedge\cdots\wedge a_{2n-2}).$$

Let $\bar{U}(A)=\widetilde{U}(A)/Q(A)$.

Any Lie module of $L(A)$ can be prolonged to n-Lie module, if and only if it is trivial $Q(A)$-module. In other words, there are one-to one correspondence between n-Lie modules and $\bar{U}(A)$-modules. In this sense $\bar{U}(A)$ can be considered as universal enveloping algebra of n-Lie algebra A.

Let M be a n-Lie module over n-Lie algebra A. Let N be a subspace of M, such that $[a_1,\dots,a_{i-1},m,a_{i+1},\dots,a_{2n-1}]\in N$, for any $m\in M_1$, $i=1,\dots,n$, and

$a_1, \dots, \hat{a}_i, \dots, a_{2n-1} \in A$. In such case we will say that N is *n-Lie submodule* of M. Any module has trivial submodules 0 and M. Call M *irreducible*, if any its submodule is trivial. Call M *completely reducible*, if it can be decomposed to a direct sum of irreducible submodules.

Proposition 2.1. *Let M be a n-Lie module over n-Lie algebra A. Then M is irreducible, if and only if M is irreducible as a Lie module over Lie algebra $L(A)$. M is completely reducible, if and only if M is completely reducible as a Lie module over Lie algebra $L(A)$.*

3. VECTOR PRODUCT n-LIE ALGEBRA AND ITS MODULES

Let V_n be a vector product n-Lie algebra over $\mathbf{C}$. It is $(n+1)$-dimensional and the multiplication on a basis $\{e_1, \dots, e_{n+1}\}$ is given by

$$[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = (-1)^i e_i, \quad i = 1, \dots, n+1.$$

For example, V_2 is the vector product algebra on $\mathbf{C}^3$ and as a Lie algebra it is isomorphic to sl_2.

Recall that the Lie algebra of skew-symmetric $n \times n$-matrices so_n, $n \geq 3$, is semi-simple over $K = \mathbf{C}$. More exactly, it is simple, if $n \neq 4$ and has type $B_{[n/2]}$, if n is odd and type $D_{n/2}$, if n is even. If $n = 4$, then $so_4 \cong A_1 \oplus A_1$. For $n = 3$, $so_3 \cong A_1$.

For $\lambda \in \mathbf{Q}$ denote by $[\lambda]$ a maximal integer, such that $[\lambda] \leq \lambda$. Let $\pi_1, \dots, \pi_{[n/2]}$ be the fundamental weights of so_n and $M(\alpha)$ be the irreducible so_n-module with highest weight α. Any highest weight can be characterized by $[n/2]$-type of nonnegative integers $\{\alpha_1, \dots, \alpha_{[n/2]}\}$, namely

$$\alpha = \sum_{i=1}^{[n/2]} \alpha_i \pi_i.$$

There is another way to describe highest weights.

Suppose that a sequence of integers or half-integers $\lambda = \{\lambda_1, \dots, \lambda_{[n/2]}\}$ satisfies the following conditions

- $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{[n/2]} \geq 0$, if n is odd and $\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_{n/2}|$, if n is even.
- $\alpha_i, i = 1, \dots, [n/2]$, are nonnegative integers, where $\alpha_i = \lambda_i - \lambda_{i+1}, i = 1, \dots, [n/2] - 1$ and $\alpha_{[n/2]} = 2\lambda_{[n/2]}$, if n is odd and $\alpha_{n/2} = \lambda_{n/2-1} + \lambda_{n/2}$, if n is even.

Then any irreducible finite-dimensional so_n-module with highest weight α can be restored by a such sequence λ.

Let M be an irreducible so_n-module. For $n \neq 4$, set $q(M) = r$, if its highest weight α satisfies the condition $\alpha_r \neq 0$, but $\alpha_{r'} = 0$, for any $r' > r$. For $n = 4$, set $q(M) = 1$, if so_4-module is isomorphic to $M_{t,t}$, for some nonnegative integer t and $q(M) = 2$, if $M \cong M_{t,r}$, for some $t \neq r$.

