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On a class of evolution algebras of "chicken" population

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This paper is devoted to the description of structure of evolution algebras of "chicken" population (EACP). Such an algebra is determined by a rectangular matrix of structural constants. Using the Jordan form of the matrix of structural constants we obtain a simple description of complex EACP. We give the classification of three-dimensional complex EACP. Moreover, some $(n+1)$-dimensional EACP are described.

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1. Introduction

An algebraic approach in Genetics consists of the study of various types of genetic algebras (like algebras of free, "self-reproduction" and bisexual populations, Bernstein algebras). The formal language of abstract algebra to study of genetics was introduced in [2–4]. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. The most comprehensive references for the mathematical research done in this area are [1, 6–9].

Let A be an arbitrary algebra and let $\{e_1, e_2, \dots\}$ be a basis of the algebra A. The table of multiplication on A is defined by the products of the basic elements, namely,

$$e_ie_j = \sum_k \gamma_{ij}^k e_k,$$

where γ_{ij}^k are the structural constants. The study of such a general algebra A is difficult, since it is determined by the cubic matrix $\{\gamma_{ij}^k\}$ of structural constants. In some evolution algebras the cubic matrix is reduced to quadratic (see [8]) or rectangular (see [5]) matrices. This simplicity of the matrix allows to obtain deeper results on the algebra.

In this paper, following [5] we consider a set $\{h_i, i = 1, \ldots, n\}$ (the set of "hen"s) and r (a "rooster").

Let $(\mathcal{C}, \cdot)$ be an algebra over a field K (with characteristic $\neq 2$). If it admits a basis $\{h_1, \ldots, h_n, r\}$ such that

$$h_i r = r h_i = \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} h_j + b_i r\right),$$
$$h_i h_j = 0, \quad i, j = 1, \ldots, n; \;\; rr = 0, \tag{1.1}$$

then this algebra is called an *evolution algebra of a "chicken" population* (EACP). The basis $\{h_1, \ldots, h_n, r\}$ is called a natural basis.

We note that an algebra $\mathcal{C}$ is defined by a rectangular $n \times (n+1)$-matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{pmatrix},$$

which is called the matrix of structural constants of the algebra $\mathcal{C}$.

For a given element x of an algebra C, define a right multiplication operator $R_x \colon C \to C$, as $R_x(y) = yx, y \in C$. It should be noted that the matrix M of EACP $\mathcal{C}$ coincides with the matrix of right multiplication operator R_r. Therefore, the multiplication of EACP $\mathcal{C}$ may be given by the operator R_r.

In [5] it is proved that the algebra $\mathcal{C}$ is commutative (and hence flexible), not associative and not necessarily power associative, in general. Moreover it is not unital. A condition is found on the structural constants of the algebra under which the algebra is associative, alternative, power associative, nilpotent, satisfies Jacobi and Jordan identities. The set of all operators of left (right) multiplications is described. Under some conditions on the structural constants it is proved that the corresponding EACP is centroidal. Moreover the classification of two-dimensional and some three-dimensional EACP are obtained.

In this paper, we continue the study of EACP. In Sec. 2 using the Jordan form of the matrix of structural constants we give a simple representation of EACP. Section 3 is devoted to the classification of three-dimensional complex EACP. In Sec. 4 we describe some $(n+1)$-dimensional EACP.

2. The Structure of EACP

Recall some notations.

Definition 1. An element x of an algebra $\mathcal{A}$ is called nil if there exists $n(a) \in \mathbb{N}$ such that $(\cdots \underbrace{((x \cdot x) \cdot x) \cdots x}_{n(a)}) = 0$. The algebra $\mathcal{A}$ is called nil if every element of the algebra is nil.

For $k \geq 1$, we introduce the following sequences:

$$\mathcal{A}^{(1)} = \mathcal{A}, \quad \mathcal{A}^{(k+1)} = \mathcal{A}^{(k)}\mathcal{A}^{(k)};$$
$$\mathcal{A}^{\langle 1 \rangle} = \mathcal{A}, \quad \mathcal{A}^{\langle k+1 \rangle} = \mathcal{A}^{\langle k \rangle}\mathcal{A};$$
$$\mathcal{A}^{1} = \mathcal{A}, \quad \mathcal{A}^{k} = \sum_{i=1}^{k-1} \mathcal{A}^{i}\mathcal{A}^{k-i}.$$

Definition 2. An algebra $\mathcal{A}$ is called

(i) solvable if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{(n)} = 0$ and the minimal such number is called index of solvability;
(ii) right nilpotent if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{\langle n \rangle} = 0$ and the minimal such number is called index of right nilpotency;
(iii) nilpotent if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{n} = 0$ and the minimal such number is called index of nilpotency.

We note that for an EACP notion as nil, nilpotent and right nilpotent algebras are equivalent. However, the indexes of nility, right nilpotency and nilpotency do not coincide in general.

In this section we consider EACP over the field of complex numbers.

Let $\mathcal{C}$ be an $(n+1)$-dimensional complex EACP and $\{h_1, h_2, \ldots, h_n, r\}$ be a basis of this algebra. Then the table of multiplications of EACP have the following form

$$h_i r = r h_i = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j + b_i r\right), \quad h_i h_j = rr = 0.$$

Let $\mathcal{C}$ and $\mathcal{D}$ be EACPs; we say that a linear homomorphism f from $\mathcal{C}$ to $\mathcal{D}$ is an evolution *homomorphism* if f is an algebraic map and for a natural basis $\{h_1, \ldots, h_n, r\}$ of $\mathcal{C}$, $\{f(r), f(h_i), i = 1, \ldots, n\}$ spans an evolution subalgebra in $\mathcal{D}$. Furthermore, if an evolution homomorphism is one-to-one and onto, it is an evolution *isomorphism*.

Theorem 1. *If $b_i = 0$ for any $i = 1, \ldots, n$, then $\mathcal{C}$ is a solvable EACP and it is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$h_{\sum_{i=1}^{s-1} n_i + 1} r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i + 1} + h_{\sum_{i=1}^{s-1} n_i + 2})$$

$$h_{\sum_{i=1}^{s-1} n_i+2}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+2} + h_{\sum_{i=1}^{s-1} n_i+3})$$
$$\vdots$$
$$h_{\sum_{i=1}^{s-1} n_i+n_s-1}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+n_s-1} + h_{\sum_{i=1}^{s-1} n_i+n_s})$$
$$h_{\sum_{i=1}^{s-1} n_i+n_s}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+n_s}),$$

where $n = n_1 + \cdots + n_k$, $s = 1, \ldots, k$ *and* $(a_1, a_2, \ldots, a_k) = (1, a_2, \ldots, a_k)$ *or* $(a_1, a_2, \ldots, a_k) = (0, 0, \ldots, 0)$.

