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ABSTRACT
We prove that assosymmetric algebras under the Jordan product are Lie triple algebras. A Lie triple algebra is called special if it is isomorphic to a subalgebra of the plus-algebra of some assosymmetric algebra. We establish that the Glennie identity of degree 8 is valid for special Lie triple algebras, but not for all Lie triple algebras.

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1. Introduction

An associative algebra under anti-commutator satisfies three remarkable polynomial identities: commutativity, Jordan identity and Glennie identity of degree 8. These identities are independent. It is not clear whether these identities form base of identities for special Jordan algebras [17].

In our paper, we establish analogues of these results for assosymmetric algebras. Assosymetric algebras play in the theory of Lie triple algebras the same role as associative algebras in the theory of Jordan algebras. Let us call algebra plus-assosymmetric if it is isomorphic to a subalgebra of the plus-algebra of an assosymmetric algebra. We prove that any assosymmetric algebra under Jordan product satisfies three polynomial identities: commutativity, Lie triple identity and Glennie identity of degree 8. In a class of plus-assosymmetric algebras, these identities are independent. Lie triple identity is a consequence of Jordan identity, but these identities are not equivalent.

To formulate our results, we need to introduce some notations and remind some definitions. Let K be a field of characteristic $p \geq 0$. We will suppose that $p \neq 2, 3$, if otherwise is not stated. For an algebra A notation, $A = (A, \cdot)$ will mean that a vector space A over a field K has multiplication $\cdot$, i.e., A is endowed by bilinear map $(a, b) \mapsto a \cdot b$.

Definition 1.1. For an algebra A with multiplication $\cdot$ its *plus-algebra* is defined as algebra $A^{(+)} = (A, \star)$, where $a \star b = a \cdot b + b \cdot a$ is anti-commutator.

Definition 1.2. Similarly *minus-algebra* $A^{(-)} = (A, [\ ,\ ])$ of A is an algebra with vector space A and multiplication given by commutator $[a, b] = a \cdot b - b \cdot a$.

For a class of algebras $\mathcal{C}$ denote by $\mathcal{C}^{(\pm)}$ class of algebras of a form $A^{(\pm)}$, where $A \in \mathcal{C}$.

Let $F = K\langle t_1, t_2, \dots\rangle$ be the *free algebra of non-associative non-commutative polynomials* with countable number of generators $t_1, t_2, \dots$. Sometimes we will restrict the number of generators and we consider F as an absolute free algebra with generators $t_1, t_2, \dots, t_k$. The algebra F sometimes is called absolute free algebra, sometimes free magmatic algebra. For an algebra $A = (A, \cdot)$ with multiplication $\cdot$ and a polynomial $f = f(t_1, \dots, t_k) \in F$ say that $f = 0$ is a *polynomial identity* of A if $f(a_1, \dots, a_k) = 0$

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for any substitution $t_i := a_i \in A, 1 \le u \le k$. Here we calculate $f(a_1, \dots, a_k)$ in terms of multiplication $\cdot$. A *variety of algebras* is defined as a class of algebras that satisfy polynomial identities. For variety of algebras generated by polynomial identities $f_1 = 0, \dots f_s = 0$, we use notation $Var(f_1, \dots, f_k)$. If $\mathcal{C} = Var(f_1, \dots, f_s)$ is a variety, denote by $F_{\mathcal{C}}$ a free algebra in this class, then $F_{\mathcal{C}}$ is isomorphic to a factor algebra of F by T-ideal generated by polynomials $f, \dots, f_s$. Denote by $F_{\mathcal{C}}(t_1, \dots, t_k)$ free algebra of a class $\mathcal{C}$ generated by elements $t_1, \dots, t_k$. About polynomial identities and varieties of non-associative algebras, see for example [17].

Let

$$[t_1, t_2] = t_1 t_2 - t_2 t_1, \qquad t_1 \star t_2 = t_1 t_2 + t_2 t_1,$$

are commutator and anti-commutator polynomials and

$$(t_1, t_2, t_3) = t_1(t_2 t_3) - (t_1 t_2) t_3$$

is associator. Note that our definition of associator differs from usual one by sign.

Associator of plus-algebra is denoted by $\langle a, b, c\rangle$,

$$\langle t_1, t_2, t_3\rangle = t_1 \star (t_2 \star t_3) - (t_1 \star t_2) \star t_3.$$

Definition 1.3. Left-symmetric and right-symmetric polynomials are defined by

$$lsym(t_1, t_2, t_3) = (t_1, t_2, t_3) - (t_2, t_1, t_3),$$
$$rsym(t_1, t_2, t_3) = (t_1, t_2, t_3) - (t_1, t_3, t_2).$$

An algebra with identities $lsym = 0$ and $rsym = 0$ is called *assosymmetric.*

So, an algebra $A = (A, \cdot)$ is assosymmetric if

$$a \cdot [b, c] = (a \cdot b) \cdot c - (a \cdot c) \cdot b,$$
$$[a, b] \cdot c = a \cdot (b \cdot c) - b \cdot (a \cdot c).$$

for any $a, b, c \in A$. Recall that *nucleus* of algebra A is a subspace of elements $x \in A$ such that $(x, a, b) = (a, x, b) = (a, b, x) = 0$ for any $a, b \in A$. Nucleus of assosymmetric algebras contains an ideal generated by commutator $[A, A]$ ([2]). Assosymmetric algebras have been studied in [2, 5, 6, 9, 13–15]. The following properties of assosymmetric algebra A,

$$([a, b], c, d) = (a, [b, c], d) = (a, b, [c, d]) = 0,$$
$$[a, b] \cdot (c, d, e) = (a, b, c) \cdot [d, e] = 0,$$

for any $a, b, c, d, e \in A$, are known (proofs one can find, for example, in [2]). Free base of assosymmetric algebras was found in [5].

