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Lie expression for multi-parameter Klyachko idempotent

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Abstract An expression for the multi-parameter Klyachko idempotent as a linear combination of Lie base elements is given.

Keywords Free Lie algebras · Lie elements · Klyachko idempotent

1 Introduction

Let \mathcal{F} be the free associative algebra with generators a_1, a_2, \dots , over the field of complex numbers \mathbf{C} . Its elements are associative polynomials in non-commuting variables a_1, a_2, \dots , with coefficients from \mathbf{C} . Let \mathcal{F}^- be the Lie algebra of \mathcal{F} under commutator $[a, b] = ab - ba$. Endow \mathcal{F} with the structure of a Hopf algebra by defining a coproduct δ such that $\delta(a_i) = a_i \otimes 1 + 1 \otimes a_i$. Let \mathcal{L} be the Lie subalgebra of \mathcal{F} generated by a_1, a_2, \dots . So \mathcal{L} is the free Lie algebra on generators a_1, a_2, \dots .

An element $X = f(a_1, \dots, a_n) \in \mathcal{F}$ is called a *Lie element* if $X \in \mathcal{L}$. There is a well-known criterion for X to be a Lie element (Friedrich's criterion): X is a Lie element if and only if X is primitive element of \mathcal{F} (considered as a Hopf algebra),

$$\delta(X) = X \otimes 1 + 1 \otimes X.$$

If X is a homogeneous Lie element of degree m , the Dynkin–Specht–Wever theorem allows us to construct its Lie expression

$$\pi X = mX,$$

where πX denotes Lie element corresponding to X constructed by

$$\pi a_i = a_i, \quad \pi(a_{i_1} \dots a_{i_m}) = [a_{i_1}, [\dots [a_{i_{m-1}}, a_{i_m}] \dots]].$$

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See [4] or [9] for details. For example, a_1^2 is not Lie element, but $X = -a_1^2 a_2 + 2a_1 a_2 a_1 - a_2 a_1 a_1$ is a Lie element with $X = -[a_1, [a_1, a_2]]$.

A.A. Klyachko in [5] has constructed the following idempotent in the group algebra of the permutation group:

$$k_n = \frac{1}{n} \sum_{\sigma \in S_n} q^{maj(\sigma)} \sigma.$$

Recall that the major-index $maj(\sigma)$ is defined as a sum of descent indices,

$$maj(\sigma) = \sum_{1 \leq i < n, \sigma(i) > \sigma(i+1)} i.$$

An important property of the Klyachko idempotent is that the element

$$k_n(a_1, \dots, a_n) = \frac{1}{n} \sum_{\sigma} q^{maj(\sigma)} a_{\sigma(1)} \cdots a_{\sigma(n)}$$

is a Lie element, assuming q is a primitive n -th root of unity. For example,

$$2k_2(a_1, a_2) = a_1 a_2 - a_2 a_1 = [a_1, a_2] \in \mathcal{L}.$$

The case $n = 3$ is not so evident. If

$$q = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2},$$

then

$$3k_3(a_1, a_2, a_3) = a_1 a_2 a_3 + q^2 a_1 a_3 a_2 + q a_2 a_1 a_3 + q^2 a_2 a_3 a_1 + q a_3 a_1 a_2 + q^3 a_3 a_2 a_1.$$

By Friedrich's criterion

$$k_3(a_1, a_2, a_3) \in \mathcal{L},$$

and by the Dynkin–Specht–Weber theorem,

$$\begin{aligned} 3k_3(a_1, a_2, a_3) &= [[a_1, a_2], a_3] + q^2 [[a_1, a_3], a_2] + q [[a_2, a_1], a_3] + q^2 [[a_2, a_3], a_1] \\ &\quad + q [[a_3, a_1], a_2] + q^3 [[a_3, a_2], a_1]. \end{aligned}$$

But one can check that the following is a simpler expression for k_3 as a Lie element,

$$k_3(a_1, a_2, a_3) = [a_1, [a_2, a_3]] + q [a_2, [a_1, a_3]] \in \mathcal{L},$$

where in all summands the last element a_3 seats in the last place.