Proof. If $b_i = 0$ for any $i = 1, \ldots, n$, then we have the multiplication

$$h_i r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j\right), \quad h_i h_j = rr = 0.$$

It follows that $\mathcal{C}^{(2)} \subseteq \langle h_1, h_2, \ldots, h_n\rangle$, which implies $\mathcal{C}^{(3)} = 0$. Hence, $\mathcal{C}$ is solvable.

Since $\mathcal{C}$ is a complex EACP, then by theorem on Jordan decomposition we conclude that there exists a basis of $\mathcal{C}$ such that the matrix of the operator R_r has Jordan form (that is, diagonal block consist of Jordan blocks). We define $R_r = J_1 \oplus J_2 \oplus \cdots \oplus J_k$, where a_i are diagonal elements of J_i and n_i are sizes of blocks $J_i, 1 \leq i \leq k$. Obviously, $n_1 + n_2 + \cdots + n_k = n$.

Then the table of multiplication of the algebra is follows:

$$h_{\sum_{i=1}^{s-1} n_i+1}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+1} + h_{\sum_{i=1}^{s-1} n_i+2})$$
$$h_{\sum_{i=1}^{s-1} n_i+2}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+2} + h_{\sum_{i=1}^{s-1} n_i+3})$$
$$\vdots$$
$$h_{\sum_{i=1}^{s-1} n_i+n_s-1}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+n_s-1} + h_{\sum_{i=1}^{s-1} n_i+n_s})$$
$$h_{\sum_{i=1}^{s-1} n_i+n_s}r = \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+n_s}),$$

where $s = 1, \ldots, k$.

If $a_{s_0} \neq 0$, for some $1 \leq s_0 \leq k$, then without loss of generality we can assume that $a_1 \neq 0$.

In this case putting: $a'_s = \frac{a_s}{a_1}$, $2 \leq s \leq k$ and

$$h'_{\sum_{i=1}^{s-1} n_i+j} = \frac{h_{\sum_{i=1}^{s-1} n_i+j}}{a_1^j}, \quad 1 \leq j \leq n_s, \ 1 \leq s \leq k,$$

one can assume that $a_1 = 1$. □

Remark 1. From the above description it is easy to see that for any $s, 1 \leq s \leq k$

$$\langle h_{\sum_{i=1}^{s-1} n_i+1}, h_{\sum_{i=1}^{s-1} n_i+2}, \ldots, h_{\sum_{i=1}^{s-1} n_i+n_s-1}, h_{\sum_{i=1}^{s-1} n_i+n_s} \rangle$$

is an ideal of the EACP.

Moreover, if $(a_1, a_2, \ldots, a_k) = (0, 0, \ldots, 0)$, then EACP is nilpotent.

Suppose that there exists $b_{i_0} \neq 0, i_0 \in \{1, \ldots, n\}$, then by shifting of the basis elements $h_i, 1 \leq i \leq n$ we can assume that $b_1 \neq 0$. Moreover, by the scaling $h_1' = \frac{1}{b_1} h_1$ we can assume that $b_1 = 1$. So, we have

$$h_1 r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{1,j} h_j + r\right), \quad h_i r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j + b_i r\right), \ 2 \leq i \leq n.$$

In obtained table of multiplication we take the following basis transformation:

$$h_i' = h_i - b_i h_1, \quad 2 \leq i \leq n.$$

Therefore, we obtain the table of multiplication

$$h_1 r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{1,j} h_j + r\right), \quad h_i r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j\right), \ 2 \leq i \leq n.$$

Thus, we obtain the following result.

Proposition 1. *Let $\mathcal{C}$ be an EACP. Then there exists a basis $\{h_1, h_2, \ldots, h_n, r\}$ such that $\mathcal{C}$ on this basis is represented by the table of multiplication as follows*:

$$h_1 r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{1,j} h_j + \delta r\right), \ \delta \in \{0, 1\}, \quad h_i r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j\right), \ 2 \leq i \leq n.$$

3. Three-Dimensional Complex EACP

In the following theorem we present the classification of three-dimensional EACP.

Theorem 2. *An arbitrary three-dimensional complex EACP $\mathcal{C}$ is isomorphic to one of the following pairwise non-isomorphic algebras*:

If $\dim \mathcal{C}^2 = 1$, *then*

$$\mathcal{C}_1 : h_1 r = \frac{1}{2} r;$$

$$\mathcal{C}_2 : h_1 r = \frac{1}{2} h_2;$$

$$\mathcal{C}_3 : h_1 r = \frac{1}{2} h_1 + \frac{1}{2} r.$$

If $\dim \mathcal{C}^2 = 2$, *then*

$$\mathcal{C}_4 : h_1 r = \frac{1}{2}(h_1 + h_2), \quad h_2 r = \frac{1}{2} h_2;$$

$$\mathcal{C}_5(\beta): h_1r = \frac{1}{2}h_1, \quad h_2r = \frac{\beta}{2}h_2, \;\; \beta \neq 0;$$

$$\mathcal{C}_6(\alpha, \beta): h_1r = \frac{1}{2}(\alpha h_1 + \beta h_2 + r), \quad h_2r = \frac{1}{2}h_1;$$

$$\mathcal{C}_7(\alpha): h_1r = \frac{1}{2}(\alpha h_1 + r), \quad h_2r = \frac{1}{2}h_2;$$

$$\mathcal{C}_8: h_1r = \frac{1}{2}(h_1 + h_2 + r), \quad h_2r = \frac{1}{2}h_2,$$

where one of non-zero parameter α, β *in the algebra* $\mathcal{C}_6(\alpha, \beta)$ *can be assumed to be equal to* 1.

Proof. Since the EACPs with condition $\dim \mathcal{C}^2 = 1$ are already classified in [5], we shall consider only algebras such that $\dim \mathcal{C}^2 = 2$.

According to Proposition 1 we have that there exists a basis $\{h_1, h_2, r\}$ such that the multiplication of $\mathcal{C}$ on this basis has the form:

$$h_1r = \frac{1}{2}(a_{1,1}h_1 + a_{1,2}h_2 + \delta r), \quad h_2r = \frac{1}{2}(a_{2,1}h_1 + a_{2,2}h_2).$$

If $\delta = 0$, then using the result of Theorem 1 we obtain the algebras

$$I: h_1r = \frac{1}{2}(\alpha h_1 + h_2), \quad h_2r = \frac{1}{2}\alpha h_2.$$

$$II: h_1r = \frac{\alpha}{2}h_1, \quad h_2r = \frac{\beta}{2}h_2.$$

Since $\dim \mathcal{C}^2 = 2$, then $\alpha\beta \neq 0$ and h_1, h_2 are symmetric in the table of multiplication, we can conclude (by scaling of r) that $\alpha = 1$ and $\beta \neq 0$, i.e. we obtain the algebras $\mathcal{C}_4, \mathcal{C}_5(\beta)$.