Denote by $F^{\pm}_{\mathcal{C}}(t_1, \dots, t_k)$ subalgebra of $F_{\mathcal{C}}(t_1, \dots, t_k)^{(\pm)}$ generated by elements $t_1, \dots, t_k$. Let As be class of associative algebras. Then $F^{-}_{As}(t_1, \dots, t_k)$ is isomorphic to free Lie algebra generated by $t_1, \dots, t_k$. In that time F^{+}_{As} is not free. For element $X \in F_{\mathcal{C}}(t_1, \dots, t_k)$ say that X is *Lie element* in the class $\mathcal{C}$ if $X \in F^{-}_{\mathcal{C}}(t_1, \dots, t_k)$. Similarly, $X \in F_{\mathcal{C}}(t_1, \dots, t_k)$ is *Jordan element* in the class $\mathcal{C}$ if $X \in F^{+}_{\mathcal{C}}(t_1, \dots, t_k)$. For a class of associative algebras, Jordan element is called *j-element.* Jordan element in a class of assosymmetric algebras is called *ja-element.* For a class of alternating algebras, Jordan element is called as *jalt-element.*

Let $\mathcal{C}^{(q)}$ be class of algebras $A^{(q)} = (A, \cdot q)$, where $a \cdot_q b = a \cdot b + q\, b \cdot a$ is q-commutator and $(A, \cdot) \in \mathcal{C}$. Note that $\mathcal{C}^{(1)}$ coincides with the class of plus-algebras and $\mathcal{C}^{(-1)}$ coincides with the class of minus-algebras. Let $\mathcal{C}$ be a variety of algebras. As we mentioned above $\mathcal{C}^{(\pm)}$ is not necessary to form variety. Theorem 2.2 of [3] states that $\mathcal{C}^{(q)}$ is a variety if $q^2 \neq 1$. Moreover, as category of algebras, it is isomorphic to the category $\mathcal{C}$. So, from categorical point of view, it is interesting to study assosymmetric algebras under Lie and Jordan commutators only.

Let

$$jor(t_1,t_2) = (t_1,t_2,t_1^2),$$
$$mjor(t_1,t_2,t_3,t_4) = (t_2,t_1,t_3t_4) + (t_3,t_1,t_4t_2) + (t_4,t_1,t_2t_3)$$

be Jordan and multilinear Jordan polynomials. If $p \neq 2,3$, then identities $jor = 0$ and $mjor = 0$ are equivalent. If $p = 3$, then the identity $mjor = 0$ is a consequence of the identity $jor = 0$, but converse is not true. Recall that Jordan algebras are defined as commutative algebras with identity $jor = 0$. Associative algebras under Jordan product satisfy the Jordan identity of degree 4,

$$\langle a,b,a^2\rangle = 0$$

and one more identity of degree 8, so called Glennie identity [4]. We give Shestakov's construction ([10], [16]) of Glennie identity. Let

$$shest(t_1,t_2,t_3) = -3\langle t_1,t_3,t_2\rangle \star (\langle t_1,t_1,t_2^2\rangle - \langle t_1,t_1,t_2\rangle \star t_2) - 2\langle t_1,\langle t_1,\langle t_1,t_3,t_2\rangle,t_2\rangle,t_2\rangle$$

be Shestakov polynomial and

$$glen(t_1,t_2,t_3) = shest(t_1,t_2,t_3\star t_3) - 2t_3 \star shest(t_1,t_2,t_3)$$

be Glennie polynomial. Shestakov has established that the derivation $D_{([a,b]\star[a,b])\star[a,b]} \in Der\, F_{As}$ is well defined on Jordan subspace: for any $a,b,c \in F_{As}$, the element

$$D = D(a,b,c) \stackrel{def}{=} [([a,b]\star[a,b])\star[a,b],c]$$

is j-element. He got exact construction of the element $D(a,b,c)$ as j-element,

$$D(a,b,c) = shest(a,b,c). \tag{1}$$

Then Glennie identity is equivalent to the Leibniz condition for derivation

$$D_{([a,b]\star[a,b])\star[a,b]} \in Der\, F_{As}^{(+)}.$$

In our paper, we prove that relation (1) holds not only for associative algebras but also for assosymmetric algebras. In other words, the element $D(a,b,c)$ is not only j-element but also ja-element. Note that relation (1) fails for a class alternative algebras. It does not mean that $D(a,b,c)$ is not $jalt$-element. We would be surprised very much if it is so. It just means that in alternative case, Jordan polynomial corresponding to the element $D(a,b,c)$ differs from Shestakov polynomial.

Let

$$lietriple(t_1,t_2,t_3) = (t_1,t_2^2,t_3) - t_2 \star (t_1,t_2,t_3)$$

be Lie triple polynomial. A commutative algebra with identity $lietriple = 0$ is called *Lie triple* ([8] , [13], [14]). So, A is Lie triple, if for any $a,b,c \in A$,

$$(a,b^2,c) = 2b(a,b,c).$$

Denote by $\mathcal{A}s^{(+)}$ class of plus-associative algebras, i.e. class of algebras $A^{(+)}$, where A runs associative algebras. Similarly, $\mathcal{A}ssym^{(+)}$ is a class of assosymmetric algebras under Jordan product.

The main results of this paper are the following.

Theorem 1.4. *Let $p \neq 2$. Any plus-algebra of assosymmetric algebra A is Lie triple,*

$$\langle a,b\star b,c\rangle = 2\,b\star\langle a,b,c\rangle, \qquad \forall a,b,c \in A,$$

and satisfies the Glennie identity of degree 8,

$$shest(a,b,c\star c) - 2c\star shest(a,b,c) = 0, \qquad \forall a,b,c \in A.$$

The identities $[t_1,t_2] = 0$, $lietriple(t_1,t_2,t_3) = 0$ and $glen(t_1,t_2,t_3) = 0$ are independent.

Theorem 1.5. *If $p \neq 2,3$, then any identity of $\mathcal{A}ssym^{(+)}$ of degree 4 follows from commutativity and Lie triple identities.*

If $p=3$, then Lie triple identity is not minimal for the class $\mathcal{A}ssym^{(+)}$. It satisfies the identity $mjor=0$ and any identity of degree 4 for the class $\mathcal{A}ssym^{(+)}$ in case $p=3$ follows from the commutativity one and the identity $mjor=0$.

2. Associators, derivations and commutators for assosymmetric algebras

In this section we assume that $A=(A,\cdot)$ is assosymmetric if otherwise is not stated. Recall that $a\star b = a\cdot b + b\cdot a$, $[a,b]=a\cdot b - b\cdot a$ and $\langle a,b,c\rangle = a\star(b\star c)-(a\star b)\star c$ be associator of anti-commutator. Here we give some preliminary results that we will use in proof of our theorems. In proof of our lemmas, we use results of [2] . This paper needs restriction $p\neq 2,3$.

Lemma 2.1. *For any algebra $A=(A,\cdot)$,*

$$\langle a,b,c\rangle - (a,b,c) + (c,b,a) = a\cdot(c\cdot b) - c\cdot(a\cdot b) - (b\cdot a)\cdot c + (b\cdot c)\cdot a.$$

Lemma 2.2. *Let A be assosymmetric. Then for any $a,b,c\in A$,*

$$\langle a,b,c\rangle = [[a,c],b].$$

Proof. By Lemma 2.1 and by assosymmetric rules

$$\begin{aligned}\langle a,b,c\rangle &= a\cdot(c\cdot b) - c\cdot(a\cdot b) - (b\cdot a)\cdot c + (b\cdot c)\cdot a\\ &= [a,c]\cdot b - b\cdot[a,c] = [[a,c],b].\end{aligned}$$

Lemma 2.3. *If A is assosymmetric, and $u=[x,y]$ for some $x,y\in A$, then*

$$(u,a,b) = (a,u,b) = (a,b,u) = 0,$$

and

$$u\cdot(a,b,c) = (a,b,c)\cdot u = 0.$$

Moreover, if a has a form $[x,y]\cdot z$ or $x\cdot[y,z]$, for some $x,y,z\in A$, then also

$$(u,a,b) = (a,u,b) = (a,b,u) = 0.$$

Proof. Follows from results of [2].

