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ON THE COHOMOLOGY OF MODULAR LIE ALGEBRAS

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The cohomology of many classes of Lie algebras over fields of characteristic zero is known well. For instance there is exhaustive information available on the cohomology of finite-dimensional semisimple complex Lie algebras. Whitehead's well-known lemma says that the cohomology of a semisimple Lie algebra with coefficients in an irreducible nontrivial finite-dimensional module is zero (cf. [1], p. 106, Exercise 12 k). The same is true of finite-dimensional nilpotent Lie algebras (Dixmier [2]). Many attempts have recently been made to compute the cohomology of infinite-dimensional Lie algebras of Cartan type. Compared with this, the cohomology of modular Lie algebras (that is, Lie algebras over prime characteristic fields) is virtually unknown. Therefore every result specially devoted to the cohomology of modular Lie algebras is of essential interest.

In this paper we prove some modular analogues of the Whitehead lemma. In contrast to the characteristic zero case, our main result (Theorem 1) remains correct for arbitrary modular Lie algebras and for a rather large class of modules (not necessarily irreducible). For example, the cohomology of a p-algebra with coefficients in an irreducible module can be nontrivial only if this module is a p-module (Theorem 2). Then we apply these results to study the cohomology of simple Lie algebras. We consider the case of a classical Lie algebra, where we take A_1, and the case of a Lie algebra of Cartan type, where we take the p^n-dimensional Zassenhaus algebra $W_1(n)$. In the former case we derive the complete description of the cohomology with coefficients in an irreducible module (Theorem 4). In the latter case we reduce the study of that cohomology to computing the cohomology of $W_1(n)$ itself and the cohomology of a (p^n-2)-dimensional nilpotent subalgebra $\mathfrak{L}_1 = \oplus_{i\geq 1} L_i$ of $W_1(n)$, where $L_i = \langle x^{(i+1)}\partial\rangle$, with coefficient in the trivial 1-dimensional module (corollary to Theorem 5).

§1. Cohomology of modular Lie algebras with coefficients in a nontrivial module

In this paper all algebras and modules are finite dimensional over a fixed field P of characteristic $p>0$. Let L be a Lie algebra, $U(L)$ the universal enveloping algebra and Z the center of $U(L)$. A polynomial of the form $f(t)=\Sigma_{i\geq 0}\lambda_i t^{p^i}\in P[t]$ is called a *p-polynomial*. With every element $l\in L$ we can associate a p-polynomial $z(t)$ such that

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replacing t by l gives a central element $z(l) \in Z$. Thus we obtain a map (in general this is ambiguous) which we denote by $z\colon L \to Z$. Let M be an L-module and

$$l \mapsto (l)_M, \qquad l \in L, (l)_M \in \operatorname{End} M,$$

its associated representation. The main result can be stated as follows.

THEOREM 1. *Let L be a Lie algebra over a field of prime characteristic and M an arbitrary L-module. Suppose for some $l \in L$ the endomorphism $(z(l))_M$ is not degenerate, $z(l)$ being an associated central element. Then the cohomology $H^*(L, M)$ is zero.*

It is useful, for the sequel, to recall some definitions concerning the cohomology of Lie algebras. We follow the notation of [1], Chapter I, §3, Exercise 12, to which we add some new notation. Let $C^k(L, M)$ denote the space of all multilinear alternating maps in k variables with $k > 0$. We also put $C^0(L, M) = M$ and $C^k(L, M) = 0$ if $k < 0$. In the cochain complex $C^*(L, M) = \oplus_k C^k(L, M)$ we introduce the coboundary map by $d\psi = \psi' + \psi''$, where $\psi', \psi'' \in C^{k+1}(L, M)$ are the cochains corresponding to $\psi \in C^k(L, M)$ and determined by the formulas

$$\psi'(l_1,\dots,l_{k+1}) = \sum_{i<j} (-1)^{i+j}\psi([l_i, l_j], l_1,\dots,\hat{l}_i,\dots,\hat{l}_j,\dots,l_{k+1}),$$

$$\psi''(l_1,\dots,l_{k+1}) = \sum_i (-1)^{i+1}(l_i)_M\psi(l_1,\dots,\hat{l}_i,\dots,l_{k+1})$$

(here and everywhere in the sequel $\hat{}$ means that the element under this sign must be omitted). The following notation is standard: $Z^k(L, M)$ is the space of k-cocycles, $B^k(L, M)$ is the space of k-coboundaries and $H^k(L, M) = Z^k(L, M)/B^k(L, M)$ is the kth cohomology space. Now the cohomology class of the cocycle $\psi \in Z^k(L, M)$ in $H^k(L, M)$ is denoted by $\bar{\psi}$. Let θ be a representation of L in $C^*(L, M)$ of the form

$$(\theta(l)\psi)(l_1,\dots,l_k) = (l)_M\psi(l_1,\dots,l_k) + \sum_i (-1)^i\psi([l, l_i]l_1,\dots,\hat{l}_i,\dots,l_k).$$

This extends to a representation of the universal enveloping algebra $U(L)$. Every element $l \in L$ determines an endomorphism of degree -1 (adjoint endomorphism) $i(l)$ of the cochain complex $C^*(L, M)$ if we put

$$(i(l)\psi)(l_1,\dots,l_{k-1}) = \psi(l, l_1,\dots,l_{k-1}), \qquad \psi \in C^k(L, M).$$

We will need the relations

$$d\theta(l) = \theta(l)d, \qquad l \in L, \tag{1}$$

$$di(l) + i(l)d = \theta(l), \qquad l \in L. \tag{2}$$

By M^L we denote the subspace of all invariants

$$M^L = \langle m \in M \mid l(m) = 0, l \in L\rangle.$$

Let L' be a subalgebra in L. We recall that the relative cohomology space $H^*(L, L', M) = \oplus_k H^k(L, L', M)$ can be defined as the cohomology of the relative cochain complex

$$C^*(L, L', M) = \langle \psi \in C^*(L, M) \mid i(l')\psi = 0, \theta(l')\psi = 0, l' \in L\rangle.$$

For more details on cohomology, see [3].