Let α be a highest weight for so_n-module and $n \neq 4$. Then $q(\alpha) = 1$, if and only if α has the form $k\pi_1$ for some nonnegative integer k.

Any finite-dimensional irreducible sl_2-module is isomorphic to $(l+1)$-dimensional irreducible module M_l with highest weight l. Recall that any highest weight of sl_2 can be identified with some nonnegative integer. As we mentioned above $so_4 \cong sl_2 \oplus sl_2$. Any irreducible so_4-module M can be characterized by two nonnegative integers (t, r). Namely, $M \cong M_{t,r} = M_t \otimes M_r$, where the action of $a + b$ on $m' \otimes m''$, where a is an element of the first copy of sl_2 and b is an element of the second copy of sl_2 and $m' \in M_t, m'' \in M_r$, is given by

$$(a+b)(m' \otimes m'') = a(m') \otimes m'' + m' \otimes b(m'').$$

Notice that in this realization to so_4-module M with $q(M) = 1$ corresponds the $sl_2 \oplus sl_2$-module $M_{t,t}$, for $t \geq 0, t \in \mathbf{Z}$.

Filippov (1985) proved that V_n is simple and any derivation of V_n is inner. Therefore, $\wedge^{n-1} V_n \cong \operatorname{Int} V_n$. More detailed observation of his proof shows that takes place the following

Theorem 3.1. *For any $n \geq 2$, $\operatorname{Der} V_n \cong \operatorname{Int} V_n \cong so_{n+1}$. The isomorphism of Lie algebras $L(V_n) \cong so_{n+1}$ can be given by*

$$e_1 \wedge \cdots \hat{e}_i \wedge \cdots \hat{e}_j \wedge \cdots \wedge e_{n+1} \mapsto (-1)^{i+j+n+1} e_{ij}, \quad i < j.$$

where e_{ij} is a skew-symmetric matrix with (i,j)th component 1, (j,i)th component -1 and other components 0.

Lemma 3.2. *Let M be so_{n+1}-module. Define quadratic elements R_{ijsk} of $U(so_{n+1})$ by*

$$R_{ijsk} = e_{ij}e_{sk} + e_{is}e_{kj} + e_{ik}e_{js}, \quad 1 \leq i, j, s, k \leq n+1.$$

Then M can be prolonged to n-Lie V_n-module, if and only if, $R_{ijsk}m = 0$, for any $m \in M$ and $1 \leq i \leq n+1, 1 \leq j < s < k \leq n+1, i \notin \{j, s, k\}$.

Proof. Below we use the following notation. If $a, b, c, \ldots$ are some vectors, then $\langle a, b, c, \ldots \rangle$ denotes their linear span and $\{a, b, c, \ldots\}$ denotes the set of these elements and by $(a, b, c, \ldots)$ we denote the vector with components $a, b, c, \ldots$.

Notice that $X_{a_1,\ldots,a_{2n-2}}$ is skew-symmetric under arguments $a_1, \ldots, a_n$ and $a_{n+1}, \ldots, a_{2n-2}$. Therefore, $X_{a_1,\ldots,a_{2n-2}} = 0$, if dimension of the subspace $\langle a_1, \ldots, a_n \rangle$ is less than n or dimension of the subspace $\langle a_{n+1}, \ldots, a_{2n-2} \rangle$ is less than $n-2$.

Suppose that $\dim\langle a_1, \ldots, a_n \rangle = n$.