Lemma 2.4. *Let $ad\,a: A\to A$ be an adjoint map, $ad\,a(b)=[a,b]=a\cdot b-b\cdot a$. Then*

$$ad\,a(b\cdot c) = ad\,a(b)\cdot c + b\cdot ad\,a(c) + (a,b,c).$$

for any $a,b,c\in A$. If a has a form $a=[x,y]$ or $a=[x,y]\cdot z$, or $a=x\cdot[y,z]$, for some $x,y,z\in A$, then $ad\,a\in Der\,A$.

In particular, $ad\,a$ is a derivation of minus-algebra $A^-=(A,[\ ,\ ])$,

$$ad\,a[b,c] = [ad\,a(b),c] + [a, ad\,a(c)].$$

The map $ad\,a$ is not derivation of plus-algebra $A^+=(A,\star)$,

$$ad\,a(b\star c) = ad\,a(b)\star c + b\star ad\,a(c) + 2(a,b,c).$$

But $ad\,a$ is a derivation of plus-algebra, if $a\in[A,A]$.

Proof. Follows from Lemma 2.3.

Lemma 2.5. *Let A be assosymmetric. Then for any $a, b, c \in A$,*

$$[(a,b,b),a]\star[[a,b],c]=0.$$

Proof. Let $u=[[a,b],c]$. Then by Lemma 2.3

$$((a,b,b)\cdot a)\cdot u=(a,b,b)\cdot[a,u]+((a,b,b)\cdot u)\cdot a=0,$$
$$u\cdot(a\cdot(a,b,b))=[u,a]\cdot(a,b,b)+a\cdot(u\cdot(a,b,b))=0.$$

Therefore,

$$\begin{aligned}[(a,b,b),a]\star u &= [(a,b,b),a]\cdot u+u\cdot[(a,b,b),a]\\ &= ((a,b,b)\cdot a)\cdot u-(a\cdot(a,b,b))\cdot u+u\cdot((a,b,b)\cdot a)-u\cdot(a\cdot(a,b,b))\\ &\quad-(a\cdot(a,b,b))\cdot u+u\cdot((a,b,b)\cdot a)\\ &= -[a\cdot(a,b,b),u]-u\cdot(a\cdot(a,b,b))+((a,b,b)\cdot a)\cdot u-[(a,b,b)\cdot a,u]\\ &= -[a\cdot(a,b,b),u]-[(a,b,b)\cdot a,u].\end{aligned}$$

So, by Lemmas 2.4 and 2.3,

$$[(a,b,b),a]\star u=-[a,u]\cdot(a,b,b)-a\cdot[(a,b,b),u]-[(a,b,b),u]\cdot a-(a,b,b)\cdot[a,u]=0.$$

Lemma 2.6. *Let $A=(A,\cdot)$ be assosymmetric algebra. For $a,b,c\in A$ set*

$$D=D(a,b,c)=[([a,b]\star[a,b])\star[a,b],c]$$

and

$$shest=shest(a,b,c)=-3\langle a,c,b\rangle\star(\langle a,a,b^2\rangle-\langle a,a,b\rangle\star b)-2\langle a,\langle a,\langle a,c,b\rangle,b\rangle,b\rangle.$$

Then

$$D(a,b,c)=shest(a,b,c).$$

In particular, $D(a,b,c)$ is Jordan element of assosymmetric algebra generated by a,b,c.

Proof. Let us set

$$x=[a,b],\qquad u=[[a,b],c].$$

Then $u=[x,c]$. We see that u and x are commutator elements.

By Lemma 2.4

$$\begin{aligned}D&=[(x\star x)\star x,c]\\ &=-2[c,(x\cdot x)\star x]\\ &=-2[c,x\cdot x]\star x-2(x\cdot x)\star[c,x]-4\,(c,x\cdot x,x)\\ &=-2([c,x]\cdot x)\star x-2(x\cdot[c,x])\star x-2(x\cdot x)\star[c,x]-2(c,x,x)\star x-4\,(c,x\cdot x,x)\\ &=-2([c,x]\star x)\star x-2(x\cdot x)\star[c,x]-2(c,x,x)\star x-4\,(c,x\cdot x,x).\end{aligned}$$

Since $x\in[A,A]$, by Lemma 2.3.

$$(c,x,x)=(c,x,x\cdot x)=0.$$

Therefore,

$$D=2\,x\star(x\star u)+(x\star x)\star u.$$

Let us set

$$D_2=[[a,b^2],a]-[[a,b],a]\star b.$$

By Lemma 2.4

$$[a, b^2] = [a, b] \cdot b + b \cdot [a, b] + (a, b, b) = x \cdot b + b \cdot x + (a, b, b)$$

Therefore,

$$D_2 = [[a, b^2], a] - [[a, b], a] \star b = [x \cdot b, a] + [b \cdot x, a] + [(a, b, b), a] - [x, a] \star b.$$

So, by Lemma 2.4

$$\begin{aligned} D_2 &= [x, a] \cdot b + x \cdot [b, a] - (a, x, b) + [b, a] \cdot x + b \cdot [x, a] - (a, b, x) \\ &\quad + [(a, b, b), a] - [x, a] \cdot b - b \cdot [x, a] \\ &= -2x \cdot x + [(b, a, b), a] - 2(a, b, x). \end{aligned}$$

Then by Lemma 2.2

$$\begin{aligned} shest &= -3[[a, b], c] \star ([[a, b^2], a] - [[a, b], a] \star b) - 2[[a, b], [[a, b], [[a, b], c]]] \\ &= -3D_2 \star u - 2[x, [x, u]]. \end{aligned}$$

Thus,

$$D - shest = D_1 + 3D_2 \star u,$$

where we set

$$D_1 = 2\, x \star (x \star u) + 2[x, [x, u]] + (x \star x) \star u.$$

We have

$$\begin{aligned} D_1 &= 2\, x \cdot (x \cdot u) + 2\, x \cdot (u \cdot x) + 2\, (x \cdot u) \cdot x + 2\, (u \cdot x) \cdot x \\ &\quad + 2\, x \cdot (x \cdot u) - 2\, x \cdot (u \cdot x) - 2\, (x \cdot u) \cdot x + 2\, (u \cdot x) \cdot x + (x \star x) \star u \\ &= 4\, x \cdot (x \cdot u) + 4\, (u \cdot x) \cdot x + (x \star x) \star u. \end{aligned}$$

