Journal of Lie Theory
Volume **23** (2013) 795–801

The Dynkin Theorem for Multilinear Lie Elements

Askar Dzhumadil'daev

Communicated by E. B. Vinberg

Abstract. We establish a generalization of the Dynkin theorem for multilinear elements. It allows us to construct the presentation of a multilinear Lie element as a linear combination of base Lie elements.
Mathematics Subject Classification 2010: 17B01, 05E99, 05A30.
Key Words and Phrases: Dynkin idempotent, Lie elements, Free Lie algebra.

1. Introduction

For elements of an associative algebra $x_1, \dots, x_n \in A$, denote by $[x_1, \dots, x_n]$ the right-bracketed commutator $[x_1, [x_2, \cdots [x_{n-1}, x_n] \cdots]]$. For example, $[x_1, x_2, x_3] = x_1x_2x_3 - x_1x_3x_2 - x_2x_3x_1 + x_3x_2x_1$.

Let F_n be the multilinear part of the free associative algebra freely generated by n elements $a_1, \dots, a_n$. Recall that a base element of the form $a_{i_1} \cdots a_{i_n}$ is called *multilinear* if each element a_i, $1 \le i \le n$, enters exactly one time. A linear combination of multilinear base elements is called multilinear. Then F_n has a base constituted by the elements $a_{\sigma(1)} \cdots a_{\sigma(n)}$, where $\sigma \in S_n$ are permutations. In particular, $\dim F_n = n!$. Let L_n be the multilinear part of the free Lie algebra generated by elements $a_1, \dots, a_n$. Then L_n has a base constituted by elements $[a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_l]$, where $\sigma \in S_n$ is such that $\sigma(n) = l$. In particular, $\dim L_n = (n-1)!$. Here we suppose that the base field K has characteristic 0. These facts are known. See details, for example, in [7].

Let $F_n^{(l)}$ be the subspace of F_n generated by elements $a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_l$, where $\sigma \in S_n$ runs through all permutations such that $\sigma(n) = l$. In particular, $\dim F_n^{(l)} = (n-1)! = \dim L_n$. Let

$$q_l : F_n \to F_n^{(l)}$$

be the natural projection of linear spaces,

$$q_l\Big(\sum_{\sigma \in S_n} \lambda_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_{\sigma(n)}\Big) = \sum_{\sigma \in S_n, \sigma(n) = l} \lambda_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_l.$$

ISSN 0949–5932 / \$2.50

Let

$$p_l = [q_l] : F_n \to L_n$$

be the linear map defined by

$$p_l(\sum_{\sigma \in S_n} \lambda_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_{\sigma(n)}) = \sum_{\sigma \in S_n, \sigma(n)=l} \lambda_\sigma [a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_l].$$

Then

$$p = p_1 + \cdots + p_n : F_n \to L_n$$

is the Dynkin map,

$$p(\sum_{\sigma \in S_n} \lambda_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_{\sigma(n)}) = \sum_{\sigma \in S_n} \lambda_\sigma [a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_{\sigma(n)}].$$

An element $X \in F_n$ is called a *Lie element* if $X \in L_n$. For example, the element $X = a_1a_2a_3 - a_1a_3a_2$ is not Lie and $Y = a_1a_2a_3 - a_1a_3a_2 - a_2a_3a_1 + a_3a_2a_1$ is Lie, $Y = [a_1, [a_2, a_3]]$. The Dynkin-Specht-Wever theorem ([1], [2], [8], [9]) states that for a homogeneous element X of degree n, X is a Lie element if and only if $pX = nX$.

The aim of our paper is to establish the following more exact version of the Dynkin theorem for multilinear elements.

Theorem 1.1. *Let $X \in F_n$ is a multilinear element of degree n. Then the following conditions are equivalent:*

(i) *X is a Lie element*

(ii) *for any $1 \le l \le n$, $p_lX = X$*

(iii) $\sum_{l=1}^{n} \lambda_l p_l(X) = \sum_{l=1}^{n} \lambda_l X$, *for any $\lambda_l \in K$, $1 \le l \le n$*

(iv) $p_1X = p_2X = \cdots = p_nX$.

For a Lie element X let us call its presentation as a linear combination of Lie base elements a *Lie expression* of X. Theorem 1.1 allows us to construct Lie expressions for known Lie elements.