Check that $X_{a_1,\ldots,a_{2n-2}} = 0$, if $V_n \neq \langle a_1, \ldots, a_{2n-2} \rangle$. We can assume that $a_1, \ldots, a_{2n-2}$ are basic vectors. Suppose that $\{a_1, \ldots, a_n\} = \{e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}\}$ for some $i \in \{1, \ldots, n+1\}$. Since V_n does not coincide with the subspace $\langle a_1, \ldots, a_{2n-2} \rangle$ and therefore, its dimension is less than $n+1$, we have $\{a_{n+1}, \ldots, a_{2n-2}\} = \{e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, \hat{e}_s, \ldots, e_{n+1}\}$ for some $j, s \neq i, j < s$. Let for simplicity $a_1 = e_1, \ldots, a_{i-1} = e_{i-1}, a_i = e_{i+1}, \ldots, a_n = e_{n+1}$ and $(a_{n+1}, \ldots, a_{2n-2}) = (e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, \hat{e}_s, \ldots, e_{n+1})$.

We have

$$[a_1, \dots, a_n] = [e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = (-1)^i e_i.$$

Further

$$a_r \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2} = 0,$$

if $a_r \neq e_i, e_j, e_s$. Therefore,

$$(a_1 \wedge \cdots \hat{a}_r \wedge \cdots \wedge a_n)\,(a_r \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2}) = 0,$$

if $a_r \neq e_i, e_j, e_s$.

Let $f : \wedge^{n-1} V_n \to so_{n+1}$ be the isomorphism of Lie algebras constructed in Theorem 3.1. Prolong it to the isomorphism of universal enveloping algebras $f : U(\wedge^{n-1} V_n) \to U(so_{n+1})$.

Thus,

$$\begin{aligned} &f([a_1, \dots, a_n] \wedge (a_{n+1} \wedge \cdots \wedge a_{2n-2})) \\ &\quad = f((-1)^i e_i \wedge e_1 \wedge \cdots \hat{e}_i \wedge \cdots \hat{e}_j \wedge \cdots \hat{e}_s \wedge \cdots \wedge e_{n+1}) \\ &\quad = -f(e_1 \wedge \cdots \hat{e}_j \wedge \cdots \hat{e}_s \wedge \cdots \wedge e_{n+1}) \\ &\quad = (-1)^{j+s+n} e_{js}. \end{aligned}$$

On the other hand

$$\begin{aligned} &\sum_{r=1}^{n} (-1)^{r+n} f(a_1 \wedge \cdots \hat{a}_r \wedge \cdots \wedge a_n)\, f(a_r \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2}) \\ &\quad = -(-1)^{n+j} f(e_1 \wedge \cdots \hat{e}_i \cdots \hat{e}_j \cdots \wedge e_{n+1}) \times f(e_j \wedge e_1 \wedge \cdots \hat{e}_i \cdots \hat{e}_j \cdots \hat{e}_s \cdots \wedge e_{n+1}) \\ &\qquad - (-1)^{n+s} f(e_1 \wedge \cdots \hat{e}_i \cdots \hat{e}_s \cdots \wedge e_{n+1}) \times f(e_s \wedge e_1 \wedge \cdots \hat{e}_i \cdots \hat{e}_j \cdots \hat{e}_s \cdots \wedge e_{n+1}) \\ &\quad = (-1)^{j+s+n+1} e_{ij} e_{is} + (-1)^{j+s+n} e_{is} e_{ij} \\ &\quad = -(-1)^{j+s+n} [e_{ij}, e_{is}] = (-1)^{j+s+n} e_{js}. \end{aligned}$$

Therefore, $f(X_{a_1, \dots, a_{2n-2}}) = 0$, and $X_{a_1, \dots, a_{2n-2}} = 0$, if the subspace generated by $a_1, \dots, a_{2n-2}$ does not coincide with V_n.