Thus,

$$\begin{aligned} D_1 + 3D_2 \star u &= 4\, x \cdot (x \cdot u) + 4\, (u \cdot x) \cdot x - 6(x \cdot x) \cdot u - 6\, u \cdot (x \cdot x) + 3[(a, b, b), a] \star u \\ &\quad - 6(a, b, x) \star u + (x \star x) \star u \\ &= 4\, (x \cdot (x \cdot u) - (x \cdot x) \cdot u) + 4\, ((u \cdot x) \cdot x - u \cdot (x \cdot x)) - 2(x \cdot x) \cdot u - 2\, u \cdot (x \cdot x) \\ &\quad + 3[(a, b, b), a] \star u - 6(a, b, x) \star u + (x \star x) \star u \\ &= 4\, (x, x, u) - 4\, (u, x, x) - (x \star x) \star u + 3[(a, b, b), a] \star u - 6(a, b, x) \star u + (x \star x) \star u \\ &= (3[(a, b, b), a] - 6(a, b, x)) \star u. \end{aligned}$$

Since by Lemma 2.3, $(a, b, x) \star u = 0$, we see that

$$D_1 + 3D_2 \star u = 3[(a, b, b), a] \star u.$$

It remains to use Lemma 2.5, to obtain that

$$D - shest = D_1 + 3\, D_2 \star u = 0.$$

Lemma 2.7. *Let A be assosymmetric. Then for any $a, b, c \in A$,*

$$\langle b, a, c \star d\rangle + \langle c, a, d \star b\rangle + \langle d, a, b \star c\rangle = -6[a, (b, c, d)].$$

Proof. We have

$$[a, b \cdot c] + [b, c \cdot a] + [c, a \cdot b] = (a, b, c) + (b, c, a) + (c, a, b) = 3(a, b, c).$$

Hence,

$$[a, b \star c] + [b, c \star a] + [c, a \star b] = 6(a, b, c).$$

Therefore, by Lemma 2.2,

$$\langle b, a, c \star d\rangle + \langle c, a, d \star b\rangle + \langle d, a, b \star c\rangle = [[b, c \star d], a] + [[c, d \star b], a] + [[d, b \star c], a]$$
$$= [[b, c \star d] + [c, d \star b] + [d, b \star c], a] = -6[a, (b, c, d)]$$

Lemma 2.8. *For any assosymmetric algebra A a multilinear map*

$$A \times A \times A \times A \to A, \qquad (a, b, c, d) \mapsto mjor(a, b, c, d)$$

is symmetric,

$$mjor(a_1, a_2, a_3, a_4) = mjor(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}),$$

for any permutation $\sigma \in S_4$.

Proof. If $\sigma \in S_4$ fixes the first element, $\sigma(1) = 1$, then

$$mjor(a_1, a_2, a_3, a_4) = mjor(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}),$$

for any algebra A, not necessary assosymmetric. The property

$$[a_1, (a_2, a_3, a_4)] = [a_2, (a_1, a_3, a_4)]$$

was established in [2], p.14, the relation (vii) or (vii'). Thus, by Lemma 2.7, $mjor(a_1, a_2, a_3, a_4)$ is symmetric by all parameters.

3. Proof of Theorem 1.4

By Lemma 2.2

$\langle a, b \star b, c\rangle = 2\,[[a, c], b^2] = 2\,[[a, c], b] \cdot b + 2\,b \cdot [[a, c], b] = 2\langle a, b, c\rangle \cdot b + 2\,b \cdot \langle a, b, c\rangle = 2\langle a, b, c\rangle \star b.$

Hence for any $a, b, c \in A$,

$$\langle a, b \star b, c\rangle = 2b \star \langle a, b, c\rangle.$$

Prove now that, if A is assosymmetric, then

$$glennie(a, b, c) = shest(a, b, c \star c) - 2\,c \star shest(a, b, c) = 0$$

is polynomial identity for plus-assosymmetric algebras. For $p \neq 2, 3$, we can apply results of Section 2.

Let $p \neq 2, 3$. By Lemma 2.6, $D(a, b, c) = shest(a, b, c)$ is well defined on plus-assosymmetric algebras and the map $c \mapsto D(a, b, c)$ is a derivation of assosymmeric algebras. In particular, it is a derivation of plus-assosymmetric algebras. The condition $shest(a, b, c \star c) = 0$ is equivalent to Leibniz condition for this derivation. As was shown by Shestakov, the identity $shest = 0$ is equivalent to Glennie identity [4]. Suppose that the identity $shest = 0$ is a consequence of commutativity and Lie-triple identities. Since Lie-triple identity is a consequence of Jordan identity, this means that $shest = 0$ will be identity for Jordan algebars also. It contradicts to Glennie's result that $glen = 0$ is a special identity that does not follow from commutativity and Jordan identities. Therefore, commutative, Jordan and Glennie identities are independent not only in a class of plus-associative algebras but also in a class of plus-assosymmetric algebras.

For $p = 3$, one can check by computer program, say by Albert [1], that Glennie identity holds for plus-assosymmetric algebras also.

4. Identities of degree 4 for commutative algebras

In this section we consider non-associative but *commutative* polynomials with variables $t_1, t_2, \dots$. Let *mjor* be multilinear Jordan polynomial

$$mjor(t_1,t_2,t_3,t_4) = (t_2,t_1,t_3t_4) + (t_3,t_1,t_4t_2) + (t_4,t_1,t_2t_3)$$

and

$$jor_2(t_1,t_2,t_3,t_4) = mjor(t_1,t_2,t_3,t_4) - mjor(t_2,t_1,t_3,t_4) + mjor(t_3,t_1,t_2,t_4) - mjor(t_4,t_1,t_2,t_3).$$

Define multilinear commutative non-associative polynomial jor_1 by

$$\begin{aligned} jor_1(t_1,t_2,t_3,t_4) &= [l_{t_1},l_{t_2}](t_3t_4) - t_3([l_{t_1},l_{t_2}](t_4)) - ([l_{t_1},l_{t_2}](t_3))t_4 \\ &= t_1(t_2(t_3t_4)) - t_3(t_1(t_2t_4)) - (t_1(t_2t_3))t_4 - t_2(t_1(t_3t_4)) - t_3(t_2(t_1t_4)) - (t_2(t_1t_3))t_4. \end{aligned}$$

Note that $jor_1(t_1,t_2,t_3,t_4)$ is skew-symmetric by variables t_1, t_2 and symmetric by variables t_3, t_4.