In this paper we show that this kind of construction exists for any n . We give expression of the Klyachko element as a Lie element in a more general context. Let $[n] = \{1, 2, \dots, n\}$ and S_n be its permutation group. We will use one-line notation

for permutations: instead of $\binom{1 \ 2 \ \dots \ n}{\sigma(1) \ \sigma(2) \ \dots \ \sigma(n)}$ we will write $\sigma(1) \cdots \sigma(n)$. More generally we will consider words in the alphabet $[n]$ that have at most one occurrence of each letter $i \in [n]$. If every element of $[n]$ appears in such a word $u = i_1 \dots i_k$, then $k = n$ and we can consider w as a permutation. For a word $u = i_1 \dots i_k$, say that $k = |u|$ is its *length* and that $s \in [n - 1]$ is a *descent index* if $i_s > i_{s+1}$. Denote by $Des(u)$ a set of descent indices of u . The sum of all descent indices, as we mentioned above, is called *major index* of u and is denoted $maj(u)$,

$$maj(u) = \sum_{j \in Des(u)} j.$$

Define the *multi-parametric q-major index* $maj_{\mathbf{q}}(u)$ of word u by

$$maj_{\mathbf{q}}(u) = \frac{\prod_{j \in Des(u)} q_{u(1)} \cdots q_{u(j)}}{\prod_{i=1}^{|u|-1} (1 - q_{u(1)} \cdots q_{u(i)})},$$

where q_1, \dots, q_n are some variables.

Let $u = i_1 \cdots i_k$ and $i_s = \max\{i_1, \dots, i_k\}$ be maximum of the letters of u . Call the subwords $L(u) = i_1 \dots i_{s-1}$ the *left part* and $R(u) = i_{s+1} \cdots i_k$ the *right part* of u . For a word $u = i_1 \cdots i_{k-1} i_k$ denote by $rev(u) = i_k i_{k-1} \cdots i_1$ the word u written in reverse order. For example, if $u = 513942$, then

$$|u| = 6, \quad Des(u) = \{1, 4, 5\}, \quad maj(u) = 1 + 4 + 5 = 10,$$

$$maj_{\mathbf{q}}(u) = \frac{q_1^2 q_3^2 q_4 q_5^3 q_9^2}{(1 - q_5)(1 - q_1 q_5)(1 - q_1 q_3 q_5)(1 - q_1 q_3 q_5 q_9)(1 - q_1 q_3 q_4 q_5 q_9)},$$

$$L(u) = 513, \quad R(u) = 42, \quad rev(u) = 249315.$$

Let $K(\mathbf{q})$ be the field of rational functions in \mathbf{q} over the field K and $K(\mathbf{q})S_n$ be group algebra of symmetric group with coefficients in $K(\mathbf{q})$. There is a natural left action of S_n on $K(\mathbf{q})$: if $f(\mathbf{q}) = f(q_1, \dots, q_n) \in K(\mathbf{q})$, then we define

$$\sigma[f(\mathbf{q})] = f(q_{\sigma(1)}, \dots, q_{\sigma(n)}).$$

The “correct” product for $K(\mathbf{q})S_n$ is not the straightforward analogue of the product for $K S_n$. Instead we need the *twisted product* of $f(\mathbf{q})\sigma$ and $g(\mathbf{q})\tau$, denoted $f(\mathbf{q})\sigma \times g(\mathbf{q})\tau$, defined by

$$f(\mathbf{q})\sigma \times g(\mathbf{q})\tau = (f(\mathbf{q})\sigma [g(\mathbf{q})]\sigma)\tau.$$

For example,

$$213 \times \frac{q_2 q_3}{(1 - q_2)(1 - q_2 q_3)} 231 = \frac{q_1 q_3}{(1 - q_1)(1 - q_1 q_3)} 132.$$

We recall the following multi-parameter generalization of Klyachko idempotent introduced in [7]. Namely, an element $k_n(\mathbf{q}) \in K(\mathbf{q})S_n$ defined by

$$k_n(\mathbf{q}) = \sum_{\sigma \in S_n} maj_{\mathbf{q}}(\sigma)\sigma$$

is an idempotent in $K(\mathbf{q})S_n$ under multiplication \times . Moreover, $k_n(\mathbf{q})(a_1, \dots, a_n)$ is a Lie element, if $q_1q_2 \cdots q_n = 1$, but $q_{i_1}q_{i_2} \cdots q_{i_r} \neq 1$, for any proper subset $\{i_1, \dots, i_r\} \subset [n]$.

The multilinear part of the free Lie algebra of degree n is $(n - 1)!$ -dimensional and any multilinear Lie element should be a linear combination of *base* elements of a form $[a_{\sigma(1)}, [\cdots [a_{\sigma(n-1)}, a_n] \cdots]]$. The existence of a Lie expression for the Klyachko element as such a linear combination was established in many papers (see for example [1–3, 8]).