Now suppose that V_n is generated by elements $a_1, \dots, a_{2n-2}$. As above we can assume that these elements are basic elements and $(a_1, \dots, a_n) = (e_1, \dots, \hat{e}_i, \dots, e_{n+1})$ and $(a_{n+1}, \dots, a_{2n-2}) = (e_1, \dots, \hat{e}_j, \dots, \hat{e}_s, \dots, \hat{e}_k, \dots, e_{n+1})$ for some $1 \leq i \leq n+1, 1 \leq j < s < k \leq n+1, i \notin \{j, s, k\}$. Then

$$[a_1, \dots, a_n] \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2} = 0,$$

since $e_i \in \{a_{n+1}, \dots, a_{2n-2}\}$. Calculations as above show that

$$\sum_{r=1}^{n} (-1)^{r+n} f(a_1 \wedge \cdots \hat{a}_r \wedge \cdots \wedge a_n)\, f(a_r \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2}) = \pm R_{ijsk}.$$

So, $f(X_{a_1,\ldots,a_{2n-2}}) \in \langle R_{ijsk} : 1 \le i \le n+1, 1 \le j,s,k \le n+1 \rangle$. It is easy to check that R_{ijsk} is skew-symmetric by arguments j,s,k and $R_{ijsk} = 0$, if $i \in \{j,s,k\}$. So, so_{n+1}-module M can be prolonged to n-Lie module, if and only if $R_{ijsk}m = 0$, for any $m \in M, 1 \le i \le n+1, 1 \le j < s < k \le n+1$. □

Below we use branching rules for irreducible modules corresponding to the imbedding $so_{n-1} \subset so_n$ given in Boerner (1955). The proof of Theorem 1.1 is based on the following:

Theorem 3.3. *Let $k > 1$.*

(i) *Let $M = M(\alpha)$ be a finite-dimensional irreducible so_{2k+1}-module with highest weight $\alpha = \sum_{i=1}^k \alpha_i\pi_i$. Then M as a module over Lie subalgebra so_{2k} has a submodule, isomorphic to $M(\bar\alpha)$, where $\bar\alpha = \sum_{i=1}^k \bar\alpha_i\pi_i$ and $\bar\alpha_i = \alpha_i$, $i = 1,\ldots,k-1$, and $\bar\alpha_k = \alpha_{k-1} + \alpha_k$.*

(ii) *Let $M = M(\alpha)$ be a finite-dimensional irreducible so_{2k}-module with highest weight $\alpha = \sum_{i=1}^k \alpha_i\pi_i$. Then M as a module over Lie subalgebra so_{2k-1} has a submodule, isomorphic to $M(\bar\alpha)$, where $\bar\alpha = \sum_{i=1}^{k-1} \bar\alpha_i\pi_i$ and $\bar\alpha_i = \alpha_i$, $i = 1,\ldots,k-2$, $\bar\alpha_{k-1} = \alpha_{k-1} + \alpha_k$.*

Proof. (i) Take

$$\lambda_k = \alpha_k/2, \quad \lambda_i = \sum_{j=i}^{k-1} \alpha_j + \alpha_k/2, \quad 1 \le i \le k-1.$$

According to branching Theorem 12.1b (Boerner, 1955), any so_{2k}-submodule of $M(\alpha)$ has weight of the form $\bar\alpha$, such that corresponding $\bar\lambda$ satisfies the following inequality

$$\lambda_1 \ge |\bar\lambda_1| \ge \lambda_2 \ge \cdots \ge \lambda_{k-1} \ge |\bar\lambda_{k-1}| \ge \lambda_k \ge |\bar\lambda_k|.$$

The $\bar\lambda_j$ are integral or half-integral according to what the λ_j are. If we take $\bar\lambda_i := \lambda_i$, then such $\bar\lambda$ satisfies these conditions. Therefore, $M(\alpha)$ has so_{2k}-submodule isomorphic to $M(\bar\alpha)$, where $\bar\alpha = \sum_{i=1}^k \bar\alpha_i\pi_i$, $\bar\alpha_i = \bar\lambda_i - \bar\lambda_{i+1} = \alpha_i$, for $i = 1,\ldots,k-1$, and $\bar\alpha_k = \bar\lambda_{k-1} + \bar\lambda_k = \lambda_{k-1} + \lambda_k = \alpha_{k-1} + \alpha_k$. So, the so_{2k+1}-module $M(\alpha)$ as so_{2k}-module has a submodule isomorphic to $M(\bar\alpha)$, where $\bar\alpha = \sum_{i=1}^{k-1} \alpha_i\pi_i + (\alpha_{k-1} + \alpha_k)\pi_k$.