Lemma 4.1. *If A is commutative algebra, then for any $a,b,c \in A$,*

$$(a,b,c) = [l_a,l_c](b).$$

In particular, for any $a,b,c \in A$,

$$(a,b,c) + (c,b,a) = 0.$$

Lemma 4.2.

$$jor_1(t_1,t_2,t_3,t_4) = (t_1,t_3t_4,t_2) - t_3(t_1,t_4,t_2) - t_4(t_1,t_3,t_2).$$

Proof. Easy calculations based on Lemma 4.1.

Lemma 4.3. $jor_1(t_1,t_2,t_3,t_4) = -mjor(t_1,t_2,t_3,t_4) + mjor(t_2,t_1,t_3,t_4)$.

Proof. We have

$$\begin{aligned} mjor(t_1,t_2,t_3,t_4) - mjor(t_2,t_1,t_3,t_4) &= (t_2,t_1,t_3t_4) + (t_3,t_1,t_4t_2) + (t_4,t_1,t_2t_3) \\ &\quad - (t_1,t_2,t_3t_4) - (t_3,t_2,t_4t_1) - (t_4,t_2,t_1t_3) \\ &= t_2(t_1(t_3t_4)) - (t_2t_1)(t_3t_4) + t_3(t_1(t_4t_2)) - (t_3t_1)(t_4t_2) \\ &\quad + t_4(t_1(t_2t_3)) - (t_4t_1)(t_2t_3) \\ &\quad - t_1(t_2(t_3t_4)) + (t_1t_2)(t_3t_4) - t_3(t_2(t_4t_1)) + (t_3t_2)(t_4t_1) \\ &\quad - t_4(t_2(t_1t_3)) + (t_4t_2)(t_1t_3) \\ &= t_2(t_1(t_3t_4)) + t_3(t_1(t_4t_2)) + t_4(t_1(t_2t_3)) \\ &\quad - t_1(t_2(t_3t_4)) - t_3(t_2(t_4t_1)) - t_4(t_2(t_1t_3)) \\ &= -t_1(t_2(t_3t_4)) + t_3(t_1(t_2t_4)) + (t_1(t_2t_3))t_4 \\ &\quad \times t_2(t_1(t_3t_4)) - t_3(t_2(t_1t_4)) - (t_2(t_1t_3))t_4 \\ &= -jor_1(t_1,t_2,t_3,t_4). \end{aligned}$$

Lemma 4.4.

$$\begin{aligned} jor_2(t_1,t_2,t_3,t_4) - jor_2(t_2,t_1,t_3,t_4) &= -2\,jor_1(t_1,t_2,t_3,t_4), \\ jor_2(t_1,t_2,t_3,t_4) - jor_2(t_1,t_2,t_4,t_3) &= -2\,jor_1(t_3,t_4,t_1,t_2), \\ jor_2(t_1,t_2,t_3,t_4) + jor_2(t_2,t_1,t_3,t_4) &= -2\,jor_1(t_3,t_4,t_1,t_2), \\ jor_2(t_1,t_2,t_3,t_4) + jor_2(t_1,t_2,t_4,t_3) &= -2\,jor_1(t_1,t_2,t_3,t_4). \end{aligned}$$

In particular,

$$jor_2(t_1,t_2,t_3,t_4)+jor_2(t_2,t_1,t_4,t_3)=0. \tag{2}$$

Proof. Since variables t_i are commuting, by Lemma 4.1

$$jor_2(t_1,t_2,t_3,t_4)-jor_2(t_2,t_1,t_3,t_4)=2(mjor(t_1,t_2,t_3,t_4)-mjor(t_2,t_1,t_3,t_4))$$

Therefore by Lemma 4.3,

$$jor_2(t_1,t_2,t_3,t_4)-jor_2(t_2,t_1,t_3,t_4)=-2\,jor_1(t_1,t_2,t_3,t_4).$$

Further, by Lemma 4.3,

$$\begin{aligned}jor_2(t_1,t_2,t_3,t_4)-jor_2(t_1,t_2,t_4,t_3)&=2\{mjor(t_3,t_1,t_2,t_4)-mjor(t_4,t_1,t_2,t_3)\}\\&=2\{mjor(t_3,t_4,t_1,t_2)-mjor(t_4,t_3,t_1,t_2)\}\\&=-2\,jor_1(t_3,t_4,t_1,t_2).\end{aligned}$$

Similarly, by Lemmas 4.1 and 4.3,

$$\begin{aligned}jor_2(t_1,t_2,t_3,t_4)+jor_2(t_2,t_1,t_3,t_4)&=2\{mjor(t_3,t_1,t_2,t_4)-mjor(t_4,t_1,t_2,t_3)\}\\&=2\{mjor(t_3,t_4,t_1,t_2)-mjor(t_4,t_3,t_1,t_2)\}\\&=-2\,jor_1(t_3,t_4,t_1,t_2),\end{aligned}$$

and

$$\begin{aligned}jor_2(t_1,t_2,t_3,t_4)+jor_2(t_1,t_2,t_4,t_3)&=2\{mjor(t_1,t_2,t_3,t_4)-mjor(t_2,t_1,t_3,t_4)\}\\&=-2\,jor_1(t_1,t_2,t_3,t_4).\end{aligned}$$

Since

$$jor_2(t_1,t_2,t_3,t_4)-jor_2(t_1,t_2,t_4,t_3)=-2\,jor_1(t_3,t_4,t_1,t_2)=jor_2(t_1,t_2,t_3,t_4)+jor_2(t_2,t_1,t_3,t_4),$$

we have

$$jor_2(t_1,t_2,t_4,t_3)+jor_2(t_2,t_1,t_3,t_4)=0.$$

□

Lemma 4.5.

$$jor_2(t_1,t_2,t_3,t_4)=-jor_1(t_1,t_2,t_3,t_4)-jor_1(t_3,t_4,t_1,t_2).$$

Proof. By Lemma 4.4

$$jor_2(t_1,t_2,t_3,t_4)-jor_2(t_2,t_1,t_3,t_4)=-2\,jor_1(t_1,t_2,t_3,t_4),$$

$$jor_2(t_1,t_2,t_3,t_4)-jor_2(t_1,t_2,t_4,t_3)=-2\,jor_1(t_3,t_4,t_1,t_2),$$

Add these two relations. By Lemma 4.4, relation (2), we receive that

$$jor_2(t_1,t_2,t_3,t_4)=-jor_1(t_1,t_2,t_3,t_4)-jor_1(t_3,t_4,t_1,t_2).$$

□

Lemma 4.6. *Let $p\neq 2$ and A be a commutative algebra. Then the following conditions are equivalent*

