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THE n-LIE PROPERTY OF THE JACOBIAN AS A CONDITION FOR COMPLETE INTEGRABILITY

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Abstract: We prove that an associative commutative algebra U with derivations $D_1, \dots, D_n \in \operatorname{Der} U$ is an n-Lie algebra with respect to the n-multiplication $D_1 \wedge \dots \wedge D_n$ if the system $\{D_1, \dots, D_n\}$ is in involution. In the case of pairwise commuting derivations this fact was established by V. T. Filippov. One more formulation of the Frobenius condition for complete integrability is obtained in terms of n-Lie multiplications. A differential system $\{D_1, \dots, D_n\}$ of rank n on a manifold M^m is in involution if and only if the space of smooth functions on M is an n-Lie algebra with respect to the Jacobian $\operatorname{Det}(D_i u_j)$.

Keywords: n-Lie algebra, Jacobian, complete integrability, differential system, Frobenius theorem

1. Introduction

Let U and V be vector spaces. Denote by $T^k(U, V)$ the space of multilinear mappings with k arguments $\psi : U \times \dots \times U \to V$. Let $C^k(U, V)$ be the subspace of $T^k(U, V)$ of the multilinear mappings with skew-symmetric arguments.

We say that U possesses *n-ary multiplication* ω if $\omega \in T^n(U, U)$. A space U with n-ary multiplication ω is an *n-algebra*. Denote it by (U, ω). Define the n-ary polynomial $\mathrm{nlie}_1 = \mathrm{nlie}_1(\omega, t_1, \dots, t_{2n-1})$ by the rule

$$\mathrm{nlie}_1(\omega, t_1, \dots, t_{2n-1}) = \omega(t_1, \dots, t_{n-1}, \omega(t_n, \dots, t_{2n-1}))$$
$$- \sum_{i=n}^{2n-1} (-1)^{i+n} \omega(\omega(t_1, \dots, t_{n-1}, t_i), t_n, \dots, \hat{t}_i, \dots, t_{2n-1}).$$

We call an n-algebra (U, ω) an *n-Lie algebra* provided that $\omega \in C^n(U, U)$ and $\mathrm{nlie}_1 = 0$ is an identity in U, i.e.

$$\omega(u_1, \dots, u_{n-1}, \omega(u_n, \dots, u_{2n-1}))$$
$$= \sum_{i=n}^{2n-1} (-1)^{i+n} \omega(\omega(u_1, \dots, u_{n-1}, u_i), u_1, \dots, \hat{u}_i, \dots, u_{2n-1})$$

for all $u_1, \dots, u_{2n-1} \in U$.

The notion of n-Lie algebra is relatively new. Y. Nambu [1] noticed the importance of studying the properties of the Jacobian as n-multiplication. For the first time, the n-Lie identity was written out by V. T. Filippov. In [2, 3], he established that the Jacobian

$$\mathrm{Jac}(u_1, \dots, u_n) = \operatorname{Det}\left(\frac{\partial u_j}{\partial x_i}\right)$$

defines n-Lie multiplication in the space of polynomials. He used the condition that the derivations $\partial_1, \dots, \partial_n$ commute pairwise.

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We demonstrate that the commutativity condition for $D_1, \dots, D_n$ may be weaken, and $(U, D_1 \wedge \dots \wedge D_n)$ remains an n-Lie algebra under a weaker condition on $D_i \in \operatorname{Der} U$. It suffices to require that the system of derivations is in involution, i.e.

$$[D_i, D_j] = \sum_{s=1}^{n} u_{i,j}^s D_s$$

holds for all D_i, D_j and some $u_{i,j}^s \in U$.

Sometimes, the n-Lie algebras are called *Nambu* and *Nambu–Takhtadjan algebras*. At present, these are often referred to as *Filippov algebras*. We also mention the article [4] which is close to our topic.

Let $(U, \cdot)$ be an associative commutative algebra with multiplication $\cdot$. A linear mapping $D : U \to U$ is called a *derivation* if

$$D(u \cdot v) = D(u) \cdot v + u \cdot D(v)$$

for all $u, v \in U$. Let $\operatorname{Der} U$ be the space of derivations of U. Note that

$$u \in U,\ D \in \operatorname{Der} U \Rightarrow u \cdot D \in \operatorname{Der} U$$

where the derivation $u \cdot D$ is defined by the rule

$$(u \cdot D)(v) = u \cdot D(v).$$

In other words, $\operatorname{Der} U$ has a U-module structure. It is said that a *differential system* $\mathscr{D} = \{D_1, \dots, D_n\}$ is given on U if $D_i \in \operatorname{Der} U$ for all $i = 1, \dots, n$. A differential system $\mathscr{D}$ has *rank* n if $D_1 \wedge \dots \wedge D_n \neq 0$. We say that a differential system $\mathscr{D}$ is *in involution* if for all $1 \le i, j \le n$ there exist $u_{i,j}^s \in U$ such that