In our paper we give an explicit such expression of the multi-parameter Klyachko element as a Lie word.

Theorem 1 *If $q_1q_2 \cdots q_n = 1$, but $q_{i_1}q_{i_2} \cdots q_{i_r} \neq 1$, for any proper subset $\{i_1, \dots, i_r\}$ of $\{1, 2, \dots, n\}$, then*

$$k_n(\mathbf{q}) = \sum_{\sigma \in S_n, \sigma(n)=n} \text{maj}_{\mathbf{q}}(\sigma)[\sigma(1), [\sigma(2), \dots, [\sigma(n-1), n] \cdots]].$$

Equivalently,

$$\sum_{\sigma \in S_n} \text{maj}_{\mathbf{q}}(\sigma)a_{\sigma(1)} \cdots a_{\sigma(n)} = \sum_{\tau \in S_{n-1}} \frac{\text{maj}_{\mathbf{q}}(\tau)[a_{\tau(1)}, [a_{\tau(2)}, \dots [a_{\tau(n-1)}, a_n] \cdots]]}{(1 - q_1 \cdots q_{n-1})}. \tag{1}$$

Example Let us demonstrate calculations for the case $n = 3$. Denote by H the right-hand side of equality (1). We have

$$\text{maj}_{\mathbf{q}}(123) = \frac{1}{(1 - q_1)(1 - q_1q_2)}, \quad \text{maj}_{\mathbf{q}}(132) = \frac{q_1q_3}{(1 - q_1)(1 - q_1q_3)},$$

$$\text{maj}_{\mathbf{q}}(213) = \frac{q_2}{(1 - q_2)(1 - q_1q_2)}, \quad \text{maj}_{\mathbf{q}}(231) = \frac{q_2q_3}{(1 - q_2)(1 - q_2q_3)},$$

$$\text{maj}_{\mathbf{q}}(312) = \frac{q_3}{(1 - q_3)(1 - q_1q_3)}, \quad \text{maj}_{\mathbf{q}}(321) = \frac{q_2q_3^2}{(1 - q_3)(1 - q_2q_3)},$$

and

$$\begin{aligned} k_3(\mathbf{q})(a, b, c) &= \frac{abc}{(1 - q_1)(1 - q_1q_2)} + \frac{q_1q_3 acb}{(1 - q_1)(1 - q_1q_3)} \\ &+ \frac{q_2 bac}{(1 - q_2)(1 - q_1q_2)} + \frac{q_2q_3 bca}{(1 - q_2)(1 - q_2q_3)} \\ &+ \frac{q_3 cab}{(1 - q_3)(1 - q_1q_3)} + \frac{q_2q_3^2 cba}{(1 - q_3)(1 - q_2q_3)}. \end{aligned}$$

On the other hand,

$$\text{maj}_{\mathbf{q}}(12) = \frac{1}{1 - q_1}, \quad \text{maj}_{\mathbf{q}}(21) = \frac{q_2}{1 - q_2},$$

and

$$\begin{aligned}
 H &= \frac{1}{1 - q_1 q_2} \left(\frac{[a, [b, c]]}{1 - q_1} + \frac{q_2 [b, [a, c]]}{1 - q_2} \right) \\
 &= \frac{1}{1 - q_1 q_2} \left(\frac{abc - acb - bca + cba}{1 - q_1} + \frac{q_2 (bac - bca - acb + cab)}{1 - q_2} \right) \\
 &= \frac{((1 - q_2)abc - (1 - q_1 q_2)acb + (q_2 - q_1 q_2)bac - (1 - q_1 q_2)bca + (q_2 - q_1 q_2)cab + (1 - q_2)cba)}{(1 - q_1)(1 - q_2)(1 - q_1 q_2)} \\
 &= \frac{abc}{(1 - q_1)(1 - q_1 q_2)} - \frac{acb}{(1 - q_1)(1 - q_2)} + \frac{q_2 bac}{(1 - q_2)(1 - q_1 q_2)} \\
 &\quad - \frac{bca}{(1 - q_1)(1 - q_2)} + \frac{q_2 cab}{(1 - q_2)(1 - q_1 q_2)} + \frac{cba}{(1 - q_1)(1 - q_1 q_2)}.
 \end{aligned}$$

Since $q_1 q_2 q_3 = 1$, this means that

$$k_3(\mathbf{q})(a, b, c) = H.$$

If q is a primitive root of 1 of degree 3 and $q_i = q$, then $(1 - q)(1 - q^2) = 1 - q - q^2 + q^3 = 3$. Therefore in this case we obtain the Klyachko element

$$k_3(\mathbf{q}) = k_3.$$

These kind of calculations can be done for any n .