$$[l_a,l_b]\in DerA \tag{3}$$

$$a(b(cd))-(a(bc))d-c(a(bd))=b(a(cd))-(b(ac))d-c(b(ad)) \tag{4}$$

$$a(b(cc)) - 2(a(bc))c = b(a(cc)) - 2c(b(ac)) \tag{5}$$

$$(a, cc, b) = 2c(a, c, b) \tag{6}$$

$$(a, bc, d) = b(a, c, d) + c(a, b, d) \tag{7}$$

$$jor_2(a, b, c, d) = 0 \tag{8}$$

$$[[l_a, l_c], l_b] = l_{(a,b,c)} \tag{9}$$

$$mjor(a, b, c, d) = mjor(b, a, c, d) \tag{10}$$

Any of these identities implies the identity

$$2((ba)a)a + b((aa)a) = 3(b(aa))a \tag{11}$$

Proof. The Leibniz rule for a derivation $[l_a, l_b]$ can be written as

$$[l_a, l_b](cd) = ([l_a, l_b]c)d + c([l_a, l_b]d)$$

Therefore (3) is equivalent to (4). Substitution in (4) $c = d$ gives us (5). Conversely, polarization of (5) gives us (4). Rewrite (5) by the following way,

$$a(b(cc)) - b(a(cc)) = 2c\{a(bc) - b(ac)\}.$$

For commutative algebra this condition is equivalent to (6). The identity (7) is a polarization of (6),

$$\begin{aligned}(a, (b+c)^2, d) - 2(b+c)(a, b+c, d) - (a, b^2, d) + 2b(a, b, d) - (a, c^2, d) + 2c(a, c, d)\\ = (a, bc, d) + (a, cb, d) - 2b(a, c, d) - 2c(a, b, d)\\ = 2\{(a, bc, d) - b(a, c, d) - c(a, b, d)\}.\end{aligned}$$

So, (6) implies (7). If $p \neq 2$, let us substitute $b = c$ in (7). We obtain (6).

By Lemma 4.3, identities $jor_1 = 0$ and $jor_2 = 0$ are equivalent. Therefore, by Lemma 4.2 identities (7) and (8) are equivalent. Further

$$jor_2(a, b, c, c) = (a, cc, b) - 2c(a, c, b).$$

So, (6), (7), (8) are equivalent. We have

$$\begin{aligned}([[l_a, l_c], l_b] - l_{(a,b,c)})d &= a(c(bd)) - c(a(bd)) - b(a(cd)) + b(c(ad)) - (a, b, c)d\\ &= (a, bd, c) - b(a, d, c) - (a, b, c)d.\end{aligned}$$

Therefore (9) is equivalent to (7). By Lemma 4.3 conditions (4) and (10) are equivalent. Take in (8) $c = d = a$,

$$\begin{aligned}jor_2(a, b, a, a) &= (a, aa, b) - 2a(a, a, b) + (a, ab, a) - a(a, b, a) - b(a, a, a)\\ &= a((aa)b) - (a(aa))b - 2a(a(ab)) + 2a((aa)b)\\ &= 3a(b(aa)) - b(a(aa)) - 2a(a(ab)).\end{aligned}$$

So, (8) implies (11).

Remark. If $p > 3$ then (11) and any of identities (3), (4), (5), (6), (7), (8) and (9) are equivalent.

Remark. In Lemma 4.6 we assume that the commutativity identity $t_1t_2 = t_2t_1$ is given. If we omit the commutativity identity, then identities (3)–(10) are not equivalent. Namely, (3) and (6) are equivalent, (4), (5), (7), (8) are equivalent, (3) and (6) imply (4), (5), (7), (8) and (4), (5), (7), (8) do not imply (3) and (6). Identities of degree 4 were studied by Osborn in series of papers [11–13].

5. Proof of Theorem 1.5

Consider identities for the class $\mathcal{A}s^{+}$. It has identity of degree 2: commutativity identity. No identity of degree 3 that does not follow from commutativity rule. In degree 4, the following commutative non-associative polynomials give us identities that are not consequences of commutativity identity (see [7], Chapter 1.1, p. 5–6):

$$g^{(1)}_{[4]}(t_1) = (t_1, t_1, t_1^2),$$

$$g^{(1)}_{[3,1]}(t_1, t_2) = (t_1, t_2, t_1^2),$$

$$g^{(2)}_{[3,1]}(t_1, t_2) = t_2(t_1t_1^2) + 2t_1(t_1(t_1t_2)) - 3t_1(t_2t_1^2),$$

$$g^{(1)}_{[2,2]}(t_1, t_2) = t_1^2t_2^2 - t_1(t_1t_2^2) - 2t_2(t_1(t_1t_2)) + 2(t_1t_2)(t_1t_2),$$

$$g^{(1)}_{[2,1,1]}(t_1, t_2, t_3) = (t_1, t_1, t_2t_3) + (t_2, t_1, t_3t_1) + (t_3, t_1, t_1t_2),$$

$$g^{(2)}_{[2,1,1]}(t_1, t_2, t_3) = 2(t_1, t_2, t_1t_3) + (t_3, t_2, t_1^2),$$

$$g^{(1)}_{[1,1,1,1]}(t_1, t_2, t_3, t_4) = (t_2, t_1, t_2t_3) + (t_3, t_1, t_1t_4) + (t_4, t_1, t_2t_3).$$

All polynomials of a form $g^{(i)}_{\alpha}$ are homogeneous of type α. This means that lower index α corresponds to the type of identity, i.e., if $\alpha = [\alpha_1, \dots, \alpha_s, \dots]$, then t_s in each monomial of $g^{(i)}_{\alpha}$ enters α_s times. Such α_s is called multiplicity of t_s. Note that permutation of indices t_s with equal multiplicities in $g^{(i)}_{\alpha}$ induces a consequence of the identity $g^{(i)}_{\alpha}$ of type α. Therefore, any consequence of these identities in degree 4 can be presented as a linear combination of polynomials with given type where variables with equal multiplicity are permuted.

Any associative algebra is assosymmetric. Therefore, any identity of type α of degree 4 for plus-assosymmetric algebras is a consequence of identities $g^{(i)}_{\alpha}$. So, polynomials of the following form should be tested for an identity of plus-assosymmetric algebras

$$\alpha = [4], \qquad f_{[4]} = g^{(1)}_{[4]},$$

$$\alpha = [3,1], \qquad f^{\mu_1,\mu_2}_{[3,1]} = \mu_1 g^{(1)}_{[3,1]} + \mu_2 g^{(2)}_{[3,1]}, \qquad \mu_i \in K,$$

$$\alpha = [2,2], \qquad f^{\mu_1,\mu_2}_{[2,2]}(t_1, t_2) = \mu_1 g^{(1)}_{[2,2]}(t_1, t_2) + \mu_2 g^{(1)}_{[2,2]}(t_2, t_1), \qquad \mu_i \in K.$$