$$[D_i, D_j] = \sum_{s=1}^{n} u_{i,j}^s \cdot D_s.$$

Suppose that U is equipped with some binary multiplication $(u, v) \mapsto u \cdot v$ which is associative and commutative as well as n-ary multiplication ω. Define the n-ary polynomials

$$\mathrm{nlie}_2 = \mathrm{nlie}_2(\omega, t_1, \dots, t_{2n}), \quad \mathrm{nlie}_3 = \mathrm{nlie}_3(\omega, t_1, t_2, \dots, t_{n+1})$$

by the rules

$$\begin{gathered}\mathrm{nlie}_2(\omega, t_1, t_2, \dots, t_{n+1}) \\ = \omega(t_1 \cdot t_2, t_3, \dots, t_{n+1}) - t_1 \cdot \omega(t_2, \dots, t_{n+1}) - \omega(t_1, t_3, \dots, t_{n+1}) \cdot t_2,\end{gathered}$$

$$\mathrm{nlie}_3(\omega, t_1, \dots, t_{2n}) = \sum_{i=n}^{2n} (-1)^{i+n} \omega(t_1, \dots, t_{n-1}, t_i) \cdot \omega(t_n, \dots, \hat{t}_i, \dots, t_{2n}).$$

An n-Lie algebra (U, ω) is called an *n-Lie–Poisson algebra* if it has two multiplications $\cdot$ and ω, and, except for the identity $\mathrm{nlie}_1 = 0$, the identity $\mathrm{nlie}_2 = 0$ holds. An n-Lie–Poisson algebra $(U, \cdot, \omega)$ is called a *strictly* n-Lie–Poisson algebra if the identity $\mathrm{nlie}_3 = 0$ holds as well.

In [5], the identities $\mathrm{nlie}_1 = 0$ and $\mathrm{nlie}_3 = 0$ were called the *fundamental identities of type* I and *type* II. It was established there that $(U, \cdot, D_1 \wedge \dots \wedge D_n)$ is an n-Lie–Poisson algebra and $(U, \mathrm{id} \wedge D_1 \wedge \dots \wedge D_n)$ is an $(n+1)$-Lie algebra if $\mathscr{D}$ is a commutative differential system. Here $\mathrm{id} : U \to U$ is the identity mapping.

We will now formulate the main results.

Theorem 1. *Let U be an associative commutative algebra and $D_1, \dots, D_n \in \operatorname{Der} U$. If the differential system $\mathscr{D} = \{D_1, \dots, D_n\}$ is in involution then $(U, D_1 \wedge \dots \wedge D_n)$ is a strictly n-Lie–Poisson algebra.*

Corollary 2 [2, 3]. *Let U be an associative commutative algebra, $D_1, \dots, D_n \in \operatorname{Der} U$ and $[D_i, D_j] = 0$ for all $1 \le i, j \le n$. Then $(U, D_1 \wedge \dots \wedge D_n)$ is an n-Lie algebra.*

Let M^m be a C^∞-manifold, and let $\mathscr{F}(M)$ be the algebra of C^∞-functions on M. A differential system on M^m can be given with the help of vector fields or differential forms (for example, see [6]). Respectively, there are two formulations of the Frobenius theorem on complete integrability of differential systems. In one of them, the theorem states that a system is complete integrable if and only if it is in involution, i.e. its vector fields form a Lie structure in the space of functions. In another form, the theorem states that the ideal of differential forms should be closed under exterior derivation. We give a third version of complete integrability in terms of n-Lie multiplications.

Locally, the notions of a vector field on M and a derivation of the algebra $\mathscr{F}(M)$ are equivalent. Therefore, for such U, the usual definitions of differential systems (distributions) on M and the conditions for them to be in involution (for example, see [7]) are compatible with our definitions.

In the case $U = \mathscr{F}(M)$, Theorem 1 is convertible.

Theorem 3. *Let M^m be a C^∞-manifold, $1 < n \le m$, and let $\mathscr{D} = \{D_1, \dots, D_n\}$ be a C^∞-differential system on M of rank n. Let $U = \mathscr{F}(M)$. The following are equivalent:*

(1) *$\mathscr{D}$ is in involution;*

(2) *$(\mathscr{F}(M), D_1 \wedge \dots \wedge D_n)$ is an n-Lie algebra;*

(3) *$(\mathscr{F}(M), \mathrm{id} \wedge D_1 \wedge \dots \wedge D_n)$ is an $(n+1)$-Lie algebra.*