Corollary 1.2. (Dynkin) *If X is a multilinear Lie element of degree n, then X is Lie if and only if $pX = nX$.*

Proof. It is clear that for any X the element pX is a Lie element. By this reason, if $pX = nX$ and characteristic of the base field is 0, then $X = 1/n\, pX$ is a Lie element.

Conversely, suppose that X is A Lie element. Since $p = \sum_{l=1}^{n} p_l$, by Theorem 1.1,

$$pX = \sum_{l=1}^{n} p_l(X) = nX. \qquad \blacksquare$$

2. Proof of Theorem 1.1

Let

$$S_{n,l} = \{\sigma \in S_n | \sigma(1) < \cdots < \sigma(l) \mid \sigma(l+1) < \cdots < \sigma(n)\}$$

and

$$S_n^{(l)} = \{\sigma \in S_n | \sigma(n) = l\}.$$

Lemma 2.1. *For any $x_1, \ldots, x_n \in A$,*

$$[x_1, \ldots, x_n] = \sum_{r=0}^{n-1} \sum_{\sigma \in S_{n-1,r}} (-1)^r x_{\sigma(1)} \cdots x_{\sigma(r)} x_n x_{\sigma(n-1)} \cdots x_{\sigma(r+1)}.$$

Proof. Follows from Theorem 8.16 [7]. ■

Lemma 2.2. *For any $1 \le l \le n$, $q_l p_l = q_l$.*

Proof. We have to prove that for any $X \in F_n$, $q_l p_l X = q_l X$. Let us present $q_l X$ as a linear combination of base elements

$$q_l X = \sum_{\sigma \in S_n^{(l)}} \mu_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_l.$$

Then

$$p_l X = [q_l X] = \sum_{\sigma \in S_n^{(l)}} \mu_\sigma [a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_l].$$

Therefore by Lemma 2.1,

$$q_l p_l X = \sum_{\sigma \in S_n^{(l)}} \mu_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_l = q_l X.$$ ■

Lemma 2.3. *Let $X \in F_n$. If $p_l X = 0$ for any $1 \le l \le n$, then $X = 0$.*

Proof. By Lemma 2.2,

$$p_l X = 0 \Rightarrow q_l p_l X = 0 \Rightarrow q_l X = 0 \Rightarrow X = \sum_{l=1}^{n} q_l X = 0.$$ ■

Proof of Theorem 1.1.

(i) $\iff$ (ii): Suppose that $X \in F_n$ is a Lie element and $1 \le l \le n$. Take base of the Lie multilinear part L_n constituted by elements $[a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_l]$, where $\sigma \in S_n^{(l)}$. Then

$$X = \sum_{\sigma \in S_n^{(l)}} \mu_\sigma [a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}, a_l],$$

for some $\mu_\sigma \in K, \sigma \in S_n^{(l)}$. Hence by Lemma 2.1,

$$q_l X = \sum_{\sigma \in S_n^{(l)}} \mu_\sigma a_{\sigma(1)} \cdots a_{\sigma(n-1)} a_l.$$

Therefore,

$$p_l X = [q_l X] = \sum_{\sigma \in S_n^{(l)}} \mu_\sigma [a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}, a_l] = X.$$

Suppose now that $p_l X = X$ for some $1 \leq l \leq n$. Since $p_l X$ is a sum of comutators, X is a Lie element.

(ii) $\iff$ (iii): If $p_l X = X$ for any $1 \leq l \leq n$, then

$$\sum_{l=1}^{n} \lambda_l p_l(X) = \sum_{l=1}^{n} \lambda_l X,$$

for any $\lambda_l \in K,\ 1 \leq l \leq n$.

Suppose conversely, that

$$\sum_{l=1}^{n} \lambda_l p_l(X) = \sum_{l=1}^{n} \lambda_l X,$$

for any $\lambda_l \in K,\ 1 \leq l \leq n$. For given $1 \leq l \leq n$ take $\lambda_l = 1$ and $\lambda_s = 0, s \neq l$. We obtain the condition (i).

So, we have proved that conditions (i), (ii), (iii) are equivalent.