As a corollary of Theorem 1 we obtain an exact expression of $k_n(a_1, \dots, a_n)$ as a Lie element.

Corollary 2 *If $q^n = 1, q^m \neq 1, 0 < m < n$, then*

$$\sum_{\sigma \in S_n} q^{maj(\sigma)} a_{\sigma(1)} \cdots a_{\sigma(n)} = \sum_{\tau \in S_{n-1}} q^{maj(\tau)} [a_{\tau(1)}, [a_{\tau(2)}, [\cdots, [a_{\tau(n-1)}, a_n] \cdots]]]$$

Let $\sqcup\sqcup$ be the shuffle product on a space of words A . For elements $u = x_1 \dots x_k, v = y_1 \dots y_l \in A$ recall that $u \sqcup\sqcup v$ is defined as the sum of elements of the form $w = z_1 \dots z_{k+l}$ such that $\{z_1, \dots, z_{k+l}\} = \{x_1, \dots, x_k, y_1, \dots, y_l\}$ and if $z_{i_1} = x_1, \dots, z_{i_k} = x_k, z_{j_1} = y_1, \dots, z_{j_l} = y_l$, then $i_1 < i_2 < \dots < i_k, j_1 < j_2 < \dots < j_l$. For $w \in A$ we say that w is a part of the shuffle product $u \sqcup\sqcup v$ and denote $w \in u \sqcup\sqcup v$ if w appears as a summand in $u \sqcup\sqcup v$. For example,

$$ab \sqcup\sqcup cd = abcd + acbd + acdb + cabd + cadb + cdab$$

and $acdb$ is a part of $ab \sqcup\sqcup cd$, but $adcb$ is not.

By Lemma 5 (see below) one can reformulate the above results in terms of major-indices and shuffle products.

Corollary 3 If $q_1 \cdots q_n = 1$, but $q_{i_1} \cdots q_{i_r} \neq 1$, for any proper subset $\{i_1, \dots, i_r\} \subset [n]$, then for any permutation $\sigma \in S_n$,

$$maj_{\mathbf{q}}(\sigma) = \frac{(-1)^{|R(\sigma)|} q_n}{(q_n - 1)} \sum_{w \in L(\sigma) \sqcup \text{rev}(R(\sigma))} maj_{\mathbf{q}}(w),$$

where $L(\sigma)$ and $R(\sigma)$ are the left and right parts of σ .

Corollary 4 If $q^n = 1, q^m \neq 1, 0 < m < n$, then for any permutation $\sigma \in S_n$,

$$q^{maj(\sigma)} = (-1)^{|R(\sigma)|} \sum_{w \in L(\sigma) \sqcup \text{rev}(R(\sigma))} q^{maj(w)},$$

where $L(\sigma)$ and $R(\sigma)$ are the left and right parts of σ .

2 Proof of Theorem 1

Let $S_{n,r}$ be the set of shuffle permutations,

$$S_{n,r} = \{ \sigma \in S_n \mid \sigma(1) < \cdots < \sigma(r), \sigma(r+1) < \cdots < \sigma(n) \}.$$

Lemma 5 For any $a_1, \dots, a_n \in A$,

$$\begin{aligned} & [a_1, [a_2, [\cdots [a_{n-1}, a_n] \cdots]]] \\ &= \sum_{r=0}^{n-1} \sum_{\sigma \in S_{n-1,r}} (-1)^r a_{\sigma(1)} \cdots a_{\sigma(r)} a_n \text{rev}(a_{\sigma(r+1)} \cdots a_{\sigma(n-1)}). \end{aligned}$$

Proof Follows from Theorem 8.16 [9]. □

Lemma 6 Let u and v be complementary words on $[n]$ with $|u| = k, |v| = l$. Then

$$\sum_{w \in u \sqcup v} maj_{\mathbf{q}}(w) = \frac{(1 - q_1 \cdots q_n)}{(1 - q_{u(1)} \cdots q_{u(k)})(1 - q_{v(1)} \cdots q_{v(l)})} maj_{\mathbf{q}}(u) maj_{\mathbf{q}}(v).$$