LEMMA 1. *Let $f(t)$ be a p-polynomial in $P[t]$ and l an element in L. Then $\theta(f(l)) = f(\theta(l))$.*

PROOF. We may restrict ourselves to the case $f(t) = t^{p^s}$. Then it suffices to prove that $(\theta(l))^{p^s}\psi = \theta(l^{p^s})\psi$, ψ being an arbitrary cochain $\psi \in C^k(L, M)$, $k \geqslant 0$. We consider representations $\theta_q\colon L \to \operatorname{End} C^k(L, M)$, $0 \leqslant q \leqslant k$, given by

$$\theta_0(l) = (l)_M,$$

$$(\theta_q(l)\psi)(l_1,\ldots,l_k) = (-1)^q\psi([l, l_q], l_1,\ldots,\hat{l}_q,\ldots,l_k), \qquad 0 < q \leqslant k.$$

These extend to representations of $U(L)$. For any $l \in L$ the endomorphisms $\theta_q(l)$, $0 \leqslant q \leqslant k$, commute pairwise, and their sum equals $\theta(l)$. Hence in raising the equation $\theta(l) = \Sigma_0^k \theta_q(l)$ to a certain power we may use the binomial expansion. Thus

$$(\theta(l))^{p^s} = \sum_{q=0}^{k} (\theta_q(l))^{p^s}.$$

But $\theta_q(l)^{p^s} = \theta_q(l^{p^s})$, and the proof is complete.

COROLLARY. *Let $z(l)$ be a central element associated with an element $l \in L$. Then $\theta(z(l)) = (z(l))_M$.*

PROOF. Since $z(\theta_q(l)) = \theta_q(z(l)) = 0$, $0 < q \leqslant k$, by Lemma 1

$$\theta(z(l)) = z(\theta(l)) = \sum_{q=0}^{k} z(\theta_q(l)) = z(\theta_0(l)) = z((l)_M) = (z(l))_M.$$

LEMMA 2. *For some $l \in L$, let the endomorphism $(z(l))_M$ be nondegenerate, and let $\bar{l} = (z(l))_M^{-1}$ be its inverse. Then $\bar{l}d = d\bar{l}$.*

PROOF. Let $\psi \in C^k(L, M)$, $k \geqslant 0$. We verify that

$$(\bar{l}\psi)' = \bar{l}(\psi'), \tag{3}$$

$$(\bar{l}\psi)'' = \bar{l}(\psi''). \tag{4}$$

(3) is obvious. Since $z(l) \in Z$ we find that $(z(l))_M(l')_M = (l')_M(z(l))_M$, whence $\bar{l}(l')_M = (l')_M\bar{l}$ for all $l' \in L$. Then

$$\bar{l}((l')_M\psi(\ldots,\hat{l}',\ldots)) = (l')_M(\bar{l}\psi(\ldots,\hat{l}',\ldots)).$$

Now (4) is clear. The proof is complete.

PROOF OF THEOREM 1. Multiplying both sides of (2) by an element of the form $\theta(l)^q$ on the right and considering (1) gives that $di_q(l) + i_q(l)d = \theta(l)^{q+1}$ for a suitable endomorphism $i_q(l)$ of degree -1 (in fact, $i_q(l) = i(l)\theta(l)^q$). Passing to linear combinations of such relations, we derive that for any p-polynomial $f(t) \in P[t]$ there exists an endomorphism $i_f\colon C^*(L, M) \to C^*(L, M)$ of degree -1 such that

$$di_f(l) + i_f(l)d = f(\theta(l)).$$

In particular, for a central element $z(l)$, making use of the corollary to Lemma 1 we find that

$$di_z(l) + i_z(l)d = (z(l))_M.$$

Now we suppose that $(z(l))_M$ is invertible. Then, by Lemma 2, $d\rho + \rho d = (\mathrm{id})_M$, where the homotopy ρ is given by $\rho = \bar{l}i_z(l)$. Thus the theorem is proved.

We have the following corollaries.

COROLLARY 1. *Let P be a field of prime characteristic and M an irreducible L-module. The cohomology $H^*(L, M)$ is nonzero only if all the endomorphisms of the form $(z(l))_M$, $l \in L$, are zero.*

PROOF. By Schur's lemma the endomorphism $(z(l))_M$ is invertible if and only if it is nonzero.

COROLLARY 2. *Let $l \in L$ be an* ad*-nilpotent element such that $(l)_M$ is not degenerate. Then $H^*(L, M)$ is trivial.*

PROOF. If $(\operatorname{ad} l)^q = 0$ then l^{p^s} is in Z as soon as $q < p^s$. Now $(l)_M$ is invertible if and only if $(l)_M^{p^s}$ is.

COROLLARY 3. *Let L be a nilpotent Lie algebra over a field of prime characteristic and M an irreducible L-module. Then the cohomology $H^*(L, M)$ is nontrivial if and only if M is a trivial* 1*-dimensional L-module.*

We remark that the latter also holds for fields of characteristic zero. This is proved in [2], and, in fact, the proof given in [2] and relying on the Serre-Hochschild spectral sequence does not depend on the characteristic of the base field. Since every finite-dimensional irreducible representation of a nilpotent Lie algebra over an algebraically closed field of characteristic zero is 1-dimensional, the result is more interesting for the modular case.