Let now $\delta = 1$. Then the table of multiplication of the algebra $\mathcal{C}$ has the form

$$h_1r = \frac{1}{2}(a_{1,1}h_1 + a_{1,2}h_2 + r), \quad h_2r = \frac{1}{2}(a_{2,1}h_1 + a_{2,2}h_2).$$

Case 1. Let $a_{2,1} \neq 0$. Then putting $r' = \frac{1}{a_{2,1}}r$ and $h_1' = h_1 + \frac{a_{1,2}}{a_{2,1}}h_2$ we get family of algebras:

$$\mathcal{C}_6(\alpha, \beta): h_1r = \frac{1}{2}(\alpha h_1 + \beta h_2 + r), \quad h_2r = \frac{1}{2}h_1.$$

Case 2. Let $a_{2,1} = 0$. Then $a_{2,2} \neq 0$ and by scaling $r' = \frac{1}{a_{2,2}}r$ we obtain the multiplication:

$$h_1r = \frac{1}{2}(a_{1,1}h_1 + a_{1,2}h_2 + r), \quad h_2r = \frac{1}{2}h_2.$$

- In the case of $a_{1,2} = 0$, we have the algebra $\mathcal{C}_7$.
- If $a_{1,2} \neq 0$, then by $h_2' = a_{1,2}h_2$ we conclude that $a_{1,2} = 1$, that is, we get family of algebras

$$h_1r = \frac{1}{2}(\alpha h_1 + h_2 + r), \quad h_2r = \frac{1}{2}h_2.$$

— In the above table of multiplication when $\alpha \neq 1$ by setting $h_1' = h_1 + \frac{1}{\alpha-1}h_2$ we get the table of multiplication $\mathcal{C}_7(\alpha \neq 1)$.
— In the case of $\alpha = 1$ we get $\mathcal{C}_8$.

Thus, we obtain the algebras

$$\mathcal{C}_4, \quad \mathcal{C}_5(\beta),\ \beta \neq 0, \quad \mathcal{C}_6(\alpha,\beta), \quad \mathcal{C}_7(\alpha), \quad \mathcal{C}_8.$$

We shall investigate isomorphisms inside of families $\mathcal{C}_6(\alpha,\beta)$ and $\mathcal{C}_7(\alpha)$.

First consider the algebra

$$\mathcal{C}_6(\alpha,\beta) : h_1 r = \frac{1}{2}(\alpha h_1 + \beta h_2 + r), \quad h_2 r = \frac{1}{2}h_1.$$

Let us take the general change of basis elements

$$h_1' = a_1h_1 + a_2h_2 + a_3r, \quad h_2' = b_1h_1 + b_2h_2 + b_3r, \quad r' = c_1h_1 + c_2h_2 + c_3r.$$

Consider the product

$$0 = h_1'h_1' = 2a_1a_3h_1r + 2a_2a_3h_2r = (a_1a_3a_{1,1} + a_2a_3)h_1 + a_1a_3a_{1,2}h_2 + a_1a_3r.$$

Hence, $a_1a_3 = a_2a_3 = 0$.

Similarly, from $0 = h_2'h_2' = r'r' = h_1'h_2'$ we derive

$$b_1b_3 = b_2b_3 = c_1c_3 = c_2c_3 = 0, \quad a_2b_3 + a_3b_2 = 0, \quad a_1b_3 + a_3b_1 = 0.$$

From the chain of equalities

$$\begin{aligned}\frac{1}{2}(a_1h_1 + a_2h_2 + a_3r) = \frac{1}{2}h_1' = h_2'r' &= (b_1c_3 + b_3c_1)h_1r + (b_2c_3 + b_3c_2)h_2r \\ &= \frac{1}{2}(b_1c_3 + b_3c_1)(\alpha h_1 + \beta h_2 + r) + \frac{1}{2}(b_2c_3 + b_3c_2)h_1,\end{aligned}$$

we deduce

$$(b_1c_3 + b_3c_1)\alpha + (b_2c_3 + b_3c_2) = a_1, \quad (b_1c_3 + b_3c_1)\beta = a_2, \quad b_1c_3 + b_3c_1 = a_3.$$

Analogously, the equality

$$h_1'r' = \frac{1}{2}(\alpha'h_1' + \beta'h_2' + r')$$

implies

$$\begin{aligned}(a_1c_3 + a_3c_1)\alpha + (a_2c_3 + a_3c_2) &= \alpha'a_1 + \beta'b_1 + c_1, \\ (a_1c_3 + a_3c_1)a_{1,2} &= \alpha'a_2 + \beta'b_2 + c_2, \\ a_1c_3 + a_3c_1 &= \alpha'a_3 + \beta'b_3 + c_3.\end{aligned}$$

Thus, we obtain the following restrictions:

$$a_1a_3 = a_2a_3 = b_1b_3 = b_2b_3 = c_1c_3 = c_2c_3 = 0, \quad a_2b_3 + a_3b_2 = 0, \quad a_1b_3 + a_3b_1 = 0,$$
$$(b_1c_3 + b_3c_1)\alpha + (b_2c_3 + b_3c_2) = a_1, \quad (b_1c_3 + b_3c_1)\beta = a_2, \quad b_1c_3 + b_3c_1 = a_3,$$

$$(a_1c_3 + a_3c_1)\alpha + (a_2c_3 + a_3c_2) = \alpha' a_1 + \beta' b_1 + c_1,$$

$$(a_1c_3 + a_3c_1)a_{1,2} = \alpha' a_2 + \beta' b_2 + c_2,$$

$$a_1c_3 + a_3c_1 = \alpha' a_3 + \beta' b_3 + c_3.$$

It is not difficult to check that $a_3 = b_3 = 0$. Therefore, $c_3 \neq 0$ (otherwise the chosen basis transformation is non-singular) and the restriction will have the following form:

$$a_1 = b_2c_3 = 1, \quad a_2 = b_1 = c_1 = c_2 = 0,$$

$$\alpha' = c_3\alpha, \quad \beta' = c_3^2\beta.$$

By choosing appropriate value of c_3 we can assert that one of non-zero parameter α', β' can be scaled to 1.