(ii) We have

$$\alpha_i = \lambda_i - \lambda_{i+1}, \quad 1 \le i \le k-1, \quad \alpha_k = \lambda_{k-1} + \lambda_k.$$

By branching Theorem 12.1a (Boerner, 1955), any so_{2k-1}-submodule of $M(\alpha)$ is isomorphic to $M(\bar\alpha)$, such that corresponding $\bar\lambda$ satisfies the following inequality

$$\lambda_1 \ge |\bar\lambda_1| \ge \lambda_2 \ge \cdots \ge \lambda_{k-1} \ge |\bar\lambda_{k-1}| \ge |\lambda_k|.$$

The $\bar{\lambda}_j$ are integral or half-integral according to what the λ_j are. Notice that a sequence $\bar{\lambda}$ constructed by the following way satisfies these conditions

$$\bar{\lambda}_i = \lambda_i, \quad 1 \le i \le n-1.$$

So, so_{2k}-module $M(\alpha)$ as so_{2k-1}-module has submodule $M(\bar{\alpha})$, where

$$\bar{\alpha}_i = \bar{\lambda}_i - \bar{\lambda}_{i+1} = \alpha_i, \quad 1 \le i \le k-2,$$
$$\bar{\alpha}_{k-1} = 2\bar{\lambda}_{k-1} = 2\lambda_{k-1} = \alpha_{k-1} + \alpha_k.$$

Corollary 3.4. *Let $n > 4$ and M be irreducible so_n-module such that $q(M) > 1$. Then M as a module over subalgebra $so_{n-1} \subset so_n$ has a submodule $\overline{M}$, such that $q(\overline{M}) > 1$.*

Proof. It is easy to see that for irreducible module M with highest weight α, the condition $q(M) > 1$, is equivalent to the condition $\sum_{i>1} \alpha_i > 0$.

Let $\bar{\alpha}$ be highest weight of so_{n-1}, defined by $\bar{\alpha}_i = \alpha_i, i = 1, \ldots, k-1$, and $\bar{\alpha}_k = \alpha_{k-1} + \alpha_k$, if $n = 2k+1$, and $\bar{\alpha}_i = \alpha_i, i = 1, \ldots, k-2$, $\bar{\alpha}_{k-1} = \alpha_{k-1} + \alpha_k$, if $n = 2k$.

Notice that

$$\sum_{i>1} \bar{\alpha}_i = \sum_{i>1} \alpha_i + 2\alpha_k \ge \sum_{i>1} \alpha_i,$$

if $n = 2k+1, k > 1$ and

$$\sum_{i>1} \bar{\alpha}_i = \sum_{i>1} \alpha_i,$$

if $n = 2k, k > 2$.

By Theorem 3.3, so_n-module $M = M(\alpha)$ as so_{n-1}-module has a submodule isomorphic to $\overline{M} = M(\bar{\alpha})$. If $q(M) > 1$, then $q(\overline{M}) > 1$. □

Notice that gl_n can be realized as a Lie algebra of derivations of $K[x_1, \ldots, x_n]$ of the form $\sum_{i,j=1}^n \lambda_{ij} x_i \partial_j$, $\lambda_{ij} \in K$. Its subalgebra so_n is generated by elements $e_{ij} = x_i \partial_j - x_j \partial_i$. The set $\{e_{ij} : 1 \le i < j \le n\}$ consists of basis of so_n. The multiplication on so_n can be given by

$$[e_{ij}, e_{sk}] = 0, \quad \text{if } |\{i, j, s, k\}| = 4,$$
$$[e_{ij}, e_{is}] = -e_{js}, \quad [e_{ij}, e_{js}] = e_{is}, \quad [e_{is}, e_{js}] = -e_{ij}.$$

Lemma 3.5. *Let $M = M_{t,r}$ be an irreducible so_4-module. Then M can be prolonged to 3-module over 3-Lie vector product algebra V_3, if and only if $t = r$.*