2. n-Lie Properties of the Jacobian

Let $(U, \cdot)$ be an associative and commutative algebra. Equip the space $T^*(U, U) = \bigoplus_k T^k(U, U)$ with two multiplications. Take $\psi \in T^k(U, U)$ and $\phi \in T^s(U, U)$. Then the multiplications $\psi \cdot \phi \in T^{k+s}(U, U)$ and $\psi \wedge \phi \in T^{k+s}(U, U)$ are defined in the following way:

$$(\psi \cdot \phi)(u_1, \dots, u_{k+s}) = \psi(u_1, \dots, u_k) \cdot \phi(u_{k+1}, \dots, u_{k+s}),$$

$$\psi \wedge \phi(u_1, \dots, u_{k+s}) = \sum_{\sigma \in \mathrm{Sym}_{k,s}} \operatorname{sgn} \sigma \, \psi(u_{\sigma(1)}, \dots, u_{\sigma(k)}) \cdot \phi(u_{\sigma(k+1)}, \dots, u_{\sigma(k+s)}),$$

where $\mathrm{Sym}_{k,s}$ is the set of all permutations $\sigma \in \mathrm{Sym}_{k+s}$ such that $\sigma(1) < \dots < \sigma(k)$, $\sigma(k+1) < \dots < \sigma(k+s)$. By linearity, these multiplications may be extended to multiplications in $T^*(U, U) = \bigoplus_k T^k(U, U)$.

The space $T^*(U, U)$ has the natural structure of a U-module:

$$(u \cdot \psi)(u_1, \dots, u_k) = u \cdot (\psi(u_1, \dots, u_k)).$$

Note that $\operatorname{Der} U \subseteq T^1(U, U)$ is a U-submodule. The algebra $(T^*(U, U), \cdot)$ is associative and commutative, and the algebra $(T^*(U, U), \wedge)$ is associative and anticommutative. Notice that

$$u \cdot (\psi \cdot \phi) = (u \cdot \psi) \cdot \phi = \psi \cdot (u \cdot \phi), \quad u \cdot (\psi \wedge \phi) = (u \cdot \psi) \wedge \phi = \psi \wedge (u \cdot \phi)$$

for all $u \in U$ and $\psi, \phi \in T^*(U, U)$.

Given $D_1, \dots, D_n \in \operatorname{Der} U$, we may define an exterior product in the following way:

$$(D_1 \wedge \dots \wedge D_n)(u_1, \dots, u_n) = \sum_{\sigma \in \mathrm{Sym}_n} \operatorname{sgn} \sigma (D_1(u_{\sigma(1)}) \dots D_n(u_{\sigma(n)}))$$

or

$$(D_1 \wedge \dots \wedge D_n)(u_1, \dots, u_n) = \sum_{\sigma \in \mathrm{Sym}_n} \operatorname{sgn} \sigma (D_{\sigma(1)}(u_1) \dots D_{\sigma(n)}(u_n)).$$

In other words,

$$(D_1 \wedge \dots \wedge D_n)(u_1, \dots, u_n) = \operatorname{Det}(D_i(u_j))$$

is the Jacobian in $u_1, \dots, u_n$ relative to the differential system $\mathscr{D} = \{D_1, \dots, D_n\}$.

Equip the space $\wedge^n \operatorname{Der} U$ with the adjoint module structure over the Lie algebra $\operatorname{Der} U$:

$$[D, D_1 \wedge \dots \wedge D_n] = \sum_{s=1}^{n} (-1)^{s+1} [D, D_s] \wedge D_1 \wedge \dots \widehat{D}_s \dots \wedge D_n$$

(here $\widehat{D}_s$ means that D_s is omitted).

Define the linear operator $D_{u_1,\dots,u_{n-1}} : U \to U$ by the rule

$$D_{u_1,\dots,u_{n-1}}(v) = (D_1 \wedge \dots \wedge D_n)(u_1, \dots, u_{n-1}, v).$$

It is easy to see that

$$D_{u_1,\dots,u_{n-1}} = \sum_{i=1}^{n} (-1)^{i+n} D_1 \wedge \dots \widehat{D}_i \dots \wedge D_n(u_1, \dots, u_{n-1}) \cdot D_i. \tag{1}$$

Given $D_1, \dots, D_n \in \operatorname{Der} U$, define the linear operator

$$R_n : \wedge^{n-1} U \to \wedge^n \operatorname{Der} U$$

by the rule

$$R_n(u_1, \dots, u_{n-1}) = \sum_{i=1}^{n} (-1)^{i+1} ((D_1 \wedge \dots \widehat{D}_i \dots \wedge D_n)(u_1, \dots, u_{n-1})) \cdot ([D_i, D_1 \wedge \dots \wedge D_n])$$
$$+ \sum_{i<j} (-1)^{i+j} (([D_i, D_j] \wedge D_1 \wedge \dots \widehat{D}_i \dots \widehat{D}_j \dots \wedge D_n)(u_1, \dots, u_{n-1})) \cdot (D_1 \wedge \dots \wedge D_n)$$

or briefly

$$R_n = \sum_{i=1}^{n} (-1)^{i+1} (D_1 \wedge \dots \widehat{D}_i \dots \wedge D_n) \cdot [D_i, D_1 \wedge \dots \wedge D_n]$$
$$+ \sum_{i<j} (-1)^{i+j} ([D_i, D_j] \wedge D_1 \wedge \dots \widehat{D}_i \dots \widehat{D}_j \dots \wedge D_n) \cdot (D_1 \wedge \dots \wedge D_n).$$