Since $g^{(1)}_{[2,2]}(t_1, t_2, t_3) = g^{(1)}_{[2,2]}(t_1, t_3, t_1)$, in case of $\alpha = [2,1,1]$, as a general form of a commutative polynomial tested for identity of plus-assosymmetric algebras, we can get

$$f^{\mu_1,\mu_2,\mu_3}_{[2,1,1]}(t_1, t_2, t_3) = \mu_1 g^{(1)}_{[2,1,1]}(t_1, t_2, t_3) + \mu_2 g^{(2)}_{[2,1,1]}(t_1, t_2, t_3) + \mu_3 g^{(2)}_{[2,1,1]}(t_1, t_3, t_2), \qquad \mu_i \in K.$$

Recall that $mjor(t_1, t_2, t_3, t_4)$ are symmetric by permutations of indices t_2, t_3, t_4. Therefore as a general form of a commutative polynomial tested for identity of plus-assosymmetric algebras in the case $\alpha = [1,1,1,1]$, we can take

$$\begin{aligned} f^{\mu_1,\mu_2,\mu_3,\mu_4}_{[1,1,1,1]}(t_1, t_2, t_3, t_4) &= \mu_1 g^{(1)}_{[1,1,1,1]}(t_1, t_2, t_3, t_4) \\ &\quad + \mu_2 g^{(1)}_{[1,1,1,1]}(t_2, t_1, t_3, t_4) + \mu_3 g^{(1)}_{[1,1,1,1]}(t_3, t_1, t_2, t_4) \\ &\quad + \mu_4 g^{(1)}_{[1,1,1,1]}(t_4, t_1, t_2, t_3), \qquad \mu_i \in K. \end{aligned}$$

By Theorem 1 of [5] free assosymmetric algebra in type α of degree 4 has the following base and dimensions

α	Base	Dim
$[4]$	$\{((aa)a)a, (aa)(aa), (a(aa))a\}$	3
$[3,1]$	$\{(aa)(ab), (b(aa))a, ((ba)a)a, ((ab)a)a, ((aa)b)a, (a(aa))b, ((aa)a)b\}$	7
$[2,2]$	$\{\{(aa)(bb), (b(ab))a, ((bb)a)a, ((ba)b)a, ((ab)b)a, (b(aa))b, ((ba)a)b, ((ab)a)b, ((aa)b)b\}$	9
$[2,1,1]$	$\{(aa)(bc), (c(ab))a, ((cb)a)a, ((bc)a)a, ((ca)b)a, ((ac)b)a, ((ba)c)a, ((ab)c)a, (c(aa))b, ((ca)a)b, ((ac)a)b, ((aa)c)b, (b(aa))c, ((ba)a)c, ((ab)a)c, ((aa)b)c\}$	16
$[1,1,1,1]$	$\{(ab)(cd), (d(bc))a, ((dc)b)a, ((cd)b)a, ((db)c)a, ((bd)c)a, ((cb)d)a, ((bc)d)a, ((da)c)b, ((dc)a)b, ((cd)a)b, ((da)c)b, ((ad)c)b, ((ca)d)b, ((ac)d)b, (d(ab))c, ((db)a)c, ((bd)a)c, ((da)b)c, ((ad)b)c, ((ba)d)c, ((ab)d)c, (c(ab))d, ((cb)a)d, ((bc)a)d, ((ca)b)d, ((ac)b)d, ((ba)c)d, ((ab)c)d\}$	29

Let us substitute in polynomials f_α instead of parameters t_i elements of free assosymmetric algebras and calculate its value in terms of assosymmetric multiplication.

We have

$$\begin{aligned} f_{[4]}(a) &= <a,a,\{a,a\}>= \{a,\{a,\{a,a\}\}\} - \{\{a,a\},\{a,a\}\} \\ &= 2(a(a(aa)) + 2a((aa)a) + 2(a(aa))a + 2((aa)a)a - 8(aa)(aa) \\ &= 2(a,a,aa) - 2(aa,a,a) + 2a((aa)a) + 2(a(aa))a - 4(aa)(aa) \\ &= 2(a,aa,a) + 4(a(aa))a - 4(aa)(aa) \\ &= 2(aa,a,a) + 4(a(aa))a - 4(aa)(aa) \\ &= 2(aa)(aa) - 2((aa)a)a + 4(a(aa))a - 4(aa)(aa) \\ &= -2((aa)a)a + 4(a(aa))a - 2(aa)(aa). \end{aligned}$$

Since elements $((aa)a)a$, $(a(aa))a$ and $(aa)(aa)$ are base elements, this means that $f_{[4]}(a) \neq 0$. So, plus-assosymmetric algebras have no identity of type [4].

Similar calculations show that $f_{[3,1]} = 0$ is identity if $\mu_1 = 0$. So, in type [3,1] plus-assosymmetric algebras have an identity $g^{(2)}_{[3,1]} = 0$.

Consider type [2,2] case. Calculations show that

$$f^{\mu_1,\mu_2}_{[2,2]}(t_1,t_2) = 6(\mu_1+\mu_2)\{(aa)(b,b) - 2(b(ab))a - ((aa)b)b + 2((ba)b)a\}.$$

So, $f^{\mu_1,\mu_2}_{[2,2]}$ is identity for plus-assosymmetric algebras if $\mu_1+\mu_2 = 0, p \neq 2,3$, and, $h_{[2,2]}(t_1,t_2) = 0$ is identity for plus-assosymmetric algebras, where $h_{[2,2]} = f^{-1,1}_{[2,2]}$. Note that

$$h_{[2,2]}(t_1,t_2) = t_1(t_1t_2^2) - t_2(t_2t_1^2) - 2t_1(t_2(t_1t_2)) + 2t_2(t_1(t_1t_2))$$

is identity for plus-assosymmetric algebras.

In the case of type [2,1,1], we have

$$f^{\mu_1,\mu_2,\mu_3}_{[2,1,1]}(a,b,c) = -6(\mu_1+\mu_2+\mu_3)\{(aa)(bc) - 2(c(ab))a - ((aa)b)c + 2((ca)b)a\}.$$

Therefore, $h^{(1)}_{[2,1,1]} = 0$ and $h^{(2)}_{[2,1,1]} = 0$ are identities for plus-assosymmetric algebras, where

$$h^{(1)}_{[2,1,1]} = f^{1,-1,0}_{[2,1,1]}, \qquad h^{(2)}_{[2,1,1]} = f^{0,1,-1}_{[2,1,1]},$$

if $p \neq 2,3$. Note that

$$h^{(1)}_{[2,1,1]}(t_1,t_2,t_3) = (t_1,t_1,t_2t_3) + (t_2,t_1,t_3t_1) + (t_3,t_1,t_1t_2) - 2(t_1,t_2,t_1t_3) - (t_3,t_2,t_1^2),$$

$$h^{(2)}_{[2,1,1]}(t_1,t_2,t_3) = 2(t_1,t_2,t_1t_3) - 2(t_1,t_3,t_1t_2) + (t_3,t_2,t_1^2) - (t_2,t_3,t_1^2).$$