Proof. The algebra so_4 has the basis $\{e_{ij} : 1 \le i < j \le 4\}$. Take here another basis $\{f_i : 1 \le i \le 6\}$, by

$$f_1 = (e_{12} + e_{34})/2, \quad f_2 = (e_{13} - e_{24})/2, \quad f_3 = (e_{14} + e_{23})/2,$$
$$f_4 = (-e_{12} + e_{34})/2, \quad f_5 = (e_{13} + e_{24})/2, \quad f_6 = (-e_{14} + e_{23})/2.$$

Then

$$[f_1, f_2] = -f_3, \quad [f_1, f_3] = f_2, \quad [f_2, f_3] = -f_1,$$
$$[f_4, f_5] = f_6, \quad [f_5, f_6] = f_4, \quad [f_6, f_4] = f_5,$$
$$[f_i, f_j] = 0, \quad i = 1, 2, 3, \ j = 4, 5, 6.$$

We see that

$$e_{12} = f_1 - f_4, \quad e_{13} = f_2 + f_5, \quad e_{14} = f_3 - f_6,$$
$$e_{23} = f_3 + f_6, \quad e_{24} = -f_2 + f_5, \quad e_{34} = f_1 + f_4,$$

and

$$R_{1234} = e_{12}e_{34} - e_{13}e_{24} + e_{14}e_{23} = C_1 - C_2,$$

where

$$C_1 = f_1^2 + f_2^2 + f_3^2, \quad C_2 = f_4^2 + f_5^2 + f_6^2,$$

are Casimir elements of subalgebras $\langle f_1, f_2, f_3\rangle \cong sl_2$ and $\langle f_4, f_5, f_6\rangle \cong sl_2$. Well known that any irreducible finite-dimensional sl_2-module is uniquely defined by eigenvalue of the Casimir operator on this module. Therefore, $M_{t,r}$ is 3-Lie module, if and only if $t = r$.

Lemma 3.6. *Let $n > 3$. Any irreducible so_{n+1}-module $M(t\pi_1)$ can be prolonged to n-Lie module of V_n. Let M be an irreducible so_{n+1}-module with $q(M) > 1$. Then M cannot be prolonged to n-Lie module over n-Lie algebra V_n.*

Proof. Let $n > 3$. Let us consider realization of $M(t\pi_1)$ as a space of homogeneous polynomials $\sum_{1\le i_1\le\cdots\le i_t\le n+1} \lambda_{i_1\cdots i_t} x_{i_1}\cdots x_{i_t}$.

By Lemma, 3.2, we need to check that

$$R_{ijsk}u = 0, \quad \text{for } u = x_{i_1}\cdots x_{i_t},$$

for any $\{i, j, s, k\}$, such that $1 \le i \le n+1$, $1 \le j \le s \le k \le n+1$, $i \notin \{j, s, k\}$ and $1 \le i_1 \le i_2 \le \cdots \le i_t \le n+1$.

Let $I = \{i, j, s, k\}$. Present u in the form vw, where $v = \prod_{l\in I\cap\{i_1,\ldots,i_t\}} x_l$ and $w = \prod_{l\in\{i_1,\ldots,i_t\}\setminus I} x_l$. Notice that

$$R_{ijsk}(vw) = R_{ijsk}(v)w.$$

Therefore it is enough to check that $R_{ijsk}(v) = 0$, for elements $v \in M(t\pi_1)$ of the form

$$v = x_i x_j x_s x_k, \quad 1 \leq i \leq n+1, \quad 1 \leq j < s < k \leq n+1, \; i \notin \{j, s, k\},$$