Lemma 4. *$D_1 \wedge \dots \wedge D_n$ is an n-Lie multiplication if and only if*

$$[D_{u_1,\dots,u_{n-1}}, D_1 \wedge \dots \wedge D_n] = 0.$$

PROOF. Note that

$$\operatorname{nlie}_1(D_1 \wedge \dots \wedge D_n, u_1, \dots, u_{2n-1}) = D_{u_1,\dots,u_{n-1}}((D_1 \wedge \dots \wedge D_n)(u_n, \dots, u_{2n-1}))$$
$$- \sum_{i=n}^{2n-1} (D_1 \wedge \dots \wedge D_n)(u_n, \dots, u_{i-1}, D_{u_1,\dots,u_{n-1}}(u_i), u_{i+1}, \dots, u_{2n-1})$$
$$= [D_{u_1,\dots,u_{n-1}}, D_1 \wedge \dots \wedge D_n](u_n, \dots, u_{2n-1}). \quad \square$$

Lemma 5. *Let $(U,\cdot)$ be an associative commutative algebra and $D_1,\dots,D_n \in \operatorname{Der} U$. Then $D_1 \wedge \dots \wedge D_n$ is an n-Lie multiplication on U if and only if $R_n = 0$.*

PROOF. By (1)

$$-[D_{u_1,\dots,u_{n-1}}, D_1\wedge\dots\wedge D_n] = \sum_{i=1}^{n}(-1)^i[D_{u_1,\dots,u_{n-1}}, D_i]\wedge D_1\wedge\dots\widehat{D}_i\dots\wedge D_n$$

$$= \sum_{i,j=1}^{n}(-1)^{i+j+n}[D_1\wedge\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1})\cdot D_j, D_i]\wedge D_1\wedge\dots\widehat{D}_i\dots\wedge D_n$$

$$= \sum_{i,j=1}^{n}(-1)^{i+j+n}(D_1\wedge\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}))\cdot([D_j,D_i]\wedge D_1\wedge\dots\widehat{D}_i\dots\wedge D_n)$$

$$- \sum_{i,j=1}^{n}(-1)^{i+j+n}(D_i(D_1\wedge\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}))\cdot(D_j\wedge D_1\wedge\dots\widehat{D}_i\dots\wedge D_n)$$

$$= \sum_{j=1}^{n}(-1)^{j+n}(D_1\wedge\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}))\Big(\sum_{i=1}^{n}(-1)^i[D_j,D_i]\wedge D_1\wedge\dots\widehat{D}_i\dots\wedge D_n\Big)$$

$$- \sum_{i=1}^{n}(-1)^n D_i(D_1\wedge\dots\widehat{D}_i\dots\wedge D_n(u_1,\dots,u_{n-1}))\cdot(D_i\wedge D_1\wedge\dots\widehat{D}_i\dots\wedge D_n) = Z_1+Z_2,$$

where

$$Z_1 = \sum_{j=1}^{n}(-1)^{j+n+1}(D_1\wedge\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}))\cdot([D_j, D_1\wedge\dots\wedge D_n]),$$

$$Z_2 = Y_2\cdot(D_1\wedge\dots\wedge D_n),$$

$$Y_2 = \sum_{i=1}^{n}(-1)^{i+n}D_i(D_1\wedge\dots\widehat{D}_i\dots\wedge D_n(u_1,\dots,u_{n-1})).$$

Note that

$$(-1)^nY_2 = \sum_{i=1}^{n}\sum_{j>i}(-1)^{i+j}[D_i,D_j]\wedge D_1\wedge\dots\widehat{D}_i\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}).$$

Thus,

$$[D_{u_1,\dots,u_{n-1}}, D_1\wedge\dots\wedge D_n] = Z_1+Z_2$$

$$= \sum_{j=1}^{n}(-1)^{j+n+1}(D_1\wedge\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}))\cdot([D_j, D_1\wedge\dots\wedge D_n])$$

$$+ \sum_{i=1}^{n}\sum_{i<j}(-1)^{i+j+n}([D_i,D_j]\wedge D_1\wedge\dots\widehat{D}_i\dots\widehat{D}_j\dots\wedge D_n(u_1,\dots,u_{n-1}))\cdot(D_1\wedge\dots\wedge D_n)$$

$$= R_n(u_1,\dots,u_{n-1}).$$

Therefore, by Lemma 4, $R_n = 0$ amounts to the fact that $D_1\wedge\dots\wedge D_n$ is an n-Lie multiplication. □

Lemma 6. *Let $(U,\cdot)$ be an associative commutative algebra and $D_i \in \operatorname{Der} U$, $i=1,\dots,n$. Then the identities $\mathrm{nlie}_2 = 0$ and $\mathrm{nlie}_3 = 0$ hold in $(U, D_1\wedge\dots\wedge D_n)$.*

PROOF. It is easy to see that $\mathrm{nlie}_2 = 0$ is an identity in $(U,\cdot,D_1\wedge\dots\wedge D_n)$ if $D_i \in \operatorname{Der} U$.