PROOF. Suppose $H^*(L, M) \neq 0$. We consider the lower central series

$$L^1 = L \supset L^2 = [L, L] \supset \cdots \supset L^q \supset 0.$$

Arguing by induction over $i = q, q-1, \ldots, 1$, we can prove that $(l)_M = 0$ for all $l \in L^i$. Then, for $i = 1$, we will find that M is a trivial L-module, hence a 1-dimensional one. Now, since L^q is in the center of L, using Corollary 1 we see that the base of the induction is true. We then assume that the assertion is true for $i+1$. Then $[(l)_M, (l')_M] = ([l, l'])_M = 0$, $l \in L^i$, $l' \in L$, because $[l, l'] \in L^{i+1}$. We see that, although l is not necessarily an element in Z, the endomorphism $(l)_M$ commutes with all endomorphisms $(l')_M$. By Schur's lemma either $(l)_M = 0$ or $(l)_M$ is not degenerate. But the latter is impossible since $z(l)$ has the form l^{p^s}; hence by theorem 1 we have $0 = (z(l))_M = (l)_M^{p^s}$. The induction step is proved.

The converse of Corollary 3 is obvious. For instance,

$$H^0(L, P) \cong P \neq 0.$$

In contrast to [2], we do not require that the base field is algebraically closed.

We recall that the nil component of $z(l) \in Z$ is an L-module M is a subspace $M_0(z(l))$ in which $(z(l))_M$ acts nilpotently; it has the form $M_0(z(l)) = \bigcup_{j \geq 1} \operatorname{Ker}(z(l))_M^j$. The subspace $M_0(Z) = \bigcap_{l \in L} M_0(z(l))$ is called the *Z-nil component* of M. The nil component $M_0(Z)$ has the structure of an L-module. We remark that, when L is nilpotent, $M_0(Z)$ coincides with Fitting's nil component

$$M_0 = \bigcap_{l \in L} \bigcup_{j \geq l} \operatorname{Ker}(l)_M^j.$$

Now we reformulate Theorem 1.

THEOREM 1′. *Let L be an arbitrary Lie algebra over a field of prime characteristic, and M an L-module. Then*

$$H^*(L, M) \cong H^*(L, M_0(Z)).$$

PROOF. If $M = M_0(Z)$, the result is obvious. If $M \neq M_0(z(l))$ for some $l \in L$, then, in $M_1(z(l)) = M/M_0(z(l))$, the endomorphism $(z(l))_{M_1(z(l))}$ is invertible; hence, due to Theorem 1, $H^*(L, M_1(z(l))) = 0$. From the long exact cohomological sequence

$$\cdots \to H^k(L, M_0(z(l))) \to H^k(L, M) \to H^k(L, M_1(z(l))) \to \cdots$$

which corresponds to the exact sequence of L-modules

$$0 \to M_0(z(l)) \to M \to M_1(z(l)) \to 0,$$

we find that

$$H^k(L, M) \cong H^k(L, M_0(z(l))), \qquad k \geqslant 0, \dim M_0(z(l)) < \dim M.$$

Using induction over the dimension of M, we complete the proof of Theorem 1′.

COROLLARY 4. *If L is a modular nilpotent Lie algebra, then $H^*(L, M)$ is isomorphic to $H^*(L, M_0)$.*

In particular, we get another proof of Corollary 3.

The following important corollary is awarded the name of a theorem.

THEOREM 2. *Let L be a Lie p-algebra, and suppose that an irreducible L-module M is not a p-module. Then $H^*(L, M)$ is trivial.*

PROOF. There exists an element $l \in L$ such that the endomorphism $(l^p - l^{[p]})_M$ is not zero. Since $l^p - l^{[p]} \in Z$, by Corollary 1 the proof is complete.

Now let L possess an invariant symmetric form $(\ ,\ )$. Let $e_1, \ldots, e_n$ be a basis in L and $e'_1, \ldots, e'_n$ its dual: $(e_i, e'_j) = \delta_{i,j}$, $i, j = 1, \ldots, n$. Since $(\ ,\ )$ is nondegenerate, the following is true.

ASSERTION. *If*

$$[e_i, l] = \sum_i \lambda_{i,j} e_j \quad \textit{and} \quad [e'_i, l] = \sum_j \lambda'_{i,j} e'_j$$

are basic decompositions for $l \in L$, then $\lambda_{i,j} + \lambda_{j,i} = 0$ for all $i, j = 1, \ldots, n$.

The Casimir element $c = \sum_i e_i e'_i$ belongs to Z. In the proof of this well-known result one uses the above assertion. The same assertion is principal in Whitehead's lemma and in the following.

THEOREM 3. *Let R be an ideal of a Lie algebra L and $(\ ,\ )$ an invariant nondegenerate symmetric form on R. Let c denote the corresponding Casimir element. If $(c)_M$ is invertible, then the cohomology $H^*(L, M)$ is trivial.*

In particular, if the trace form $(l, l')_M = \operatorname{tr}((l)_M(l')_M)$ corresponding to an irreducible L-module M is not degenerate and the dimension of R is not divisible by the characteristic of the base field, then $H^*(L, M) = 0$ (see [1], Chapter I, §3, Exercise 12j).

We briefly recall the proof of this theorem. Let ρ be an endomorphism of degree -1 in $C^*(L, M)$ such that

$$(\rho\psi)(l_1,\ldots,l_{k-1}) = \sum_i (e_i)_M \psi(e_i', l_1,\ldots,l_{k-1}), \qquad \psi \in C^k(L, M).$$

Then, using the above assertion, we find that

$$d\rho + \rho d = (c)_M. \tag{5}$$

The beginning of the proof of Theorem 3 is precisely the same. To complete the proof after formula (5) we apply Lemma 2.

Although Whitehead's lemma and Theorem 3 differ in minor details, the latter has wider application. Indeed, every simple Lie algebra over a field of characteristic zero has a nondegenerate trace form. This is not the case when the characteristic is prime. Nevertheless, every classical modular simple Lie algebra possesses a nondegenerate invariant form. In fact this is true of Lie algebras which are not necessarily classical or even p-algebras. For example, every Hamiltonian Lie algebra has a nondegenerate invariant form which is not a trace form [7].