Proof Let \langle , \rangle be inner (scalar) product on \mathcal{F} defined by $\langle u, v \rangle = \delta_{u,v}$ for any words u and v and extended to \mathcal{F} by linearity. By the following statement (see end of Sect. 2 of the paper [7]),

$$\begin{aligned} & \langle k_n(\mathbf{q}), u \sqcup v \rangle \\ &= (1 - q_1 \cdots q_n) \frac{(\prod_{j \in D(u)} q_{u(1)} \cdots q_{u(n)}) (\prod_{j \in D(v)} q_{v(1)} \cdots q_{v(n)})}{(\prod_{i=1}^k q_{u(1)} \cdots q_{u(i)}) (\prod_{j=1}^l q_{v(1)} \cdots q_{v(j)})} \end{aligned}$$

we have

$$\langle k_n(\mathbf{q}), u \sqcup v \rangle = \frac{(1 - q_1 \cdots q_n) maj_{\mathbf{q}}(u) maj_{\mathbf{q}}(v)}{(1 - q_{u(1)} \cdots q_{u(k)}) (1 - q_{v(1)} \cdots q_{v(l)})}.$$

It remains to note that

$$\langle k_n(\mathbf{q}), u \sqcup v \rangle = \sum_{w \in u \sqcup v} \text{maj}_{\mathbf{q}}(w). \quad \square$$

Lemma 7 For any $\tau \in S_n$,

$$\text{maj}_{\mathbf{q}}(\tau) = \frac{(-1)^{|\tau|+1} (\prod_{j \in \text{Des}(\text{rev}(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)})^{-1}}{\prod_{i=1}^{n-1} (1 - (q_{\tau(1)} \cdots q_{\tau(i)})^{-1})}$$

Proof For $\tau \in S_n$ let

$$\text{Des}(\tau) = \{i \in [n - 1] \mid \tau(i) > \tau(i + 1)\},$$

$$\text{Rise}(\tau) = \{i \in [n - 1] \mid \tau(i) < \tau(i + 1)\}$$

be sets of descent and rise indices. Then

$$\text{Des}(\tau) \cup \text{Rise}(\tau) = \{1, 2, \dots, n - 1\},$$

$$\text{Des}(\tau) \cap \text{Rise}(\tau) = \emptyset,$$

$$|\text{Rise}(\tau)| = n - 1 - |\text{Des}(\tau)|,$$

$$\text{Des}(\text{rev}(\tau)) = \{n, \dots, n\} - \text{rev}(\text{Rise}(\tau))$$

(here (n, \dots, n) is a sequence with components n and length $|\text{Rise}(\tau)|$). Thus,

$$\text{Rise}(\tau) = \{n - j \mid j \in \text{Des}(\text{rev}(\tau))\}.$$

Therefore,

$$\begin{aligned} & \frac{(-1)^{n+1}}{\prod_{j \in \text{Des}(\text{rev}(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)} \prod_{i=1}^{n-1} (1 - (q_{\tau(1)} \cdots q_{\tau(i)})^{-1})} \\ &= \frac{(-1)^{n+1} \prod_{i=1}^{n-1} (q_{\tau(1)} \cdots q_{\tau(i)})}{\prod_{j \in \text{Des}(\text{rev}(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)} \prod_{i=1}^{n-1} (-1 + (q_{\tau(1)} \cdots q_{\tau(i)}))} \\ &= \frac{\prod_{i=1}^{n-1} (q_{\tau(1)} \cdots q_{\tau(i)})}{\prod_{j \in \text{Des}(\text{rev}(\tau))} q_{\tau(1)} \cdots q_{\tau(n-j)} \prod_{i=1}^{n-1} (1 - q_{\tau(1)} \cdots q_{\tau(i)})} \\ &= \frac{\prod_{i \in \text{Des}(\tau)} q_{\tau(1)} \cdots q_{\tau(i)}}{\prod_{i=1}^{n-1} (1 - q_{\tau(1)} \cdots q_{\tau(i)})} \\ &= \text{maj}_{\mathbf{q}}(\tau). \quad \square \end{aligned}$$

Lemma 8 Let u and v be complementary nonempty words on $[n - 1]$. Let $q_1 \cdots q_n = 1$, but $q_{i_1} \cdots q_{i_r} \neq 1$, for any proper subset $\{i_1, \dots, i_r\} \subset \{1, \dots, n\}$. Then

$$maj_{\mathbf{q}}(u \, n \, rev(v)) = \frac{(-1)^{|v|} maj_{\mathbf{q}}(u) maj_{\mathbf{q}}(v)}{(1 - \prod_{i=1}^{|u|} q_{u(i)}) (1 - \prod_{j=1}^{|v|} q_{v(j)})}.$$