Note that $X = \mathrm{nlie}_3(D_1\wedge\dots\wedge D_n, u_1,\dots,u_{2n})$ is skew-symmetric with respect to $n+1$ arguments $(u_n, \dots, u_{2n})$. Moreover, X is a skew-symmetric sum of some elements of the shape $a\cdot D_{i_1}(u_n)\dots D_{i_{n+1}}(u_{2n})$, where $a = a(u_1,\dots,u_{n-1}) \in U$ and $i_1,\dots,i_{n+1}$ range over the n-element set $\{1,\dots,n\}$. It means that $\mathrm{nlie}_3(D_1\wedge\dots\wedge D_n, u_1,\dots,u_{2n}) = 0$ for all $D_1,\dots,D_n \in \operatorname{Der} U$, $u_1,\dots,u_{2n} \in U$. □

Lemma 7. *Let $(U, \cdot, \omega)$ be a strictly n-Lie–Poisson algebra and $a \in U$. Define the new multiplication $a \cdot \omega : \wedge^n U \to U$ by the rule*

$$(a \cdot \omega)(u_1, \ldots, u_n) = a \cdot (\omega(u_1, \ldots, u_n)).$$

Then $(U, \cdot, a \cdot \omega)$ is a strictly n-Lie–Poisson algebra.

PROOF. Since

$$\mathrm{nlie}_2(a \cdot \omega, u_1, \ldots, u_{n+1}) = a \cdot \mathrm{nlie}_2(\omega, u_1, \ldots, u_{n+1}),$$

the identity $\mathrm{nlie}_2 = 0$ is obvious for the multiplication $a \cdot \omega$.

According to the identity $\mathrm{nlie}_2 = 0$, we have

$$\begin{aligned}
\mathrm{nlie}_1(a \cdot \omega, u_1, \ldots, u_{2n-1}) &= a \cdot \omega(u_1, \ldots, u_{n-1}, a \cdot \omega(u_n, \ldots, u_{2n-1})) \\
&- \sum_{i=n}^{2n-1} (-1)^{i+n} a \cdot \omega(a \cdot \omega(u_1, \ldots, u_{n-1}, u_i), u_n, \ldots, \hat{u}_i, \ldots, u_{2n-1}) \\
&= (a \cdot a) \cdot \big(\omega(u_1, \ldots, u_{n-1}, \omega(u_n, \ldots, u_{2n-1})) \\
&- \sum_{i=n}^{2n-1} (-1)^{i+n} (a \cdot a) \cdot \omega(\omega(u_1, \ldots, u_{n-1}, u_i), u_n, \ldots, \hat{u}_i, \ldots, u_{2n-1})) \\
&+ a \cdot \omega(u_1, \ldots, u_{n-1}, a) \cdot \omega(u_n, \ldots, u_{2n-1})\big) \\
&- \sum_{i=n}^{2n-1} (-1)^{i+n} a \cdot \omega(u_1, \ldots, u_{n-1}, u_i) \cdot \omega(a, u_n, \ldots, \hat{u}_i, \ldots, u_{2n-1}) \\
&= (a \cdot a) \cdot \mathrm{nlie}_1(\omega, u_1, \ldots, u_{2n-1}) + a \cdot \mathrm{nlie}_3(\omega, u_1, \ldots, u_{n-1}, a, u_n, \ldots, u_{2n-1}).
\end{aligned}$$

Hence,

$$\mathrm{nlie}_1(a \cdot \omega, u_1, \ldots, u_{2n-1}) = 0$$

for all $a, u_1, \ldots, u_{2n-1} \in U$. Thus, $(U, a \cdot \omega)$ is an n-Lie algebra for every $a \in U$ if $(U, \cdot, \omega)$ is a strictly n-Lie–Poisson algebra.