Now consider the case [1,1,1,1]. By Lemma 2.7 the polynomial *mjor* can not give an identity on $\mathcal{A}ssym^+$, if $p \neq 2,3$. For any assosymmetric algebra A and for any its four elements $a,b,c,d \in A$ by Lemma 2.8,

$$f^{\mu_1,\mu_2,\mu_3,\mu_4}_{[1,1,1,1]}(a,b,c,d) = (\mu_1+\mu_2+\mu_3+\mu_4)mjor(a,b,c,d).$$

Thus, $f^{\mu_1,\mu_2,\mu_3,\mu_4}_{[1,1,1,1]} = 0$ is identity on $\mathcal{A}ssym^+$ if and only if

$$\mu_1+\mu_2+\mu_3+\mu_4 = 0$$

Therefore,

$$\mu_4 = -\mu_1-\mu_2-\mu_3,$$

and

$$\begin{aligned} f^{\mu_1,\mu_2,\mu_3,\mu_4}_{[1,1,1,1]}(t_1,t_2,t_3,t_4) &= \mu_1(mjor(t_1,t_2,t_3,t_4) - mjor(t_4,t_1,t_2,t_3)) \\ &+ \mu_2(mjor(t_2,t_1,t_3,t_4) - mjor(t_4,t_1,t_2,t_3)) \\ &+ \mu_3(mjor(t_3,t_1,t_2,t_4) - mjor(t_4,t_1,t_2,t_3)). \end{aligned}$$

In other words, by Lemma 4.3,

$$-f^{\mu_1,\mu_2,\mu_3,\mu_4}_{[1,1,1,1]}(t_1,t_2,t_3,t_4) = \mu_1\, jor_1(t_1,t_4,t_2,t_3) + \mu_2\, jor_1(t_2,t_4,t_1,t_3) + \mu_3\, jor_1(t_3,t_4,t_1,t_2).$$

By Lemma 2.8 $jor_1 = 0$ is identity on $\mathcal{A}ssym^+$. So, $jor_1 = 0$ is identity for $\mathcal{A}ssym^+$.

It remains to prove that all identities appeared for types [2,2] and [2,1,1] are consequences of the identity $jor_1 = 0$ if $p \neq 2,3$.

It is easy too see that

$$\begin{aligned} h_{[2,2]}(t_1,t_2) &= (t_1,t_1,t_2^2) - (t_2,t_2,t_1^2) - 2(t_1,t_2,t_1t_2) + 2(t_2,t_1,t_1t_2) \\ &= t_1(t_1t_2^2) - t_2(t_2t_1^2) - 2\{t_1(t_2(t_1t_2)) - t_2(t_1(t_1t_2))\} \\ &= -t_1(t_2,t_2,t_1) + t_2(t_1,t_1,t_2) - t_1(t_2(t_1t_2)) + t_2(t_1(t_1t_2)). \end{aligned}$$

Therefore by Lemma 4.1,

$$\begin{aligned} h_{[2,2]}(t_1,t_2) &= t_1(t_1,t_2,t_2) + t_2(t_1,t_1,t_2) - t_1((t_1t_2)t_2) + (t_1(t_1t_2))t_2 \\ &= t_1(t_1,t_2,t_2) + t_2(t_1,t_1,t_2) - (t_1,t_1t_2,t_2). \end{aligned}$$

Hence the identity $h_{[2,2]} = 0$ is a consequence of the identity

$$(t_1,t_2t_3,t_4) - t_1(t_1,t_3,t_4) - (t_1,t_2,t_4)t_3 = 0.$$

By Lemma 4.6, (7), this means that the identity $h_{[2,2]} = 0$ is a consequence of identity $jor_1 = 0$. By Lemma 4.3

$$h^{(1)}_{[2,1,1]} = mjor(t_1,t_2,t_3,t_1) - mjor(t_2,t_1,t_3,t_1) = -jor_1(t_1,t_2,t_3,t_1),$$

$$h^{(2)}_{[2,1,1]} = mjor(t_2,t_3,t_1,t_1) - mjor(t_3,t_2,t_1,t_1) = -jor_1(t_2,t_3,t_1,t_1).$$

So, $jor_1 = 0$ is a minimal identity for $\mathcal{A}ssym^+$ that does not follow from commutativity identity if $p = char\, K \neq 2,3$.

Now consider the case $p = 3$. By Lemma 2.7 $mjor = 0$ is an identity for plus-assosymmetric algebras. As we have checked above $g^{(1)}_{[2,2]} = 0$ and $g^{(1)}_{[2,1,1]} = g^{(2)}_{[2,1,1]} = 0$ are identities for $p = 3$. We have

$$g^{(1)}_{[2,2]}(t_1, t_2) = -(t_1, t_1, t_2^2) - 2(t_2, t_1, t_1 t_2) = -mjor(t_1, t_1, t_2, t_2),$$

$$g^{(1)}_{[2,1,1]}(t_1, t_2, t_3) = mjor(t_1, t_1, t_2, t_3),$$

$$g^{(2)}_{[2,1,1]}(t_1, t_2, t_3) = 2(t_1, t_2, t_1 t_3) + (t_3, t_2, t_1^2) = mjor(t_2, t_3, t_1, t_1).$$

So, $mjor = 0$ is a minimal identity for $\mathcal{A}ssym^{+}$, $p = 3$.

Since by Lemma 4.6 Lie triple identity and the identity $jor_1 = 0$ are equivalent, Theorem 1.5 is proved completely.

6. Additional remarks

Remark 1. concerns Theorem 1.5 in case $p = 2$. If $p = 2$, then $\mathcal{A}ssym^{(+)} = \mathcal{A}ssym^{(-)}$. Therefore, plus-assosymmetric algebras are Lie, and all identities for $\mathcal{A}ssym^{(+)}$ follow from commutativity and Jacobi identities. In particular, Jordan identity and Glennie identity are consequences of commutativity one and Jacobi identity.

Remark 2. Recall that an Jordan algebra is called *special* Jordan, if it is isomorphic to a subalgebra of algebra $A^{(+)}$ for some associative algebra A. Well known that Jordan algebra of 3×3 hermitian matrices over octonians M_3^8 is not special, and Glennie identity is an example of special Jordan identity. Say that a Lie triple algebra is *special* if it is isomorphic to a subalgebra of algebra $A^{(+)}$ for some Lie triple algebra A. Since for commutative algebras, Jordan identity implies Lie triple identity, by Theorem 1.4 M_3^8 as Lie triple algebra is exceptional and Glennie identity of degree 8 is special Lie triple identity.

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