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SHORT COMMUNICATIONS

On the Hesse–Muir Formula for the Determinant of the Matrix $A^{n-1}B^2$

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In [1], Hesse presented the identity

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 = \begin{vmatrix} \begin{vmatrix} a & b & x_1 \\ b & c & x_2 \\ x_1 & x_2 & 0 \end{vmatrix} & \begin{vmatrix} a & b & x_1 \\ b & c & x_2 \\ y_1 & y_2 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & x_1 \\ b & c & x_2 \\ y_1 & y_2 & 0 \end{vmatrix} & \begin{vmatrix} a & b & y_1 \\ b & c & y_2 \\ y_1 & y_2 & 0 \end{vmatrix} \end{vmatrix}.$$

In [2], Muir generalized this result. He showed that, for any $n \times n$-symmetric matrix $A = (a_{i,j})$ and for any $n \times n$ matrix $B = (x_{i,j})$, the following formula for the determinants holds:

$$|A^{n-1}B^2| = (-1)^n \begin{vmatrix} \begin{vmatrix} A & x_1^t \\ x_1 & 0 \end{vmatrix} & \begin{vmatrix} A & x_1^t \\ x_2 & 0 \end{vmatrix} & \dots & \begin{vmatrix} A & x_1^t \\ x_n & 0 \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} A & x_n^t \\ x_1 & 0 \end{vmatrix} & \begin{vmatrix} A & x_n^t \\ x_2 & 0 \end{vmatrix} & \dots & \begin{vmatrix} A & x_n^t \\ x_n & 0 \end{vmatrix} \end{vmatrix}, \tag{1}$$

where $x_i = (x_{i1}, \dots, x_{in})$ is the ith row of the matrix B and x_i^t is this row written as a column.

In our paper, we give a simple proof of this fact for any, not necessarily symmetric, matrix A.

Theorem. *Formula* (1) *holds for all $n \times n$ matrices A and B.*

Proof. Denote the right-hand side of formula (1) by F. For the matrix A, let $\operatorname{adj} A = (A_{j,i})$ denote its associated matrix and let A^t be its transpose.

For any row u and column v of length n and for any scalar λ, the following formula is valid:

$$\begin{vmatrix} A & v \\ u & \lambda \end{vmatrix} = \lambda|A| - u(\operatorname{adj} A)v$$

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(see [3, Theorem 3.1.3]). Therefore,

$$\begin{vmatrix} A & x_i^t \\ x_j & 0 \end{vmatrix} = -x_j(\operatorname{adj} A)x_i^t.$$

Hence

$$F = |G|,$$

where

$$G = \begin{pmatrix} x_1(\operatorname{adj} A)x_1^t & x_2(\operatorname{adj} A)x_1^t & \cdots & x_n(\operatorname{adj} A)x_1^t \\ \vdots & \vdots & \ddots & \vdots \\ x_1(\operatorname{adj} A)x_n^t & x_2(\operatorname{adj} A)x_n^t & \cdots & x_n(\operatorname{adj} A)x_n^t \end{pmatrix} = B(\operatorname{adj} A)^t B^t.$$

It remains to use the well-known formulas for determinants

$$|(\operatorname{adj} A)^t| = |\operatorname{adj} A| = |A|^{n-1}, \qquad |B^t| = |B|$$

to obtain

$$F = |A|^{n-1}|B|^2. \quad \square$$

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