Proof Let $\sigma = u \, n \, rev(v)$. Note that $\sigma^{-1}(n) = |u| + 1$ and $|u| \notin Des(\sigma)$. Therefore,

$$\begin{aligned} & maj_{\mathbf{q}}(\sigma) \\ &= \frac{\prod_{j \in Des(\sigma)} q_{\sigma(1)} \cdots q_{\sigma(j)}}{\prod_{i=1}^{n-1} (1 - q_{\sigma(1)} \cdots q_{\sigma(i)})} \\ &= \frac{\prod_{j \in Des(\sigma), j < |u|} q_{u(1)} \cdots q_{u(j)}}{\prod_{i=1}^{|u|-1} (1 - q_{u(1)} \cdots q_{u(i)})} \\ &\quad \times \frac{(q_n \prod_{j=1}^{|u|} q_{\sigma(j)})}{(1 - q_{u(1)} \cdots q_{u(|u|)}) (1 - q_{u(1)} \cdots q_{u(|u|)} q_n)} \\ &\quad \times \frac{\prod_{j \in Des(rev(v))} q_{u(1)} \cdots q_{u(|u|)} q_n q_{v(|v|-j+1)} \cdots q_{v(|v|)}}{\prod_{i=1}^{|v|-1} (1 - q_{u(1)} \cdots q_{u(|u|)} q_n q_{v(1+i)} \cdots q_{v(|v|)})}. \end{aligned}$$

Now use the condition

$$q_{u(1)} \cdots q_{u(|u|)} q_n q_{v(1)} \cdots q_{v(|v|)} = q_1 \cdots q_n = 1.$$

We have

$$\begin{aligned} & maj_{\mathbf{q}}(\sigma) \\ &= maj_{\mathbf{q}}(u) \times \frac{1}{(1 - q_{u(1)} \cdots q_{u(|u|)}) (-1 + q_{v(1)} \cdots q_{v(|v|)})} \\ &\quad \times \frac{\prod_{j \in Des(rev(v))} (q_{v(1)} \cdots q_{v(|v|-j)})^{-1}}{\prod_{i=1}^{|v|-1} (1 - (q_{v(1)} \cdots q_{v(i)})^{-1})}. \end{aligned}$$

It remains to use Lemma 7. □

Proof of Theorem 1 For any $\sigma \in S_n$ let us calculate the coefficient of $a_{\sigma(1)} \cdots a_{\sigma(n)}$. On the left hand side of (1) it is $maj_{\mathbf{q}}(\sigma)$.

Now consider the coefficient on the right hand side of (1). Denote it as λ_{σ} . If $r = \sigma^{-1}(n)$, then $u = L(\sigma) = \sigma(1) \cdots \sigma(r - 1)$ and $v = R(\sigma) = \sigma(r + 1) \cdots \sigma(n)$. Then $a_{\sigma(1)} \cdots a_{\sigma(n)}$ appears only in terms of a form

$$\frac{maj_{\mathbf{q}}(w)[a_{w(1)}, [\cdots [a_{w(n-1)}, a_n] \cdots]]}{(1 - q_1 \cdots q_{n-1})},$$

where $w \in u \sqcup \text{rev}(v)$, with coefficient $(-1)^{|v|}$. Hence, by Lemma 5,

$$\lambda_\sigma = \sum_{w \in u \sqcup \text{rev}(v)} \frac{(-1)^{|v|}}{(1 - q_1 \cdots q_{n-1})} \text{maj}_{\mathbf{q}} w.$$

Since u and v are complementary words on $[n - 1]$, by Lemma 6

$$\lambda_\sigma = \frac{(-1)^{|v|}}{(1 - q_{u(1)} \cdots q_{u(k)})(1 - q_{v(1)} \cdots q_{v(l)})} \text{maj}_{\mathbf{q}}(u) \text{maj}_{\mathbf{q}}(\text{rev}(v)).$$

By Lemma 8 we see that

$$\lambda_\sigma = \text{maj}_{\mathbf{q}}(u n v)$$

So, coefficients at $a_{\sigma(1)} \cdots a_{\sigma(n)}$ on the left and right hand sides of (1) are equal for any $\sigma \in S_n$. Theorem 1 is proved. □