Furthermore,

$$\begin{aligned}
&\mathrm{nlie}_3(a \cdot \omega, u_1, \ldots, u_{2n}) = \\
&(a \cdot a) \cdot \sum_{i=n}^{2n} (-1)^{i+n} \omega(u_1, \ldots, u_{n-1}, u_i) \omega(u_n, \ldots, \hat{u}_i, \ldots, u_{2n}) \\
&= (a \cdot a) \cdot \mathrm{nlie}_3(\omega, u_1, \ldots, u_{2n}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{nlie}_3(a \cdot \omega, u_1, \ldots, u_{n+1}) \\
&= a \cdot \omega(u_1 \cdot u_2, u_3, \ldots, u_{n+1}) - u_1 \cdot (a \cdot \omega(u_2, \ldots, u_{n+1})) - a \cdot (\omega(u_1, u_3, \ldots, u_{n+1}) \cdot u_2) \\
&= (a \cdot \omega)(u_1 \cdot u_2, u_3, \ldots, u_{n+1}) - u_1 \cdot (a \cdot \omega(u_2, \ldots, u_{n+1})) - (a \cdot \omega)(u_1, u_3, \ldots, u_{n+1}) \cdot u_2 \\
&= a \cdot \mathrm{nlie}_3(\omega, u_1, u_2, \ldots, u_{n+1}).
\end{aligned}$$

In other words, $(U, \cdot, a \cdot \omega)$ is a strictly n-Lie–Poisson algebra if $(U, \cdot, \omega)$ is a strictly n-Lie–Poisson algebra. □

Lemma 8. *Let $(U, \cdot)$ be an associative commutative algebra with some derivations $D_1, \dots, D_n$. Assume that $(U, \cdot, D_1 \wedge \cdots \wedge D_n)$ is an n-Lie algebra. Given $u_{i,j} \in U$, we construct some new derivations D'_i by the rule*

$$D'_i = \sum_{j=1}^{n} u_{i,j} D_j.$$

Then $(U, \cdot, D'_1 \wedge \cdots \wedge D'_n)$ is a strictly n-Lie–Poisson algebra.

Proof. By Lemma 6, it suffices to verify that $\mathrm{nlie}_1 = 0$ is an identity for the multiplication $D'_1 \wedge \cdots \wedge D'_n$.

Notice that

$$D'_1 \wedge \cdots \wedge D'_n = a \cdot D_1 \wedge \cdots \wedge D_n$$

for $a = \mathrm{Det}(u_{i,j}) \in U$. By Lemma 6, $(U, \cdot, D_1 \wedge \cdots \wedge D_n)$ is a strictly n-Lie–Poisson algebra. Therefore, by Lemma 7, $(U, \cdot, D'_1 \wedge \cdots \wedge D'_n)$ is an n-Lie–Poisson algebra. $\square$

Let $S_{k,m}$ be the set of all ordered indexes $\tau = (i_1, \dots, i_k)$ such that $1 \le i_1 < \cdots < i_k \le m$. Put $\tau(1) = i_1, \dots, \tau(k) = i_k$ in this case. Given $\tau = (i_1, \dots, i_k) \in S_{k,m}$, we put $\tau^* = (j_1, \dots, j_{m-k}) \in S_{m-k,m}$, where $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{m-k}\} = \{1, \dots, m\}$. Note that $S_{m,m}$ consists of one element $(1, 2, \dots, m)$.

Lemma 9. *Let $(U, \cdot)$ be an associative commutative algebra without zero divisors. Assume that $D_i \in \mathrm{Der}\, U$, $1 \le i \le m$, and $D_1 \wedge \cdots \wedge D_m \ne 0$. Then the exterior forms $D_\tau = D_{\tau(1)} \wedge \cdots \wedge D_{\tau(k)}$ with $\tau \in S_{k,m}$ are U-linearly independent for all $k \le m$.*

Proof. It is clear that

$$D_\tau \wedge D_\sigma = 0, \text{ if } \tau^* \ne \sigma; \quad D_\tau \wedge D_\sigma = \pm D_1 \wedge \cdots \wedge D_n, \text{ if } \tau^* = \sigma.$$

Admit that

$$\sum_{\tau \in S_{k,m}} u_\tau D_{\tau(1)} \wedge \cdots \wedge D_{\tau(k)} = 0$$

and multiply both parts by $D_{\tau_0^*}$ for every $\tau_0 \in S_{k,m}$. We infer that

$$u_\tau D_1 \wedge \cdots \wedge D_n = 0.$$

Thus, $u_\tau = 0$ for every $\tau \in S_{k,m}$.

Lemma 10. *Let U be an associative commutative algebra without zero divisors, and let $D_1, \dots, D_m$ be some derivations of U such that $D_1 \wedge \cdots \wedge D_m \ne 0$. Let $n \le m$. Suppose that for every $1 \le i, j \le n$ there exist $u^s_{i,j} \in U$, $1 \le s \le m$, such that*

$$[D_i, D_j] = \sum_{s=1}^{m} u^s_{i,j} D_s$$

and $D_1 \wedge \cdots \wedge D_n$ is an n-Lie multiplication on U. Then the system $\mathscr{D} = \{D_1, \dots, D_n\}$ is in involution.