We apply Theorems 2 and 3 to study the cohomology of the 3-dimensional simple Lie algebra of the type A_1. We start with a proposition of independent interest. We recall that an abelian subalgebra T in a Lie algebra L is called a *torus* if there is a basis $\langle e_\alpha \mid \alpha \in T^* \rangle$ such that the action of every element $h \in T$ is semisimple, i.e. $[h, e_\alpha] = \alpha(h)e_\alpha$, $\alpha \in T^*$.

PROPOSITION 1. *Let L be a finite-dimensional Lie algebra over an arbitrary field, and let the action of a torus T in a finite-dimensional L-module M be semisimple. Let a coboundary $d\psi \in B^*(L, M)$ be invariant under the action of the torus: $\theta(h)\, d\psi = 0\ \forall\, h \in T$. Then, in the cohomology class of the cochain ψ, there exists a representative φ which is also invariant with respect to the action of the torus: $\theta(h)\varphi = 0\ \forall\, h \in T$, $\varphi - \psi \in B^*(L, M)$.*

PROOF. Since all the T-modules under consideration are semisimple, there exists a basis in the finite-dimensional cochain space $C^*(L, M) \cong \Lambda^*L \otimes M$ whose elements are eigenvectors with respect to all endomorphisms $\theta(h)$, $h \in T$. Let $\psi = \Sigma_0^k \psi_j$, where $\psi_0, \psi_1,\ldots,\psi_k$ are eigenvectors with pairwise different eigenvalues $\lambda_0, \lambda_1,\ldots,\lambda_k$. Let, say, $\lambda_0 = 0$. We apply the endomorphism $\theta(h)^q$ to the equation $d\psi = \Sigma_0^k d\psi_j$. Using (1), we get

$$\sum_{j=0}^{k} \lambda_j^q d\psi_j = 0, \qquad 0 \leqslant q \leqslant k.$$

The Vandermonde determinant is nonzero:

$$\begin{bmatrix} \lambda_1 & \cdots & \lambda_k \\ \lambda_1^2 & \cdots & \lambda_k^2 \\ \vdots & & \vdots \\ \lambda_1^k & \cdots & \lambda_k^k \end{bmatrix}$$

Hence $d\psi_j = 0$ for all $1 \leqslant j \leqslant k$. Then, according to (2),

$$d\big(i(h)\psi_j\big) = \theta(h)\psi_j = \lambda_j\psi_j, \qquad 1 \leqslant j \leqslant k,$$

which gives $\psi_j = d(i(h)\psi_j/\lambda_j) \in B^*(L, M)$. To finish we put $\varphi = \psi + d\sigma$, where $\sigma = \Sigma_1^k -i(h)\psi_j/\lambda_j$. The proof is complete.

COROLLARY. *Let L, M and T denote the same as in Proposition* 1. *Then* $H^*(L, M) \cong H^*(L, M)^T$.

Now let

$$L = \langle e_-, h, e_+ \mid [e_+, e_-] = h, [h, e_\pm] = \pm 2e_\pm \rangle$$

denote the Lie algebra of type A_1 over an algebraically closed field of characteristic $p > 3$. All p-representations of this Lie algebra can be obtained by reduction modulo p from standard irreducible representations of dimensions $1 \leqslant i + 1 \leqslant p$ with highest weight i (see [5]). The endomorphism $(c)_M$ corresponding to the Casimir element $c = e_+ e_- + e_- e_+ + h^2/2$ is not degenerate if $\dim M \neq 1, p - 1$. According to Theorems 2 and 3, then $H^*(L, M) = 0$ as soon as M is any irreducible L-module whose dimension is not 1 or $p - 1$. Now let $M = \langle v_i \mid 1 \leqslant i \leqslant p - 1 \rangle$ be a $(p - 1)$-dimensional irreducible L-module with maximal vector v_{p-1}: $e_+ v_{p-1} = 0$ and with minimal vector v_1: $e_- v_1 = 0$. The classes of the cocycles $\psi^1_+, \psi^1_- \in Z^1(L, M)$ and $\psi^2_+, \psi^2_- \in Z^2(L, M)$ with nonzero components satisfying the relations

$$\psi^1_+(e_+) = v_1, \quad \psi^1_-(e_-) = v_{p-1},$$
$$\psi^2_+(e_+, h) = v_1, \quad \psi^2_-(e_-, h) = v_{p-1},$$

form a basis of $H^*(L, M)$. This follows from the corollary to Proposition 1. Thus we have proved the following.

THEOREM 4. *Let L be a Lie algebra of the type A_1 over an algebraically closed field of characteristic $p \geqslant 3$, and let M be an irreducible L-module. Then*

$$H^0(L, P) \cong H^3(L, P) \cong P,$$
$$H^1(L, M) \cong H^2(L, M) \cong P \oplus P, \qquad \dim M = p - 1.$$

In all the other cases $H^k(L, M)$ is trivial.

A deeper application of Theorem 1 is given in the following section, where we study the cohomology of the Zassenhaus algebra.

§2. The cohomology of the Zassenhaus algebra

All finite-dimensional simple Lie algebras over fields of prime characteristic known so far split into two classes of simple Lie algebras, called *classical* and *Cartan* Lie algebras. While the lowest-dimensional representative of the first class is the 3-dimensional Lie algebra of the type A_1, the lowest-dimensional representative in the class of Cartan Lie algebras is the p-dimensional Witt algebra $W_1(1)$. Our argument concerning the cohomology of the Witt algebra works for the Zassenhaus algebra as well.