$$v = x_j x_s x_k, \quad 1 \leq j \leq s \leq k \leq n+1,$$

$$v = x_i x_s x_k, \quad 1 \leq i \leq n+1, \quad 1 \leq s \leq k \leq n+1,$$

$$v = x_j x_k, \quad 1 \leq j \leq k \leq n+1,$$

$$v = x_i x_k, \quad 1 \leq i \leq n+1, \quad 1 \leq k \leq n+1,$$

$$v = x_k, \quad 1 \leq k \leq n+1, \quad v = x_i, \quad 1 \leq i \leq n+1.$$

Let $i \neq j, s, k$. Then

$$\begin{aligned}
&R_{ijsk}(x_i x_j x_s x_k) \\
&\quad = e_{ij}(x_i x_j x_s^2 - x_i x_j x_k^2) + e_{is}(x_i x_s x_k^2 - x_i x_j^2 x_s) + e_{ik}(x_i x_j^2 x_k - x_i x_s^2 x_k) \\
&\quad = e_{ij}(x_i x_j) x_s^2 - e_{ij}(x_i x_j) x_k^2 + e_{is}(x_i x_s) x_k^2 - e_{is}(x_i x_s) x_j^2 + e_{ik}(x_i x_k) x_j^2 - e_{ik}(x_i x_k) x_s^2 \\
&\quad = (x_i^2 - x_j^2) x_s^2 - (x_i^2 - x_j^2) x_k^2 + (x_i^2 - x_s^2) x_k^2 - (x_i^2 - x_s^2) x_j^2 \\
&\qquad + (x_i^2 - x_k^2) x_j^2 - (x_i^2 - x_k^2) x_s^2 = 0,
\end{aligned}$$

Similarly,

$$\begin{aligned}
&R_{ijsk}(x_j x_s x_k) = 0, \quad R_{ijsk}(x_i x_s x_k) = 0, \quad R_{ijsk}(x_s x_k) = 0, \\
&R_{ijsk}(x_i x_k) = 0, \quad R_{ijsk}(x_k) = 0, \quad R_{ijsk}(x_i) = 0.
\end{aligned}$$

So, we have checked that $Q(V_n) M(t\pi_1) = 0$, if $n > 3$.

Suppose now that $q(M) > 1$. We need to prove that $R_{ijsk} m \neq 0$, for some $1 \leq i \leq n+1, 1 \leq j < s < k \leq n+1$ and $m \in M$.

Let us use induction on $n \geq 3$. If $n = 3$, then by Lemma 3.5 any irreducible so_{n+1}-module M with $q(M) > 1$ cannot be prolonged to n-Lie module. Suppose that the statement is true for $n - 1 \geq 3$. If $q(M) > 1$ for so_{n+1}-module M, then by Corollary 3.4 there exists its so_n-submodule $\overline{M}$, such that $q(\overline{M}) > 1$. Then by inductive suggestion there exists some $R_{ijsk} \in Q(V_{n-1}) \subset U(so_n)$ and $m \in \overline{M}$, such that $R_{ijsk} m \neq 0$. Since $m \in \overline{M} \subseteq M$ and $R_{ijsk} \in U(so_n) \subset U(so_{n+1})$, this means that $R_{ijsk} m \neq 0$ as elements of M. So, we have proved that our statement for n. □

Proof of Theorem 1.1. (i) By Theorem 3.1, Lie algebra $\wedge^{n-1} V_n \cong so_{n+1}$ is semi-simple. Therefore, by Weyl theorem and Proposition 2.1, any finite-dimensional n-Lie representation of V_n is completely reducible.

(ii) and (iii) For $n = 2$ our statements are evident. Let $n > 2$. By Lemmas 3.6 and 3.5, $M(t\pi_1), n > 3$, or $M_{t,t}, n = 3$, is V_n-module for any nonnegative integer t and any module with $q(M) > 1$ cannot be n-Lie module.

ACKNOWLEDGMENTS

I am grateful to INTAS foundation for support. I am thankful to referee for paying of my attention to the preprint Sh. M. Kasymov, E. N. Kuz'min, "Classification of representations of split simple n-Lie algebra of type A_1", preprint No. 23, IM SO RAN, Novosibirsk, 1987, where was obtained results similar to ours.

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Received October 2002
Accepted January 2004