3 Additional remarks

Remark 1 It will be interesting to study associative algebra as an n -algebra under k_n as an n -ary multiplication. For example, for $n = 3$ the 3-multiplication $[a, b, c]$ defined by

$$[a_1, a_2, a_3] = \sum_{\sigma \in \mathcal{S}_3} q^{\text{maj}(\sigma)} a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)}, \quad q = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2},$$

can be re-written as

$$[a, b, c] = (ab)c + (cb)a + q((ba)c + (ca)b) + q^2((ac)b + (bc)a)$$

It satisfies the cyclic q -identity of 3-degree 1,

$$[a, b, c] = q[c, a, b],$$

and the following 25 terms identity of 3-degree 2

$$\begin{aligned} & -q^2[a, b, [c, d, e]] - q[a, b, [d, c, e]] + [a, c, [b, d, e]] + q[a, c, [d, b, e]] \\ & - [a, d, [b, c, e]] + q^2[a, d, [c, b, e]] - [a, e, [b, c, d]] + q^2[a, e, [c, b, d]] \\ & + q^2[b, a, [c, d, e]] + [b, a, [d, c, e]] - q[b, c, [a, d, e]] - [b, c, [d, a, e]] \\ & + q[b, d, [a, c, e]] - q^2[b, d, [c, a, e]] + q[b, e, [a, c, d]] - q^2[b, e, [c, a, d]] \\ & - [c, a, [b, d, e]] - q^2[c, a, [d, b, e]] + q[c, b, [a, d, e]] + q^2[c, b, [d, a, e]] \\ & - q[c, d, [a, b, e]] + [c, d, [b, a, e]] - q[c, e, [a, b, d]] + [c, e, [b, a, d]] \\ & + (q - q^2)[d, e, [a, b, c]] = 0. \end{aligned}$$

Remark 2 The Dynkin–Specht–Wever theorem allows us to check whether an element $X \in \mathcal{F}$ can be presented as a linear combination of Lie commutators. Let us give some modification of this theorem that allows to write a multilinear Lie element as a linear combination of Lie *base* elements. The algorithm is the following:

- Step 1** Write multilinear element X as a linear combination of Lie commutators. One can use here the Dynkin–Specht–Wever theorem.
- Step 2** Write all Lie commutators in a right-bracketed form by the rule (Jacobi identity) $[[a, b]c] := [a, [b, c]] - [b, [a, c]]$.
- Step 3** Write any right-bracketed Lie commutator $[a_{i_1}, [\dots [a_{i_{n-1}}, a_{i_n}] \dots]]$ as a linear combination of multilinear base elements $[a_{\sigma(1)}, [\dots [a_{\sigma(n-1)}, a_n] \dots]]$, where $\sigma \in S_{n-1}$. Here one can use the following formula:

$$\begin{aligned}
 & [a_n, [a_1, [\dots, [a_{n-2}, a_{n-1}]]]] \\
 &= \sum_{r=0}^{n-2} \sum_{\sigma \in S_{n-1}, r+1, \sigma(r+1)=n-1} [a_{\sigma(1)}, [\dots [a_{\sigma(r)}, [a_{n-1}, [a_{\sigma(n-2)}, \\
 & \quad [\dots [a_{\sigma(r+1)}, a_n]]]]]]]. \tag{2}
 \end{aligned}$$

Example Let us present an element $X = [[a_2, a_5], [[a_3, a_4], a_1]]$ as a linear combination of Lie base elements. We can begin from step 2,

$$\begin{aligned}
 X &= [a_2, [a_5, [a_3, [a_4, a_1]]]] - [a_2, [a_5, [a_4, [a_3, a_1]]]] - [a_5, [a_2, [a_3, [a_4, a_1]]]] \\
 & \quad + [a_5, [a_2, [a_4, [a_3, a_1]]]].
 \end{aligned}$$

By (2) we have

$$\begin{aligned}
 [a_5, [a_3, [a_4, a_1]]] &= -[a_3, [a_4, [a_1, a_5]]] + [a_3, [a_1, [a_4, a_5]]] \\
 & \quad + [a_4, [a_1, [a_3, a_5]]] - [a_1, [a_4, [a_3, a_5]]],
 \end{aligned}$$

and,

$$\begin{aligned}
 & [a_2, [a_5, [a_3, [a_4, a_1]]]] \\
 &= -[a_2, [a_3, [a_4, [a_1, a_5]]]] + [a_2, [a_3, [a_1, [a_4, a_5]]]] + [a_2, [a_4, [a_1, [a_3, a_5]]]] \\
 & \quad - [a_2, [a_1, [a_4, [a_3, a_5]]]].
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & [a_2, [a_5, [a_4, [a_3, a_1]]]] \\
 &= -[a_2, [a_4, [a_3, [a_1, a_5]]]] + [a_2, [a_4, [a_1, [a_3, a_5]]]] + [a_2, [a_3, [a_1, [a_4, a_5]]]] \\
 & \quad - [a_2, [a_1, [a_3, [a_4, a_5]]]].
 \end{aligned}$$