We sketch the definition of the Zassenhaus algebra $W_1(n)$ (for the details see [4]). We recall that the multiplication in the divided power algebra $O_1(n) = \langle x^{(i)} \mid 0 \leqslant i \leqslant p^n - 1 \rangle$ is given by

$$x^{(i)} x^{(j)} = \binom{i + j}{i} x^{(i+j)}, \qquad \binom{i + j}{i} = \frac{(i + j)!}{i! j!}.$$

The derivation algebra $W_1(n) = \langle u\partial \mid u \in O_1(n) \rangle$ of $O_1(n)$

$$u\partial\colon v \mapsto u(\partial(v)), \qquad u, v \in O_1(n),$$
$$\partial\colon x^{(i)} \mapsto x^{(i-1)} \quad (i > 0), \qquad \partial\colon x^{(0)} \mapsto 0,$$

is called *the general Cartan type Lie algebra in one variable* (in the terminology due to Kostrikin and Shafarevich), or the *Zassenhaus algebra*. One can choose a basis $\{e_i = x^{(i+1)} | -1 \leq i \leq p^n - 2\}$ such that

$$[e_i, e_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) e_{i+j}.$$

$L = W_1(n)$ has a grading of the form

$$L = \bigoplus_{i=-1}^{p^n-2} L_i, \qquad L_i = \langle e_i \rangle.$$

The associated filtration will be denoted as follows:

$$L = \mathfrak{L}_{-1} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \cdots \supset \mathfrak{L}_{p^n-2} \supset 0, \qquad \mathfrak{L}_i = \bigoplus_{j \geq i} L_j.$$

It should be remarked that the subalgebra $L_0 = \langle e_0 \rangle$ is a torus in L.

The divided power algebra $U = O_1(n)$ has a natural grading of the form

$$U = \bigoplus_{i=0}^{p^n-1} U_i, \qquad U_i = \langle x^{(i)} \rangle.$$

Thus the associative algebra U has a natural structure of a graded L-module. This module is reducible and possesses a one dimensional trivial submodule P. We introduce new graded L-module structures in $O_1(n)$ by putting

$$(l, v) \mapsto l(v) + t(\operatorname{Div} l)v, \qquad t \in P,$$

where $\operatorname{Div}(u\partial) = \partial(u)$ is the divergence of the derivation $u\partial \in W_1(n)$. The L-module thus obtained is denoted by U_t. In particular, U_0 is the natural L-module U.

Now the nilpotent subalgebra $\mathfrak{L}_0$ is endowed by a p-structure of the form $e_0^{[p]} = e_0$, $e_i^{[p]} = 0$ and $(\operatorname{ad} e_{-1})^{p^n} = 0$. In particular, Witt's algebra $W_1(1)$ is a Lie p-algebra. According to Corollary 1 of Theorem 1 the cohomology $H^*(L, M)$ of the Zassenhaus algebra L with coefficients in an irreducible module M is nontrivial only in the cases where

$$(e_{-1})_M^{p^n} = 0, \quad (e_0)_M^p = (e_0)_M, \quad (e_i)_M^p = 0, \qquad i > 0.$$

Although the structure of irreducible representations of $W_1(n)$ in the general cases is rather complicated (see [6]), the "almost p-representations" as above admit a good realization. Their corresponding modules are exhausted by the following list: the 1-dimensional trivial L-module P, the $(p^n - 1)$-dimensional L-quotient module U/P and the p^n-dimensional L-modules U_t, where $t \in \mathbf{Z}/p\mathbf{Z}$, $t \neq 0, 1$, in number $p - 2$.

The rest of the paper is devoted to computing the cohomology $H^*(L, U_t)$ modulo the cohomology $H^*(\mathfrak{L}_1, P)$. To formulate our result we introduce a one-dimensional $\mathfrak{L}_0$-module $\langle 1_t \rangle$ such that $e_0 1 = t1$ and $e_i 1 = 0$ for $i > 0$. We abbreviate $H^*(L, P)$ and $H^*(\mathfrak{L}_0, \langle 1_t \rangle)$ to $H^*(L)$ and $H^*(\mathfrak{L}_0, 1_t)$ respectively. We recall that, given a module V over a Lie algebra L', we denote by $V^{L'}$ the invariant subspace of V.

THEOREM 5. *Let $L = W_1(n)$. For every $t \in P$ there exists an isomorphism of spaces $(k \geq 0)$*

$$H^k(L, U_t) \cong \left(\left(H^k(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1) \oplus H^{k-2}(\mathfrak{L}_1)\right) \otimes \langle 1_t \rangle\right)^{L_0}.$$

An immediate corollary of Theorem 5 and Corollary 1 to Theorem 1 is the following.

COROLLARY 1. *Let P be an algebraically closed field of characteristic $p > 3$, and let M be a finite-dimensional irreducible module over $L = W_1(n)$. Then the cohomology $H^*(L, M)$ is trivial except for the following cases.*

(i) *M is a p^n-dimensional L-module U_t, where $t \in \mathbf{Z}/p\mathbf{Z}$, $t \neq 0, 1$. In this case*

$$H^k(L, M) \cong \left(\left(H^k(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1) \oplus H^{k-2}(\mathfrak{L}_1)\right) \otimes \langle 1_t \rangle\right)^{L_0}, \qquad k \geqslant 0.$$

(ii) *M is the $(p^n - 1)$-dimensional L-quotient module U/P. Then there is an exact sequence of the form*

$$0 \to H^0(L) \to H^0(L, U) \to H^0(L, M) \to H^1(L)$$
$$\to \cdots \to H^k(L) \to H^k(L, U) \to H^k(L, M) \to H^{k+1}(L) \to \cdots,$$

where

$$H^k(L, U) \cong \left(H^k(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1) \oplus H^{k-2}(\mathfrak{L}_1)\right)^{L_0}, \qquad k \geqslant 0.$$

(iii) *M is the one-dimensional trivial L-module.*

It is worth explaining the origin of the exact sequence in case (ii). It arises as the long exact cohomological sequence associated with the short exact sequence of L-modules of the form $0 \to P \to U \to M \to 0$.