In the case of the family of algebras $\mathcal{C}_7(\alpha)$ taking general change of basis and considering products in new basis, we obtain restrictions:

$$a_1a_3 = a_2a_3 = b_1b_3 = b_2b_3 = c_1c_3 = c_2c_3 = 0, \quad a_2b_3 + a_3b_2 = 0,$$

$$a_1b_3 + a_3b_1 = 0, \quad b_1 = (b_1c_3 + b_3c_1)\alpha, \quad b_2 = (b_2c_3 + b_3c_2),$$

$$b_3 = b_1c_3 + b_3c_1, \quad (a_1c_3 + a_3c_1)\alpha = \alpha' a_1 + c_1,$$

$$a_2c_3 + a_3c_2 = \alpha' a_2 + c_2, \quad a_1c_3 + a_3c_1 = \alpha' a_3 + c_3.$$

Simple study of the restrictions lead to

$$a_1 = c_3 = 1, \quad a_3 = b_1 = b_3 = c_1 = c_2 = 0, \ \ b_2 \neq 0, \ \ \alpha = \alpha'.$$

Therefore, for different values of parameter α in the family $\mathcal{C}_7(\alpha)$ we get non-isomorphic algebras.

Applying argumentations similar as above we derive that there is no algebra of the family $\mathcal{C}_7(\alpha)$ which is isomorphic to the algebra $\mathcal{C}_8$.

Since any algebra of the family $\mathcal{C}_6(\alpha, \beta)$ has not one-dimensional non-abelian ideal and $\langle h_2 \rangle$ is non-abelian ideal in algebras $\mathcal{C}_7(\alpha), \mathcal{C}_8$, we conclude that there is no algebra of the family $\mathcal{C}_6(\alpha, \beta)$ which is isomorphic to $\mathcal{C}_7(\alpha)$ and $\mathcal{C}_8$. □

Using the explicit form of algebras $\mathcal{C}_i$, $i = 1, \dots, 8$ mentioned in Theorem 2 one can easily check the following proposition.

Proposition 2.

- *The algebras* $\mathcal{C}_i$, $i = 1, 4, 5$, *are solvable.*
- *The algebra* $\mathcal{C}_2$ *is nilpotent.*
- *The algebras* $\mathcal{C}_i$, $i = 3, 6, 7, 8$, *are non-solvable.*

4. The Description of Complex EACP

In this section we give the description of some $(n + 1)$-dimensional EACP.

According to Proposition 1, we obtain that any $(n+1)$-dimensional EACP admits a basis $\{h_1, h_2, \ldots, h_n, r\}$ such that the table of multiplication of this basis has the form

$$h_1 r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{1,j} h_j + \delta r\right), \quad \delta \in \{0,1\}, \quad h_i r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j\right), \quad 2 \le i \le n.$$

Since the result of Theorem 1 gives a description in the case of $\delta = 0$, we shall consider only the case $\delta = 1$.

Let us introduce denotations

$$V_2 = \langle h_2, \ldots, h_n \rangle, \quad R_{\overline{r}} = R_r|_{V_2}.$$

Then we have

$$h_i \overline{r} = \frac{1}{2}\left(\sum_{j=1}^{n} a_{i,j} h_j\right), \quad 2 \le i \le n. \tag{4.1}$$

Considering the equality (4.1) by modulo space $\langle h_1 \rangle$, we obtain

$$h_i \overline{r} = \frac{1}{2}\left(\sum_{j=2}^{n} a_{i,j} h_j\right), \quad 2 \le i \le n \quad \mathrm{mod}(\langle h_1 \rangle). \tag{4.2}$$

Similar to the proof of Theorem 1, we obtain the following products:

$$\begin{aligned}
h_{\sum_{i=1}^{s-1} n_i+2} r &= \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+2} + h_{\sum_{i=1}^{s-1} n_i+3}) \\
h_{\sum_{i=1}^{s-1} n_i+3} r &= \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+3} + h_{\sum_{i=1}^{s-1} n_i+4}) \\
&\vdots \\
h_{\sum_{i=1}^{s-1} n_i+n_s-1} r &= \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+n_s-1} + h_{\sum_{i=1}^{s-1} n_i+n_s}) \\
h_{\sum_{i=1}^{s-1} n_i+n_s} r &= \frac{1}{2}(a_s h_{\sum_{i=1}^{s-1} n_i+n_s}),
\end{aligned}$$

where $s = 1, \ldots, k$, $n_1 + n_2 + \cdots + n_k = n - 1$ and $(a_1, a_2, \ldots, a_k) = (1, a_2, \ldots, a_k)$ or $(a_1, a_2, \ldots, a_k) = (0, 0, \ldots, 0)$.

Now go up to the space V_1, we obtain the following result.

Theorem 3. *There exists a basis $\{h_1, h_2, \ldots, h_n, r\}$ of EACP, such that the table of multiplication of this basis has the following form*:

$$h_1 r = \frac{1}{2}\left(\sum_{j=1}^{n} a_{1,j} h_j + r\right)$$

$$h_{\sum_{i=1}^{s-1} n_i+2} r = \frac{1}{2}(a_{\sum_{i=1}^{s-1} n_i+2,1} h_1 + a_s h_{\sum_{i=1}^{s-1} n_i+2} + h_{\sum_{i=1}^{s-1} n_i+3})$$

$$h_{\sum_{i=1}^{s-1} n_i+3}r = \frac{1}{2}(a_{\sum_{i=1}^{s-1} n_i+3,1}h_1 + a_s h_{\sum_{i=1}^{s-1} n_i+3} + h_{\sum_{i=1}^{s-1} n_i+4})$$
$$\vdots$$
$$h_{\sum_{i=1}^{s-1} n_i+n_s-1}r = \frac{1}{2}(a_{\sum_{i=1}^{s-1} n_i+n_s-1,1}h_1 + a_s h_{\sum_{i=1}^{s-1} n_i+n_s-1} + h_{\sum_{i=1}^{s-1} n_i+n_s})$$
$$h_{\sum_{i=1}^{s-1} n_i+n_s}r = \frac{1}{2}(a_{\sum_{i=1}^{s-1} n_i+n_s,1}h_1 + a_s h_{\sum_{i=1}^{s-1} n_i+n_s}),$$

where $s = 1, \dots, k$ *and* $a_1 = 1$.

4.1. *The case of* $k = 1$

Now we shall investigate the case of $k = 1$. Thanks to Theorem 3 we obtain the multiplication:

$$\begin{cases} h_1 r = \dfrac{1}{2}\left(\displaystyle\sum_{j=1}^{n} a_{1,j}h_j + r\right), \\ h_i r = \dfrac{1}{2}(a_{i,1}h_1 + \alpha h_i + h_{i+1}), \quad 2 \le i \le n-1, \\ h_n r = \dfrac{1}{2}(a_{n,1}h + \alpha h_n), \end{cases}$$

where $\alpha \in \{0; 1\}$.

Assume there exists some m, $2 \le m \le n$ such that $a_{1,m} \ne 0$ and $a_{1,m}$ is the first non-zero parameter. Putting

$$h_i' = \sum_{j=m}^{n} a_{1,j}h_{j-m+i}, \quad 2 \le i \le n,$$

we can assume that the table of multiplication has the form

$$\mathcal{C}^{\alpha}(a_i, m_1) : \begin{cases} h_1 r = a_1 h_1 + \alpha h_{m_1} + r, & 2 \le m_1 \le n, \\ h_i r = a_i h_1 + \alpha h_i + h_{i+1}, & 2 \le i \le n-1, \\ h_n r = a_n h_1 + h_n, \end{cases}$$

where $\alpha \in \{0; 1\}$.