(i), (ii), (iii) $\iff$ (iv)

Suppose that

$$\sum_{l=1}^{n} \lambda_l p_l(X) = \sum_{l=1}^{n} \lambda_l X,$$

for any $\lambda_l \in K,\ 1 \leq l \leq n$. Take $\lambda_l = 1, \lambda_{l+1} = -1$, and $\lambda_s = 0,\ s \neq l, l+1$, for $1 \leq l < n$. Then by (iii)

$$p_l X - p_{l+1} X = 0$$

for any $1 \leq l < n$. This is the condition (iv).

Now suppose that we have the condition (iv). Then

$$p_1 X = \cdots = p_n X = Y,$$

for some $Y \in F_n$. Moreover, since $p_l X$ is a sum of commutators, Y is a Lie element. Then (since conditions (i) and (ii) are equivalent),

$$p_l Y = Y, \qquad 1 \leq l \leq n.$$

Let $Z = X - Y$. Then

$$p_l Z = p_l X - p_l Y = Y - Y = 0,$$

for any $1 \leq l \leq n$. By Lemma 2.3, $Z = 0$. So, $X = Y$, and

$$p_1 X = \cdots = p_n X = X.$$

We obtain the condition (ii).

Theorem 1.1 is proved completely.

3. Applications of Theorem 1.1

For a Lie element X let us call its presentation as a linear combination of Lie base elements a *Lie expression* of X. In this section we construct Lie expressions for known Lie elements. Since these elements are multilinear, all constructions follow from Theorem 1.1.

For a word $u = i_1 \dots i_k$ with components in $[n] = \{1, 2, \dots, n\}$, let us say that $k = |u|$ is its *length*, and that $s \in [n-1]$ is *a descent index* if $i_s > i_{s+1}$. Denote by $Des(u)$ the set of descent indices of u. The sum of all descent indices is called the *major index* of u and is denoted as $maj(u)$,

$$maj(u) = \sum_{j \in Des(u)} j.$$

Define the *multi-parametric* $\mathbf{q}$*-major index* $maj_{\mathbf{q}}(u)$ of a word u by

$$maj_{\mathbf{q}}(u) = \frac{\prod_{j \in Des(u)} q_{u(1)} \cdots q_{u(j)}}{\prod_{i=1}^{|u|-1} (1 - q_{u(1)} \cdots q_{u(i)})},$$

where $\mathbf{q} = (q_1, \dots, q_n)$ are some variables.

For a primitive n-th root of unity $q \in K$, A.A. Klyachko has constructed in [4] the following Lie element:

$$k_n = \frac{1}{n} \sum_{\sigma \in S_n} q^{maj(\sigma)} a_{\sigma(1)} \cdots a_{\sigma(n)}.$$

The Klyachko element has the following multi-parameter generalisation [6]. Let

$$k_n(\mathbf{q}) = \sum_{\sigma \in S_n} maj_{\mathbf{q}}(\sigma) a_{\sigma(1)} \dots a_{\sigma(n)}.$$

Then $k_n(\mathbf{q})$ is a Lie element if $q_1 q_2 \cdots q_n = 1$, but $q_{i_1} q_{i_2} \cdots q_{i_r} \neq 1$ for any proper subset $\{i_1, \dots, i_r\} \subset [n]$.

The Klyachko element has one more generalization. In [5] a Lie idempotent was constructed that generalises three other well-known idempotents. This generalization concerns the so called q-Solomon Lie element,

$$\phi_n(q) = \frac{1}{n} \sum_{\sigma \in S_n} \frac{(-1)^{des(\sigma)} q^{maj(\sigma) - \binom{d(\sigma)+1}{2}}}{\left[\begin{array}{c} n-1 \\ d(\sigma) \end{array} \right]_q} a_{\sigma(1)} \cdots a_{\sigma(n)}.$$

Here $\left[\begin{array}{c} n-1 \\ p \end{array} \right]_q$ denotes the q-binomial coefficient. Recall that

$$[n]_q! = 1 + q + \cdots + q^{n-1}$$

and

$$\left[\begin{array}{c} n \\ p \end{array} \right]_q = \frac{[n]_q!}{[p]_q! [n-p]_q!}.$$

q-Solomon elements have the following properties:

$$\phi_n(\omega) = k_n(\omega)$$

is the Klyachko element if ω is a primitive root of degree n,

$$\phi_n(0) = [\cdots[a_1, a_2], \cdots, a_n]$$

is the Dynkin Lie element in the case $q = 0$, and

$$\phi_n(1) = \sum_{\sigma\in S_n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} a_{\sigma(1)} \cdots a_{\sigma(n)}$$

gives us the (first) Euler element if $q = 1$.