Before proving the theorem we introduce some additional notation. If $V = \bigoplus_{i \in \mathbf{Z}} V_i$ is a graded space with homogeneous components V_i, then we write $|v| = i$ if $v \in V_i$. We denote by $\mathrm{pr}_{V'}$ the natural projection of V onto a subspace V' of V. If $v_1, \ldots, v_n$ is a basis in V, then every $v \in V$ can be represented as $v = \sum_1^n \mathrm{pr}_j v$, where $\mathrm{pr}_j v = \mathrm{pr}_{\langle v_j \rangle} v$. Let A_j denote the coefficient at v_j for the projection $\mathrm{pr}_j v$. The following convention will be observed: the elements of L will be denoted by $l, l_1, \ldots,$ the elements of the natural L-module U will be denoted by $u, v, u_1, v_1, \ldots,$ and L-module U_t will be denoted by M and its elements by $m, m', \ldots$. Finally, the usual $(l)_M(m)$ will be shortened to $l(m)$. This convention will be useful in determining from the context which subspaces the elements l and m belong to.

Now we endow the L-module $M = U_t$ with a structure of a module over the associative algebra U by naturally putting $(u, m) \mapsto um$, $u \in U$, $m \in M$. It is obvious that M is a unitary U-module (that is, $1m = m$ for $m \in M$), and that M is a free U-module with a basis $\langle 1 \rangle$ which is the invariant subspace $M^{L_{-1}} = \langle m \in M \mid e_{-1}(m) = 0 \rangle$. Besides, M is a graded module both over L and U, and these structures agree in that

$$l(u, m) = l(u)m + ul(m), \qquad l \in L, u \in U, m \in M.$$

Therefore the natural pairing of the cochain spaces

$$C^*(L, U) \cup C^*(L, M) \to C^*(L, M)$$

extends to a pairing of the cohomology spaces

$$H^*(L, U) \cup H^*(L, M) \to H^*(L, M).$$

In particular, the first cohomology space $H^1(L, U)$ acts in the cohomology space $H^*(L, M)$. The following is the explicit form of the action of an element $\psi \in C^1(L, U)$ on an element $\varphi \in C^k(L, M)$:

$$(\psi \cup \varphi)(l_1, \ldots, l_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \psi(l_i) \cup \varphi(l_1, \ldots, \hat{l}_i, \ldots, l_{k+1}).$$

LEMMA 3. *The relative cohomology $H^*(L, L_{-1}, M)$ is a direct summand in $H^*(L, M)$. There exists an isomorphism $H^*(L, L_{-1}, U_t) \cong H^*(\mathfrak{L}_0, 1_t)$, $t \in P$.*

PROOF. Let

$$\mathcal{A}: C^*(L, U_t) \to C^*(\mathfrak{L}_0, 1_t)$$

denote the product of two linear maps, one of them being the restriction map to $\mathfrak{L}_0$ and the other being the projection onto the subspace $(U_t)^{L_{-1}} = \langle 1 \rangle$:

$$(A\psi)(l_1, \ldots, l_k) = \mathrm{pr}_{\langle 1 \rangle}(\psi(l_1, \ldots, l_k)), \qquad \psi \in C^k(L, U_t), l_1, \ldots, l_k \in \mathfrak{L}_0, k > 0,$$

$$Am = \mathrm{pr}_{\langle 1 \rangle}(m), \qquad m \in C^0(L, U_t) = U_t.$$

The following diagram is commutative:

$$\begin{array}{ccccc} C^k(L, U_t) & \xrightarrow{d} & C^{k+1}(L, U_t) & & \\ \mathcal{A}\downarrow & & \mathcal{A}\downarrow & & k \geqslant 0. \\ C^k(\mathfrak{L}_0, 1_t) & \xrightarrow{d} & C^{k+1}(\mathfrak{L}_0, 1_t) & & \end{array}$$

Indeed, since U_t is a graded L-module, we have $\mathrm{pr}_{\langle 1 \rangle}(l(m)) = \mathrm{pr}_{\langle 1 \rangle}(l(\mathrm{pr}_{\langle 1 \rangle}(m)))$ for all $l \in \mathfrak{L}_0$ and $m \in U_t$. Thus

$$l(A\psi(\ldots, \hat{l}, \ldots)) = A(l(\psi(\ldots, \hat{l}, \ldots)))$$

for all $l \in \mathfrak{L}_0$ and $\psi \in C^k(L, U_t)$. Therefore $(\mathcal{A}\psi)'' = \mathcal{A}(\psi'')$. But $(\mathcal{A}\psi)' = \mathcal{A}(\psi')$ is obvious. Finally, $d\mathcal{A}\psi = \mathcal{A}d\psi$. In other words, $\mathcal{A}$ is a projection of the cochain complexes.

Now we verify that the subspace $C^*(L, L_{-1}, U_t)$ is mapped by $\mathcal{A}$ injectively. Suppose not. Then $\mathcal{A}\psi = 0$ but $\mathrm{pr}_{\langle x^{(j)} \rangle}(\psi(\cdots)) \neq 0$, where $j > 0$ is assumed to be the least such number. Let $\psi \in C^k(L, L_{-1}, U_t)$. We rewrite the condition $\theta(e_{-1})\psi = 0$ in a more transparent form. We have

$$e_{-1}(\psi(l_1, \ldots, l_k)) = \sum_{i=1}^{k} (-1)^i \psi([e_{-1}, l_1], l_1, \ldots, \hat{l}_i, \ldots, l_i). \tag{6}$$

The projection of the left-hand side of (6) onto the subspace $\langle x^{(j-1)} \rangle$ for some $l_1, \ldots, l_k \in \mathfrak{L}_0$ is different from zero. Hence the same is true of one of the summands in the left-hand side of (6). This, however, contradicts the choice of j. Finally, we see that the restriction of $\mathcal{A}$ to $C^*(L, L_{-1}, U_t)$ has trivial kernel.