Consider two algebras $\mathcal{C}^1(a_i, m_1)$ and $\mathcal{C}^1(b_i, m_2)$, with basis $\{h_1, h_2, \dots, h_n, r\}$ and $\{h_1', h_2', \dots, h_n', r'\}$, respectively. If $m_1 \ne m_2$, then without loss of generality we can suppose $m_1 < m_2$.

Theorem 4. *Two algebras* $\mathcal{C}^1(a_i, m_1)$ *and* $\mathcal{C}^1(b_i, m_2)$ *with* $m_1 < m_2$ *are isomorphic if and only if*

(1) $a_i = b_i = 0$, $m_1 \le i \le n$;
(2) $a_1 = b_1$ *and* $b_1 \ne 1$, *and* $b_1 \ne a_{m_1-1} + 1$ *in the case of* $b_{m_1-1} \ne 0$; $b_1 \ne 1$, *or* $a_{m_1-2} \ne -1$ *in the case of* $b_{m_1-1} = 0$, $b_{m_1-2} \ne 0$;

(3) *there exist* $B_2, B_3, \ldots, B_{m_1-1} \in \mathbb{C}$, *with* $B_2 \neq 0$, *such that*

$$b_i = \sum_{k=i}^{m_1-1} B_{k-i+2} a_k, \quad 2 \leq i \leq m_1 - 1.$$

Proof. *Necessary.* Let f be an isomorphism $f : \mathcal{C}^1(b_i, m_2) \to \mathcal{C}^1(a_i, m_1)$. Then

$$f(h_i') = \sum_{j=1}^{n} A_{i,j} h_j + C_i r, \; 1 \leq i \leq n, \quad f(r') = \sum_{j=1}^{n} D_j h_j + C_{n+1} r.$$

Note that, without loss of generality we can assume $C_{n+1} \neq 0$.

From the following chain of equalities

$$0 = f(h_i')f(h_i') = \left(\sum_{j=1}^{n} A_{i,j} h_j + C_i r\right)\left(\sum_{j=1}^{n} A_{i,j} h_j + C_i r\right) = 2C_i \sum_{j=1}^{n} A_{i,j} h_j r$$

$$= C_i \left[A_{i,2} h_2 + A_{i,1} h_{m_1} + A_{i,1} r + \sum_{j=1}^{n} A_{i,j} a_j h_1 + \sum_{j=3}^{n} (A_{i,j-1} + A_{i,j}) h_j\right],$$

$$0 = f(r')f(r') = \left(\sum_{j=1}^{n} D_j h_j + D_{n+1} r\right)\left(\sum_{j=1}^{n} D_j h_j + C_{n+1} r\right) = 2C_{n+1} \sum_{j=1}^{n} D_j h_j r$$

$$= C_{n+1} \left[D_2 h_2 + D_1 h_{m_1} + D_1 r + \sum_{j=1}^{n} D_j a_j h_1 + \sum_{j=3}^{n} (D_{j-1} + D_j) h_j\right],$$

we obtain the restrictions:

$$C_i \sum_{j=1}^{n} A_{i,j} a_j = 0, \;\; C_i A_{i,k} = 0, \quad 1 \leq i \leq n, \;\; 1 \leq k \leq n,$$

$$C_{n+1} \sum_{j=1}^{n} D_j a_j = 0, \;\; C_{n+1} D_k = 0, \quad 1 \leq k \leq n.$$

Since $C_{n+1} \neq 0$, we obtain $D_i = 0$ for $1 \leq i \leq n$. Moreover, it is not difficult to obtain that $C_i = 0$ for $1 \leq i \leq n$. Indeed, if there exists $C_{i_0} \neq 0$, for $2 \leq i_0 \leq n$, then it implies that $A_{i_0,j} = 0$ for $1 \leq j \leq n$. It is a contradiction with the condition of the matrix of the general change is not singular, since $D_i = 0$ for $1 \leq i \leq n$.

Thus, we have

$$f(h_i') = \sum_{j=1}^{n} A_{i,j} h_j, \;\; 1 \leq i \leq n, \quad f(r') = r.$$

Consider the multiplication

$$f(h'_n)f(r') = \sum_{j=1}^{n} A_{n,j}h_j r = A_{n,2}h_2 + A_{n,1}h_{m_1} + A_{n,1}r + \sum_{j=1}^{n} A_{n,j}a_j h_1 + \sum_{j=3}^{n}(A_{n,j-2} + A_{n,j})h_j.$$

On the other hand,

$$f(h'_n)f(r') = b_n f(h'_1) + f(h'_n) = \sum_{j=1}^{n}(b_n A_{1,j} + A_{n,j})h_j.$$

Comparing the coefficients at the basis elements we obtain

$$\begin{cases} A_{n,1} = 0, & \sum_{j=2}^{n} A_{n,j}a_j = b_n A_{1,1}, \\ 0 = b_n A_{1,2}, & \\ A_{n,k} = b_{n,1}A_{1,k+1}, & 2 \le k \le n-1. \end{cases} \tag{4.3}$$

Analogously, considering products

$$f(h'_i)f(r') = \sum_{j=1}^{n} A_{i,j}h_j r = A_{i,2}h_2 + A_{i,1}h_{m_1} + A_{i,1}r + \sum_{j=1}^{n} A_{i,j}a_j h_1 + \sum_{j=3}^{n}(A_{i,j-1} + A_{i,j})h_j$$

$$= b_i f(h'_1) + f(h'_i) + f(h'_{i+1}) = \sum_{j=1}^{n}(b_i A_{1,j} + A_{i,j} + A_{i+1,j})h_j,$$

for $2 \le i \le n-1$, we get

$$\begin{cases} A_{i,1} = 0, \\ \sum_{j=2}^{n} A_{i,j}a_j = b_i A_{1,1}, \\ b_i A_{1,2} + A_{i+1,2} = 0, \\ A_{i,k} = b_i A_{1,k+1} + A_{i+1,k+1}, \quad 2 \le k \le n-1. \end{cases} \tag{4.4}$$

Consider the product

$$f(h'_1)f(r') = \sum_{j=1}^{n} A_{1,j}h_j r = A_{1,2}h_2 + A_{1,1}h_{m_1} + A_{1,1}r + \sum_{j=1}^{n} A_{1,j}a_j h_1 + \sum_{j=3}^{n}(A_{1,j-1} + A_{1,j})h_j$$

$$= b_1 f(h'_1) + f(h'_{m_2}) + f(r') = r + \sum_{j=1}^{n}(b_1 A_{1,j} + A_{m_2,j})h_j.$$