Proof. We have

$$\sum_{i=1}^{n} (-1)^{i+1} (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot ([D_i, D_1 \wedge \cdots \wedge D_n])$$
$$= \sum_{i,j=1}^{n} (-1)^{i+j} (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot ([D_i, D_j] \wedge D_1 \wedge \cdots \widehat{D}_j \cdots \wedge D_n)$$
$$= \sum_{i,j=1}^{n} \sum_{s=1}^{m} (-1)^{i+j} u^s_{i,j} \cdot (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_s \wedge D_1 \wedge \cdots \widehat{D}_j \cdots \wedge D_n) = Z_1 + Z_2,$$

where

$$Z_1 = \sum_{i,j=1}^{n} (-1)^{i+1} u_{i,j}^{j} \cdot (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n),$$

$$Z_2 = \sum_{i,j=1}^{n} \sum_{s=n+1}^{m} (-1)^{i+1} u_{i,j}^{s} \cdot (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_{j-1} \wedge D_s \wedge D_{j+1} \wedge \cdots \wedge D_n).$$

Analogously,

$$\sum_{i<j} (-1)^{i+j} ([D_i, D_j] \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$

$$= \sum_{i<j} \sum_{s=1}^{m} (-1)^{i+j} u_{i,j}^{s} \cdot (D_s \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n) = T_1 + T_2,$$

where

$$T_1 = \sum_{i} (-1)^{i} \sum_{j} u_{i,j}^{j} (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n),$$

$$T_2 = \sum_{i<j} \sum_{s=n+1}^{m} (-1)^{i+j} u_{i,j}^{s} \cdot (D_s \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n).$$

Then $Z_1 + T_1 = 0$ and $R_n = Z_2 + T_2$. By Lemma 5, $R_n = 0$ if $D_1 \wedge \cdots \wedge D_n$ is an n-Lie multiplication. In particular, for all $v_1, \ldots, v_{n-1} \in U$,

$$\sum_{j=1}^{n} \sum_{s=n+1}^{m} a_{j,s} \cdot (D_1 \wedge \cdots \wedge D_{j-1} \wedge D_s \wedge D_{j+1} \wedge \cdots \wedge D_n) + b_1 \cdot (D_1 \wedge \cdots \wedge D_n) = 0,$$

where

$$a_{j,s} = \sum_{i=1}^{n} (-1)^{i+1} u_{i,j}^{s} \cdot (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n)(v_1, \ldots, v_{n-1}) \in U, \quad n < s \leq m,$$

$$b_1 = \sum_{i<j} \sum_{s=n+1}^{m} (-1)^{i+j} u_{i,j}^{s} \cdot (D_s \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n)(v_1, \ldots, v_{n-1}) \in U.$$

Therefore, by Lemma 9, we have $a_{j,s} = 0$ for all (j, s) such that $j \leq n < s \leq m$. In other words,

$$\sum_{i=1}^{n} (-1)^{i+1} u_{i,j}^{s} \cdot (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) = 0.$$

Use Lemma 9 again. We conclude that $u_{i,j}^{s} = 0$ for (i, j, s) such that $i \leq n$, $j \leq n$, $n < s \leq m$. In other words,

$$[D_i, D_j] = \sum_{s=1}^{n} u_{i,j}^{s} D_s, \quad 1 \leq i, j \leq n. \quad \square$$

3. Proof of Theorem 1

By Lemma 6, it suffices to verify the condition for $D_1 \wedge \cdots \wedge D_n$ to be an n-Lie multiplication. The involution property will be needed to verify the condition $\mathrm{nlie}_1 = 0$.

We have

$$[D_i, D_1 \wedge \cdots \wedge D_n] = \sum_{j=1}^{n} D_1 \wedge \cdots \wedge \underbrace{[D_i, D_j]}_{j} \wedge \cdots \wedge D_n$$

$$= \sum_{j,s=1}^{n} u_{i,j}^s D_1 \wedge \cdots \wedge D_{j-1} \wedge D_s \wedge D_{j+1} \wedge \cdots \wedge D_n = \sum_{j=1}^{n} u_{i,j}^j D_1 \wedge \cdots \wedge D_n.$$

Therefore,

$$\sum_{i=1}^{n} (-1)^{i+1} (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot [D_i, D_1 \wedge \cdots \wedge D_n]$$
$$= \sum_{i,j=1}^{n} (-1)^{i+1} u_{i,j}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n).$$

Furthermore, for $i < j$,

$$[D_i, D_j] \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n = \sum_{s=1}^{n} u_{i,j}^s D_s \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n$$
$$= u_{i,j}^i D_i \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n + u_{i,j}^j D_j \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n$$
$$= (-1)^{i+1} u_{i,j}^i D_1 \wedge \cdots \widehat{D}_j \cdots \wedge D_n + (-1)^j u_{i,j}^j D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n.$$