Now we construct a splitting map $\mathcal{A}'$: $C^*(\mathfrak{L}_0, 1_t) \to C^*(L, U_t)$ for $\mathcal{A}$. We put $\mathcal{A}'1 = 1$ for $k = 0$. For $k > 0$ we put

$$(\mathcal{A}'\varphi)(x^{(i_1)}\partial, \ldots, x^{(i_k)}\partial) = \sum_{j_1, \ldots, j_k = 1}^{p^n - 1} \partial^{j_1}(x^{(j_1)}) \cdots \partial^{j_k}(x^{(i_k)}) \varphi(x^{(j_1)}\partial, \ldots, x^{(j_k)}\partial).$$

It is easy to check that $\theta(e_{-1})\mathcal{A}'\varphi = 0$ and $i(e_{-1})\mathcal{A}'\varphi = 0$, which means that, in fact, $\mathcal{A}'$ maps $C^*(\mathfrak{L}_0, 1_t)$ into $C^*(L, L_{-1}, U_t)$.

Now we verify that the product of the maps

$$C^*(\mathfrak{L}_0, 1_t) \xrightarrow{\mathcal{A}'} C^*(L, U_t) \xrightarrow{\mathcal{A}} C^*(\mathfrak{L}_0, 1_t)$$

is an identity map. Let $C^k_r(\mathfrak{L}_0, 1_t)$ denote a subspace in $C^k(\mathfrak{L}_0, 1_t)$ whose elements are the cochains φ such that $\varphi(l_1, \ldots, l_k) = 0$ if $|l_1| + \cdots + |l_k| \neq r$, $r \geqslant 0$, $k > 0$. Then the graded space $C_r^*(\mathfrak{L}_0, 1_t) = \oplus_k C_r^k(\mathfrak{L}_0, 1_t)$ is a cochain complex. Moreover, there is a direct decomposition

$$C^*(\mathfrak{L}_0, 1_t) = \bigoplus_{r \geqslant 0} C_r^*(\mathfrak{L}_0, 1_t).$$

Therefore the above assertion reduces to cochains in $C_r^*(\mathfrak{L}_0, 1_t)$, $r \geqslant 0$.

Now let $\varphi \in C_r^k(\mathfrak{L}_0, 1_t)$, $k > 0$. We must show that

$$\mathfrak{A}\mathfrak{A}'\varphi(l_1,\ldots,l_k) = \varphi(l_1,\ldots,l_k) \tag{7}$$

for all $l_1,\ldots,l_k \in \mathfrak{L}_0$. Since U_t is a graded U-module with basis $\langle 1 \rangle$, the cochain $\mathfrak{A}'\varphi$ is a homogeneous map of degree $-r$; that is,

$$|\mathfrak{A}'\varphi(l_1,\ldots,l_k)| = |l_1| + \cdots + |l_k| - r.$$

Therefore

$$\mathfrak{A}'\varphi(l_1,\ldots,l_k) = 0, \qquad |l_1| + \cdots + |l_k| < r,$$
$$\mathfrak{A}\mathfrak{A}'\varphi(l_1,\ldots,l_k) = 0, \qquad |l_1| + \cdots + |l_n| > r.$$

A careful inspection of the definition of $\mathfrak{A}'$ shows that $\mathfrak{A}\mathfrak{A}'\varphi(l_1,\ldots,l_k) = \varphi(l_1,\ldots,l_k)$ as soon as $|l_1| + \cdots + |l_k| = r$. Now (7) is proved.

So, $\mathfrak{A}'$ is a splitting map for the projection $\mathfrak{A}$ and $\mathfrak{A}'$ induces the isomorphism of the cochain complexes $C^*(L, L_{-1}, U_t)$ and $C^*(\mathfrak{L}_0, 1_t)$. Now the proof of the lemma is complete.

It is convenient, for the sequel, to give an explicit formula for the coboundary map d: $C^*(\mathfrak{L}_0, 1_t) \to C^*(\mathfrak{L}_0, 1_t)$. It is obvious that $d\varphi = \varphi' + \varphi''$, where the cochain $\varphi'' \in C^{k+1}(\mathfrak{L}_0, 1_t)$ associated with the cochain $\varphi \in C^k(\mathfrak{L}_0, 1_t)$ is given by the rule

$$\varphi'(l_1,\ldots,l_{k+1}) = \sum_{i=1}^{k+1} \sum_{|l_i|=0} (-1)^{i+1} l_i\big(\varphi(l_1,\ldots,\hat{l}_i, l_{k+1})\big),$$

and φ' is defined as previously (see §1).

It is useful to recall some facts concerning the structure of the cohomology space $H^1(L, U)$. The cochains $\alpha, \beta \in C^1(L, U)$ such that

$$\alpha(u\partial) = ux^{(p^n-1)}, \quad \beta(u\partial) = \partial(u),$$

are the cocycles. Indeed, $\alpha = d(x^{(p^n)})$ is an "almost" coboundary. Moreover, the cohomology class of α is nontrivial since $x^{(p^n)} \notin U$. We remark that the subspace $\langle \bar{\alpha} \rangle \subset H^1(L, U)$ may be considered as the cohomology space $H^1(L_{-1}, U)$ of the 1-dimensional subalgebra L_{-1}.

We also give another interpretation of this subspace. Let $\Omega^* = \oplus_k \Omega^k$ denote the de Rham cochain complex (that is, $\Omega^0 = U$ and $\Omega^k = 0$ for $k > 1$), and let $\Omega^1 \cong U \otimes \Lambda^1$ be the space of outer differential forms with coefficients in the divided power algebra U. Then the 1-cohomology de Rham space $H^1(\Omega^*)$ is isomorphic to $U/\partial(U)$. This latter space is isomorphic to the subspace spanned by the cohomology class of the cocycle α in the cohomology space $H^1(L, U)$.

We will need an explicit definition of the cocycle α:

$$\alpha(e_{-1}) = x^{(p^n-1)}, \quad \alpha(e_i) = 0, \qquad i \geqslant 0.$$

Now β being a cocycle is equivalent to a well-known property of the divergence operator:

$$\operatorname{Div}[l_1, l_2] = l_1(\operatorname{Div} l_2) - l_2(\operatorname{Div} l_1).$$

In fact this result and the nontriviality of the cohomology class of β follow immediately from Lemma 3, since $\beta \in Z^1(L, L_{-1}, U)$ and since the projection $\mathcal{P}\beta$ uniquely determines a basic cocycle in the cocycle space $Z^1(L_0)$ which is a subspace in $Z^1(\mathfrak{L}_0)$. It will be seen later that the classes of α and β form a basis in the cohomology space $H^1(L, U)$.