From this we derive

$$\begin{cases} A_{1,1} = 1, & \sum_{j=1}^{n} A_{1,j}a_j = b_1, \\ A_{1,2} = b_1 A_{1,2} + A_{m_2,2}, & \\ A_{1,k} + A_{1,k+1} = b_1 A_{1,k+1} + A_{m_2,k+1}, & 2 \le k \le n-1, \ \ k \ne m_1 - 1, \\ 1 + A_{1,m_1-1} + A_{1,m_1} = b_1 A_{1,m_1} + A_{m_2,m_1}. & \end{cases} \tag{4.5}$$

From $b_n A_{1,2} = 0$, we obtain $b_n = 0$. Indeed, if $b_n \ne 0$ then $A_{1,2} = 0$ and according to (4.4) we get $A_{i,2} = 0$ for $3 \le i \le n$. From $A_{n,2} = b_n A_{1,3}$ and by (4.4) we get

$$A_{1,3} = 0, \ \ A_{2,2} = A_{3,3}, \ \ A_{i,3} = 0, \quad 4 \le i \le n.$$

Recurrently, we obtain

$$A_{1,k} = 0, \ \ A_{k,k} = A_{2,2}, \ \ A_{i,k} = 0, \quad 2 \le k \le m_1, \ \ k+1 \le i \le n.$$

But since $m_1 < m_2$, from $1 + A_{1,m_1-1} + A_{1,m_1} = b_1 A_{1,m_1} + A_{m_2,m_1}$ we get incorrect equality $1 = 0$. It is a contradiction with assumption $b_n \ne 0$.

Thus, $b_n = 0$. From this we have $A_{n,i} = 0, 2 \le i \le n-1$ and $a_n = 0$.

Continuing this process we obtain the condition (1), i.e.

$$b_i = a_i = 0, \quad m_1 \le i \le n.$$

- If $b_{m_1-1,1} \ne 0$, then it is not difficult to obtain that

$$\begin{aligned} &A_{1,j} = 0, \quad 2 \le j \le m_1 - 1, \\ &A_{i,j} = 0, \quad 3 \le i \le m_1 - 1, \ \ 2 \le j \le i - 1. \end{aligned} \tag{4.6}$$

  Indeed, from $b_{m_1-1,1} A_{1,2} = 0$ we have $A_{1,2} = 0$, which implies $A_{i,2} = 0$ for $3 \le i \le m_1 - 1$. So, the equality (4.6) is true for $j = 2$. Recurrently we obtain the equality (4.6) for any j $(2 \le j \le m_1 - 1)$.

  Thus, we obtain $a_1 = b_1$, $\sum_{k=i}^{m_1-1} A_{i,k}a_k = b_i, 2 \le i \le m_1 - 1$, and

$$\begin{cases} 1 + A_{1,m_1} = b_1 A_{1,m_1}, & \\ A_{1,k} + A_{1,k+1} = b_1 A_{1,k+1}, & m_1 \le k \le m_2 - 2, \\ A_{1,k} + A_{1,k+1} & \\ \qquad = b_1 A_{1,k+1} + A_{m_1,m_1-m_2+k+1}, & m_2 - 1 \le k \le n-1, \\ A_{i,k} = A_{i+1,k+1}, & 2 \le i \le m_1 - 2, \ \ i \le k \le m_1 - 2, \\ A_{i,k} = b_i A_{1,k+1} + A_{i+1,k+1}, & 2 \le i \le m_1 - 1, \ \ m_1 - 1 \le k \le n-1, \\ A_{i,k} = A_{i+1,k+1}, & m_1 \le i \le n-1, \ \ i \le k \le n-1. \end{cases} \tag{4.7}$$

Taking into account the equality $1 + A_{1,m_1} = b_1 A_{1,m_1}$, we have $b_1 \neq 1$, and $A_{1,m_1} = \frac{1}{b_1-1}$. From the fifth equality of (4.7) for $k = m_1 - 1$, we have

$$A_{m_1,m_1} = A_{m_1-1,m_1-1} - b_{m_1-1}A_{1,m_1} = b_{m_1-1}\left(\frac{1}{a_{m_1-1}} - \frac{1}{b_1-1}\right).$$

Since $A_{m_1,m_1} \neq 0$, we obtain $b_1 \neq a_{m_1-1} + 1$, i.e. the condition (2) is satisfied.

Taking into account, the fourth equation of the system (4.7), putting $B_k = A_{2,k}$, $2 \leq k \leq m_1 - 1$ we obtain the condition (3).

- In the case of $b_{m_1-1} = 0$, using the similar argument for the first non-zero element b_t from the set $\{b_{m_1-2}, \ldots, b_2\}$ we obtain the equality $a_1 = b_1$, $\sum_{k=i}^{t} A_{i,k}a_k = b_i, 2 \leq i \leq t$, and

$$\begin{cases} A_{1,t+1} = b_1 A_{1,t+1}, & \\ A_{1,k} + A_{1,k+1} = b_1 A_{1,k+1}, & t+1 \leq k \leq m_1 - 2, \\ 1 + A_{1,m_1-1} + A_{1,m_1} = b_1 A_{1,m_1}, & \\ A_{1,k} + A_{1,k+1} = b_1 A_{1,k+1}, & m_1 \leq k \leq m_2 - 2, \\ A_{1,k} + A_{1,k+1} & \\ \qquad = b_1 A_{1,k+1} + A_{t+1,t-m_2+k+2}, & m_2 - 1 \leq k \leq n - 1, \\ A_{i,k} = A_{i+1,k+1}, & 2 \leq i \leq t-1, \;\; i \leq k \leq t-1, \\ A_{i,k} = b_i A_{1,k+1} + A_{i+1,k+1}, & 2 \leq i \leq t, \;\; t \leq k \leq n-1, \\ A_{i,k} = A_{i+1,k+1}, & t+1 \leq i \leq n-1, \;\; i \leq k \leq n-1. \end{cases} \tag{4.8}$$

If $t = m_1 - 2$, then in the case of $b_1 = 1$ and $a_{m_1-2} = -1$, we obtain $A_{m_1-1,m_1-1} = b_{m_1-2}(\frac{1}{a_{m_1-2}} + 1) = 0$, which is a contradiction with the existence of isomorphism f. Therefore, if $t = m_1 - 2$, then $b_1 \neq 1$ or $a_{m_1-2} \neq -1$, i.e. the condition (2) is satisfied.

Putting $B_k = A_{2,k}$, $2 \leq k \leq m_1 - 1$ we obtain the condition (3).