By (2),

$$[a_5, [a_2, [a_3, [a_4, a_1]]]]$$

$$\begin{aligned}
 &= [a_1, [a_4, [a_3, [a_2, a_5]]]] - [a_2, [a_1, [a_4, [a_3, a_5]]]] + [a_2, [a_3, [a_1, [a_4, a_5]]]] \\
 &\quad - [a_2, [a_3, [a_4, [a_1, a_5]]]] + [a_2, [a_4, [a_1, [a_3, a_5]]]] \\
 &\quad - [a_3, [a_1, [a_4, [a_2, a_5]]]] + [a_3, [a_4, [a_1, [a_2, a_5]]]] \\
 &\quad - [a_4, [a_1, [a_3, [a_2, a_5]]]], \\
 &[a_5, [a_2, [a_4, [a_3, a_1]]]] \\
 &= [a_1, [a_3, [a_4, [a_2, a_5]]]] - [a_2, [a_1, [a_3, [a_4, a_5]]]] + [a_2, [a_3, [a_1, [a_4, a_5]]]] \\
 &\quad + [a_2, [a_4, [a_1, [a_3, a_5]]]] - [a_2, [a_4, [a_3, [a_1, a_5]]]] \\
 &\quad - [a_3, [a_1, [a_4, [a_2, a_5]]]] - [a_4, [a_1, [a_3, [a_2, a_5]]]] \\
 &\quad + [a_4, [a_3, [a_1, [a_2, a_5]]]].
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 X &= [a_1, [a_3, [a_4, [a_2, a_5]]]] - [a_1, [a_4, [a_3, [a_2, a_5]]]] - [a_3, [a_4, [a_1, [a_2, a_5]]]] \\
 &\quad + [a_4, [a_3, [a_1, [a_2, a_5]]]].
 \end{aligned}$$

Remark 3 Klyachko element has one more generalization. In [6] a Lie idempotent was constructed that generalises three other well-known idempotents. This generalization is the q -Solomon idempotent

$$\phi_n(q) = \frac{1}{n} \sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)} q^{maj(\sigma) - \binom{d(\sigma)+1}{2}}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} \sigma.$$

It has the following properties:

$$\phi_n(\omega) = k_n(\omega),$$

is the Klyachko element if ω is a primitive root of degree n ,

$$\phi_n(0) = [\cdots [1, 2], \dots, n]$$

is the Dynkin idempotent in case of $q = 0$ and

$$\phi_n(1) = \sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} \sigma$$

gives us the (first) Euler idempotent if $q = 1$. Here $\begin{bmatrix} n-1 \\ p \end{bmatrix}_q$ denotes a q -binomial coefficient.

One can establish that the q -Solomon idempotent has a similar Lie presentation by base elements,

$$\phi_n(q) = \frac{1}{n} \sum_{\sigma \in S_n, \sigma(n)=n} \frac{(-1)^{des(\sigma)} q^{maj(\sigma) - \binom{d(\sigma)+1}{2}}}{[d(\sigma)]_q} [\sigma],$$

where $[\sigma]$ denotes the Lie commutator $[\sigma(1), [\dots [\sigma(n-1), n] \dots]]$. It follows from the following property of major indices

$$\begin{aligned} & \sum_{w \in u \sqcup \perp \perp rev(v)} \frac{(-1)^{|v|} q^{maj(w) - \binom{des(w)+1}{2}}}{[|w|]_q} \\ &= \frac{(-1)^{des(u)+des(v)} q^{maj(u)+maj(v)+(|u|+1)(des(v)+1) - \binom{des(u)+des(v)+2}{2}}{[des(u)+des(v)+1]_q}. \end{aligned}$$

In particular, the Lie presentation for Dynkin element is given by

$$\begin{aligned} & [[\dots [a_1, a_2], \dots], a_n] \\ &= \sum_{\sigma \in S_n, maj(\sigma) = \binom{des(\sigma)+1}{2}, \sigma(n)=n} (-1)^{des(\sigma)} [a_{\sigma(1)}, [\dots [a_{\sigma(n-1)}, a_n] \dots]] \end{aligned}$$

and the Euler element has the following Lie presentation:

$$\sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} a_{\sigma(1)} \dots a_{\sigma(n)} = \sum_{\sigma \in S_n, \sigma(n)=n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} [a_{\sigma(1)}, [\dots [a_{\sigma(n-1)}, a_n] \dots]].$$

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