Corollary 3.1. *A Lie expression for the multi-parameter Klyachko element is*

$$k_n(\mathbf{q}) = \sum_{\sigma\in S_n, \sigma(n)=n} maj_{\mathbf{q}}(\sigma)[a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}, a_n].$$

Corollary 3.2. *A Lie expression for the Klyachko element is*

$$k_n = \frac{1}{n} \sum_{\sigma\in S_n, \sigma(n)=n} q^{maj(\sigma)}[a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}, a_n].$$

These facts was established in [3].

Corollary 3.3. *A Lie expression for the q-Solomon element is given by*

$$\phi_n(q) = \frac{1}{n} \sum_{\sigma\in S_n, \sigma(n)=n} \frac{(-1)^{des(\sigma)} q^{maj(\sigma) - \binom{d(\sigma)+1}{2}}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} [a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_n].$$

Corollary 3.4. *A Lie expression for the Dynkin element is*

$$[[\cdots[a_1, a_2], \cdots], a_n] = \sum_{\sigma\in M_n} (-1)^{des(\sigma)}[a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_n].$$

Here summation is taken over the set of permutations

$$M_n = \{\sigma \in S_n | maj(\sigma) = \binom{des(\sigma)+1}{2}, \sigma(n) = n\}.$$

Let

$$M'_n = \{\sigma \in S_{n-1} | maj(\sigma) = \binom{des(\sigma)+1}{2}\}.$$

Then

$$M_n = \{\sigma\, n | \sigma \in M'_n\}.$$

Note that the set M'_n can be easily construced by induction. Set $M'_2 = \{1\}$. Then M'_{n+1}, for $n > 1$, consists of permutations of the forms $\sigma\, n$ (append n at the end) and $n\, \sigma$ (prepend n at the beginning), where $\sigma \in M'_n$. Note also that for any $\sigma \in M'_n$,

$$des(n\, \sigma) = des(\sigma) + 1, \quad des(\sigma\, n) = des(\sigma).$$

Corollary 3.5. *The Euler element has the following Lie expression*

$$\phi_n(1) = \sum_{\sigma \in S_n, \sigma(n)=n} \frac{(-1)^{des(\sigma)}}{\binom{n-1}{des(\sigma)}} [a_{\sigma(1)}, \cdots, a_{\sigma(n-1)}, a_n].$$

References

[1] Dynkin, E. B., *Computation of the coefficients in the Campbell-Hausdroff formula,* Doklady Akad. Nauk SSSR **57** (1947), 323–326.

[2] —, *On the representation of the series* $\log(e^x e^y)$ *in non-commutating* x *and* y *via the commutators,* Mat. Sbornik **25** (1949), 155–162.

[3] Dzhumadil'daev, A. S., *Lie expression for multi-parameter Klyachko idempotent,* J. Alg. Comb. **33** (2011), 531–542.

[4] Klyachko, A. A., *Lie elements in a tensor algebra,* Sibirsk. Mat. Zh. **15** (1974), 1296–1304.

[5] Krob, D., B. Leclerc, and J.-Y. Thibon, *Noncommutative symmetric functions II: Transformations of alphabets,* Int. J. Algebra and Computation **7** (1997), 181–264.

[6] McNamara, P., and C. Reutenauer, *P-partitions and multi-parameter Klyachko idempotent*, Elect. J. Comb. **11(2)** (2005), #R21.

[7] Reutenauer, C., "Free Lie algebras," Clarendon Press, Oxford, 1993.

[8] W. Specht, *Die linearen Beziehungen zwischen höheren Kommutatoren,* Math. Z. **51** (1948), 367–376.

[9] F. Wever, *Über Invarianten in Lieschen Ringen,* Math. Annalen, **120** (1949), 563–580.

Askar Dzhumadil'daev
Kazakh-British Technical University
Tole bi 59
Almaty,050000, Kazakhstan
dzhuma@hotmail.com

Received November 9, 2012
and in final form December 19, 2012