Sufficient. Let the conditions (1), (2) and (3) are satisfied. From the previous proof it follows that the existence of an isomorphism $f : \mathcal{C}^1(b_i, m_2) \to \mathcal{C}^1(a_i, m_1)$ is equivalent to the solvability of (4.7) ((4.8) if $b_{m_1-1} = 0$).

In the case of $b_{m_1-1} \neq 0$, and $b_1 \neq 1$, $b_1 \neq a_{m_1-1} + 1$ we find a solution of (4.7) as follows:

$$A_{i,k} = B_{k-i+2}, \quad 2 \leq i \leq m_1 - 1, \;\; i \leq k \leq m_1 - 1,$$

$$A_{1,k} = \frac{1}{(b_1-1)^{k-m_1+1}}, \quad m_1 \leq k \leq m_2 - 1,$$

$$A_{i,i} = b_{m_1-1}\left(\frac{1}{a_{m_1-1}} - \frac{1}{b_1-1}\right), \quad m_1 \leq i \leq n.$$

Then from (4.7) we obtain other parameters $A_{i,k}$.

The case $b_{m_1-1} = 0$ is similar to the case $b_{m_1-1} \neq 0$. $\square$

Corollary 1. *$\mathcal{C}^1(0,m) \cong \mathcal{C}^1(0,2)$ for any $m(3 \le m \le n)$. And as an isomorphism we can take*

$$f(h_1') = h_1 + \sum_{k=1}^{m-1}(-1)^k h_k + \sum_{k=m}^{m}((-1)^{k-1} + (-1)^{k-m})h_k,$$

$$f(h_i') = h_i, \; 2 \le i \le n, \quad f(r') = r.$$

Analogously, we obtain the following theorem for the class of algebras $\mathcal{C}^0(a_i, m)$.

Theorem 5. *Two algebras $\mathcal{C}^0(a_i, m_1)$ and $\mathcal{C}^0(b_i, m_2)$ with $m_1 < m_2$ are isomorphic if and only if*

(1) $a_i = b_i = 0, \; m_1 \le i \le n$;
(2) $a_1 = b_1$ *and* $b_1 \ne 0$, *and* $b_1 \ne a_{m_1-1}$ *in the case of* $b_{m_1-1} \ne 0$; $b_1 \ne 0$, *or* $a_{m_1-2} \ne -1$ *in the case of* $b_{m_1-1} = 0$, $b_{m_1-2} \ne 0$;
(3) *there exist* $B_2, B_3, \ldots, B_{m_1-1} \in \mathbb{C}$, *with* $B_2 \ne 0$, *such that*

$$b_i = \sum_{k=i}^{m_1-1} B_{k-i+2}a_k, \quad 2 \le i \le m_1 - 1.$$

Now we investigate the criteria of isomorphism inside the class of $\mathcal{C}^1(a_i, m)$.

According to Theorem 4, if $a_i = 0$ for $m-1 \le i \le n$, then there exists an algebra $\mathcal{C}^1(c_i, m-1)$ which is isomorphic to $\mathcal{C}^1(a_i, m)$, we consider the case of $a_i \ne 0$ for some $i(m-1 \le i \le n)$.

For this purpose consider two algebras $\mathcal{C}^1(a_i, m)$ and $\mathcal{C}^1(b_i, m)$, i.e. case of $m_1 = m_2$.

Similar to the proof of Theorem 4, we consider the isomorphism $f : \mathcal{C}^1(b_i, m) \to \mathcal{C}^1(a_i, m)$ as follows:

$$f(h_i') = \sum_{j=1}^{n} A_{i,j}h_j, \; 1 \le i \le n, \quad f(r') = r.$$

We obtain $A_{1,1} = 1$, $A_{i,1} = 0$, for $2 \le i \le n$ and the following restrictions:

$$\begin{cases} b_1 = a_1 + \sum_{j=2}^{n} A_{1,j}a_j, & \\ b_i = \sum_{j=2}^{n} A_{i,j}a_j, & 2 \le i \le n, \\ b_nA_{1,2} = 0, \; b_iA_{1,2} + A_{i+1,2} = 0, & 2 \le i \le n, \\ A_{i,k} = b_iA_{1,k+1} + A_{i+1,k+1}, & 2 \le i \le n-1, \; 2 \le k \le n-1, \\ A_{n,k} = b_nA_{1,k+1}, & 2 \le k \le n-1, \\ A_{1,2} = b_1A_{1,2} + A_{m,2}, & \\ A_{1,k} + A_{1,k+1} = b_1A_{1,k+1} + A_{m,k+1}, & 2 \le k \le n-1, \; k \ne m-1, \\ 1 + A_{1,m-1} + A_{1,m} = b_1A_{1,m} + A_{m,m}. & \end{cases} \tag{4.9}$$

From (4.9) it is not difficult to obtain the following recurrent formula:

$$A_{i+1,k+1} = A_{2,k-i+2} - \sum_{j=2}^{i} b_i A_{1,k-i+1+j}. \tag{4.10}$$

Let p be the first non-zero element from the set $\{b_n, b_{n-1}, \ldots, b_{m-1}\}$.

In the case of $p = m-1$, i.e. $b_n = b_{n-1} = \cdots = b_m = 0$, $b_{m-1} \neq 0$, from (4.9) we obtain $a_n = a_{n-1} = \cdots = a_m = 0$, and

$$\begin{cases} A_{1,i} = 0, & 2 \le i \le m-1, \\ A_{i,i} = A_{2,2}, & 3 \le i \le m-1, \\ A_{i,i} = A_{2,2} - b_{m-1}A_{1,m}, & m \le i \le n, \\ b_1 = a_1, b_i = \displaystyle\sum_{j=2}^{m+1-i} A_{2,j}a_{j-2+i}, & 2 \le i \le m-1, \\ A_{2,2} = 1 + (1 - b_1 + b_{m-1,1})A_{1,m}, & \\ A_{2,3} = (1 - b_1 + b_{m-1})A_{1,m+1} + (1 + b_{m-2})A_{1,m}, & \\ A_{2,i} = (1 - b_1 + b_{m-1})A_{1,m-2+i} + (1 + b_{m-2})A_{1,m-3+i} & \\ \qquad + \displaystyle\sum_{j=4}^{i} b_{m+1-j}A_{1,m+i-j}, & 4 \le i \le m-1, \\ A_{2,i} = (1 - b_1 + b_{m-1})A_{1,m-2+i} + (1 + b_{m-2})A_{1,m-3+i} & \\ \qquad + \displaystyle\sum_{j=4}^{m-1} b_{m+1-j}A_{1,m+i-j}, & m \le i \le n-m+2, \end{cases} \tag{4.11}$$

From this we obtain the following result.