Hence,

$$\sum_{i<j} (-1)^{i+j} ([D_i, D_j] \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$= \sum_{i<j} (-1)^{j+1} u_{i,j}^i (D_1 \wedge \cdots \widehat{D}_j \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$+ \sum_{i<j} (-1)^i u_{i,j}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$= \sum_{i>j} (-1)^{i+1} u_{j,i}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$+ \sum_{i<j} (-1)^i u_{i,j}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$\text{(since } u_{i,j}^s = -u_{j,i}^s\text{)}$$
$$= \sum_{i>j} (-1)^i u_{i,j}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$+ \sum_{i<j} (-1)^i u_{i,j}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n)$$
$$= \sum_{i} (-1)^i \sum_{j} u_{i,j}^j (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot (D_1 \wedge \cdots \wedge D_n).$$

Thus,

$$\sum_{i=1}^{n} (-1)^{i+1} (D_1 \wedge \cdots \widehat{D}_i \cdots \wedge D_n) \cdot [D_i, D_1 \wedge \cdots \wedge D_n]$$
$$+ \sum_{i<j} (-1)^{i+j} ([D_i, D_j] \wedge D_1 \wedge \cdots \widehat{D}_i \cdots \widehat{D}_j \cdots \wedge D_n) \cdot D_1 \wedge \cdots \wedge D_n = 0.$$

Therefore, by Lemma 5, $D_1 \wedge \cdots \wedge D_n$ is an n-Lie multiplication. $\square$

4. Proof of Theorem 3

Our considerations are local. The notions of a vector field on M and a derivation on $U = \mathscr{F}(M)$ are known to be equivalent locally.

If $\mathscr{D}$ is in involution then $(U, D_1 \wedge \cdots \wedge D_n)$ is an n-Lie algebra by Theorem 1.

Conversely, suppose that $(U, D_1 \wedge \cdots \wedge D_n)$ is an n-Lie algebra. Prove that the system $\mathscr{D}$ is in involution.

We will follow the argument of the proof of the Frobenius theorem [6, Chapter VII, Theorem 2.1]. Let x_0 be a point of M. Consider the local coordinates $x^1, \dots, x^m$ equal to 0 in x_0 and such that the vector fields $\partial/\partial x^1, \dots, \partial/\partial x^n$ generate the fiber V_{x_0} which is an n-dimensional subspace of $T_{x_0}M$. We may consider on open neighborhood of the point x_0 such that the C^∞-vector fields $D_1, \dots, D_n$ have the shape

$$D_j = \frac{\partial}{\partial x^j} + \sum_{k=1}^{m} \alpha_j^k(x) \frac{\partial}{\partial x^k}, \quad j = 1, \dots, n,$$

where $\alpha_j^k(x_0) = 0$ for all j, k. Then the $n \times n$-matrix $I_n + \left(\alpha_j^k(x)\right)_{1 \le j,k \le n}$ is invertible for all $x \in W$ and some infinitesimal W. Let $\left(\beta_j^k(x)\right)_{1 \le j,k \le n}$ be its inverse matrix. Then the vector fields $D'_j = \sum_{k=1}^n \beta_j^k D_k$ have the shape

$$D'_j = \frac{\partial}{\partial x^j} + \sum_{k=n+1}^{m} \lambda_{i,j}^k(x) \frac{\partial}{\partial x^k}, \quad 1 \le j \le n.$$

Put

$$D'_j = \frac{\partial}{\partial x^j}, \quad n < j \le m.$$

Notice that $(D'_1 \wedge \cdots \wedge D'_m)(x^1, \dots, x^m) = 1$. Therefore, $D'_1 \wedge \cdots \wedge D'_m \ne 0$. Furthermore, the commutator $[D'_i, D'_j]$ is a U-linear combination of $D'_1, \dots, D'_m$ for all $1 \le i, j \le n$. Since $(U, D_1 \wedge \cdots \wedge D_n)$ is an n-Lie algebra, $(U, D'_1 \wedge \cdots \wedge D'_n)$ is an n-Lie algebra by Lemma 10. By Lemma 10, $[D'_i, D'_j]$ is a U-linear combination of $D'_1, \dots, D'_n$.

Note that the expression of D_i begins from $\partial/\partial x^i$ but $[D'_i, D'_j]$ has no $\partial/\partial x^s$-component if $s \le n$. Hence, $[D'_i, D'_j] = 0$, $1 \le i, j \le n$. Thus, the system $D_i, 1 \le i \le n$, as a U-linear combination of commuting vector fields $D'_j, 1 \le j \le n$, is in involution.

Let (U, ω) be an $(n+1)$-Lie algebra with $\omega = \mathrm{id} \wedge D_1 \wedge \cdots \wedge D_n$. Then $i(1)\omega = D_1 \wedge \cdots \wedge D_n$, and $(U, D_1 \wedge \cdots \wedge D_n)$ is an n-Lie algebra by [2]. Conversely, if $(U, D_1 \wedge \cdots \wedge D_n)$ is an n-Lie algebra then $(U, \mathrm{id} \wedge D_1 \wedge \cdots \wedge D_n)$ is also an $(n+1)$-Lie algebra by [5, Theorem 6.3]. $\square$

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