Theorem 6. *Any algebra $\mathcal{C}^1(a_i, m)$ with $a_{m-1} \neq 0$, $a_m = a_{m+1} = \cdots = a_n = 0$ is isomorphic to one of the following non-isomorphic algebras*:

$$\mathcal{C}_1^1(a_i, m) : \begin{cases} a_1 = a, \\ a_{m-2} = -1, \\ a_{m-1} = a - 1, \\ a_j = 0 & \textit{otherwise}, \end{cases}$$

$$\mathcal{C}_2^1(a_i, m) : \begin{cases} a_1 = a, \\ a_{m-1} = a - 1, \\ a_j = 0 & \textit{otherwise}, \end{cases}$$

$$\mathcal{C}_3^1(a_i, m) : \begin{cases} a_1 \neq 2, \\ a_{m-1} = 1, \\ a_j = 0 & \textit{otherwise}. \end{cases}$$

Proof. From (4.11) it is not difficult to see that if $b_{m-1} = b_1 - 1$ and $b_{m-2} = -1$, then

$$\begin{cases} A_{2,2} = 1, \\ A_{2,3} = 0, \\ A_{2,i} = \displaystyle\sum_{j=4}^{i} b_{m+1-j} A_{1,m+i-j}, \quad 4 \le i \le m-1, \\ A_{2,i} = \displaystyle\sum_{j=4}^{m-1} b_{m+1-j} A_{1,m+i-j}, \quad m \le i \le n-m+2, \end{cases} \tag{4.12}$$

which implies $a_{m-1} = b_{m-1} = b_1 - 1$, $a_{m-2} = -1$, and $b_i = a_i + \sum_{j=4}^{m+1-i} A_{2,j} a_{j-2+i}$, for $2 \le i \le m-3$.

Putting

$$A_{2,4} = -\frac{a_{m-3}}{b_{m-1}}, \quad A_{2,5} = -\frac{1}{b_{m-1}}(a_{m-4} - a_{m-1}A_{2,4}),$$

$$A_{2,i} = -\frac{1}{b_{m-1}}\left(a_{m+1-i} - \sum_{j=4}^{i-1} a_{m-1-i+j} A_{2,j}\right), \quad 5 \le i \le m-1,$$

we can suppose $b_2 = b_3 = \cdots = b_{m-3} = 0$. Other parameters of the isomorphism f can be found by (4.12) and (4.10). Thus, for any parameters $a_1, a_2, a_3, \ldots, a_{m-1}$ with conditions $a_{m-1} = b_{m-1} = b_1 - 1$, $a_{m-2} = -1$, there exists an isomorphism f from the algebra $\mathcal{C}^1(a_i, m)$ to the algebra $\mathcal{C}_1^1$.

If $b_{m-1} = b_1 - 1$ and $b_{m-2} \neq -1$, then we have $A_{2,2} = 1$ and $a_{m-1} = b_{m-1} = b_1 - 1$.

Putting

$$A_{2,3} = -\frac{a_{m-2}}{b_{m-1}}, \quad A_{2,4} = -\frac{a_{m-3}}{b_{m-1}}, \quad A_{2,5} = -\frac{1}{b_{m-1}}(a_{m-4} - a_{m-1}A_{2,4}),$$

$$A_{2,i} = -\frac{1}{b_{m-1}}\left(a_{m+1-i} - \sum_{j=3}^{i-1} a_{m-1-i+j} A_{2,j}\right), \quad 4 \le i \le m-1,$$

we can suppose $b_2 = b_3 = \cdots = b_{m-2} = 0$. Other parameters of the isomorphism f can be found by (4.12) and (4.11). Thus, for any parameters $a_1, a_2, a_3, \ldots, a_{m-1}$ with conditions $a_{m-1} = a_1 - 1$ there exists an isomorphism f from the algebra $\mathcal{C}_1^1(a_i, m)$ to $\mathcal{C}_2^1$.

Analogously, in the case of $b_{m-1} \neq b_1 - 1$ and $b_{m-2} \neq -1$, putting $A_{2,j}$ we can suppose $b_2 = b_3 = \cdots = b_{m-2} = 0$ and $b_{m-1} = 1$, which derive the algebra L_3^1. □

Now consider the case $p \geq m$, i.e. $b_n = b_{n-1} = \cdots = b_{p+1} = 0$, $b_p \neq 0$, then from (4.9) we obtain $a_n = a_{n-1} = \cdots = a_{p+1} = 0$, and

$$\begin{cases} A_{1,i} = 0, & 2 \leq i \leq p, \\ A_{i,i} = 1, & 2 \leq i \leq p, \\ A_{i,i} = 1 - b_p A_{1,p+1}, & p+1 \leq i \leq n, \\ b_1 = a_1,\ b_i = a_i + \displaystyle\sum_{j=p+3-m}^{p+2-i} A_{2,j} a_{j-2+i}, & 2 \leq i \leq m-1, \\ A_{2,p-m+3} = (1 - b_1 + b_{m-1}) A_{1,p+1}, & \\ A_{2,p-m+4} = (1 - b_1 + b_{m-1}) A_{1,p+2} + (1 + b_{m-2}) A_{1,p+1}, & \\ A_{2,i} = (1 - b_1 + b_{m-1}) A_{1,p-1+i} + (1 + b_{m-2}) A_{1,p-2+i} & \\ \qquad + \displaystyle\sum_{j=4}^{i} b_{m+1-j} A_{1,p+1+i-j}, & 4 \leq i \leq m-1, \\ A_{2,i} = (1 - b_1 + b_{m-1}) A_{1,p-1+i} + (1 + b_{m-2}) A_{1,p-2+i} & \\ \qquad + \displaystyle\sum_{j=4}^{m-1} b_{m+1-j} A_{1,p+1+i-j}, & m \leq i \leq n-m+2. \end{cases}$$

Similar to Theorem 6, we obtain the following result.

Theorem 7. *Any algebra $\mathcal{C}^1(a_i, m)$ with $a_p \neq 0$, $a_{p+1} = a_{p+2} = \cdots = a_n = 0$, $p \geq n$ is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$\mathcal{C}_1^1(a_i, m, p) : \begin{cases} a_1 = a, & \\ a_{m-2} = -1, & \\ a_{m-1} = a - 1, & \\ a_j \in \mathbb{C}, & m \leq j \leq p, \\ a_j = 0 & \textit{otherwise}, \end{cases}$$

$$\mathcal{C}_2^1(a_i, m, p) : \begin{cases} a_1 = a, & \\ a_{m-1} = a - 1, & \\ a_j \in \mathbb{C}, & m \leq j \leq p, \\ a_j = 0 & \textit{otherwise}, \end{cases}$$

$$\mathcal{C}_3^1(a_i, m, p) : \begin{cases} a_1 \neq 2, & \\ a_{m-1} = 1, & \\ a_j \in \mathbb{C}, & m \leq j \leq p, \\ a_j = 0 & \textit{otherwise}. \end{cases}$$

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