As we remarked above, for all $t \in P$ and $k \geqslant 0$ there exist pairings

$$H^1(L, U) \cup H^k(L, U_t) \to H^{k+1}(L, U_t).$$

In particular,

$$H^1(L, U) \cup H^k(L, L_{-1}, U_t) \to H^{k+1}(L, U_t).$$

An explicit formula is

$$(\alpha \cup \psi)(l_1, \ldots, l_{k+1}) = \sum_{i=1}^{k+1} \sum_{|l_i|=-1} (-1)^{i+1} x^{(p^n-1)} \psi(l_1, \ldots, \hat{l}_i, \ldots, l_{k+1}),$$

where $\psi \in C^k(L, L_{-1}, U_t)$. An explicit form of the pairing

$$C^1(L_0) \cup C^k(\mathfrak{L}_0, L_0, 1_t) \to C^{k+1}(\mathfrak{L}_0, 1_t),$$

is

$$(\beta \cup \varphi)(l_1, \ldots, l_{k+1}) = \sum_{i=1}^{k+1} \sum_{|l_i|=0} (-1)^{i+1} \varphi(l_1, \ldots, \hat{l}_i, \ldots, l_{k+1}).$$

Obviously this pairing induces a pairing of the cohomology spaces

$$H^1(L_0) \cup H^k(\mathfrak{L}_0, L_0, 1_t) \to H^{k+1}(\mathfrak{L}_0, 1_t).$$

LEMMA 4. *The following isomorphisms are valid*:

$$H^*(\mathfrak{L}_0, L_0, 1_t) \cong (H^*(\mathfrak{L}_1) \otimes \langle 1_k \rangle)^{L_0}, \qquad t \in P,$$
$$H^*(\mathfrak{L}_0, 1_t) \cong H^*(L_0) \otimes H^*(\mathfrak{L}_0, L_0, 1_t), \qquad t \in P.$$

In particular,

$$H^k(\mathfrak{L}_0, 1_t) \cong ((H^k(\mathfrak{L}_1) \oplus H^{k-1}(\mathfrak{L}_1)) \otimes \langle 1_t \rangle)^{L_0}, \qquad k \geqslant 0.$$

PROOF. It is easily seen that the linear map

$$C^k(\mathfrak{L}_0, L_0, 1_t) \to C^k(\mathfrak{L}_1, 1_t),$$

which is the natural restriction to the subalgebra $\mathfrak{L}_1$, gives an isomorphism $C^k(\mathfrak{L}_0, L_0, 1_t) \to C^k(\mathfrak{L}_1, 1_t)^{L_0}$. It is obvious that this map commutes with the coboundary map. Hence

$$H^k(\mathfrak{L}_0, L_0, 1_t) \cong H^k(\mathfrak{L}_1, 1_t)^{L_0} \cong (H^k(\mathfrak{L}_1) \otimes \langle 1_t \rangle)^{L_0}.$$

From Proposition 1 we have the isomorphism

$$H^*(\mathfrak{L}_0, 1_t) \cong H^*(\mathfrak{L}_0, 1_t)^{L_0}.$$

Let $\varphi \in Z^k(\mathfrak{L}_0, 1_t)^{L_0}$. Then $i(e_0)\varphi \in Z^{k-1}(\mathfrak{L}_0, L_0, 1_t)$. If $\varphi = d\omega$ is a coboundary with $\omega \in C^{k-1}(\mathfrak{L}_0, 1_t)^{L_0}$, then $i(e_0)\varphi$ is a coboundary in $Z^{k-1}(\mathfrak{L}_0, L_0, 1_t)$. For,

$$i(e_0)\varphi = i(e_0)\, d\omega = -d(i(e_0)\omega), \quad i(e_0)\omega \in C^{k-2}(\mathfrak{L}_0, L_0, 1_t).$$

Thus we have proved the correctness of the map

$$\mathfrak{B}\colon H^*(\mathfrak{L}_0, 1_t)^{L_0} \to H^{*-1}(\mathfrak{L}_0, L_0, 1_t),$$
$$\bar{\varphi} \mapsto \overline{i(e_0)\varphi}\,.$$

Using the above formulas for pairings, we can easily prove the correctness of the map

$$\mathfrak{B}'\colon H^{*-1}(\mathfrak{L}_0, L_0, 1_t) \to H^*(\mathfrak{L}_0, 1_t)^{L_0},$$

given by $\mathfrak{B}'\colon \bar{\varphi} \to \overline{\beta \cup \varphi}$, and to show that this is a splitting map for $\mathfrak{B}$. Another obvious argument computes the kernel of $\mathfrak{B}$:

$$\operatorname{Ker} \mathfrak{B} = \left\langle \bar{\varphi} \in H^*(\mathfrak{L}_0, 1_t)^{L_0} \,|\, i(e_0)\varphi = 0 \right\rangle \cong H^*(\mathfrak{L}_1, 1_t)^{L_0}.$$

Now the proof of Lemma 4 is complete.

From Lemma 3 we know that the relative cohomology $H^*(L, L_{-1}, U_t)$ is a direct summand in $H^*(L, U_t)$. The complement is described in what follows.

LEMMA 5. *The following space isomorphism holds*:

$$H^*(L, U_t) \cong H^*(L_{-1}, U) \otimes H^*(L, L_{-1}, U_t), \qquad t \in P.$$

In particular,

$$H^k(L, U_t) \cong H^k(L, L_{-1}, U_t) \oplus H^{k-1}(L, L_{-1}, U_t), \qquad k \geqslant 0.$$

PROOF. We introduce an endomorphism of the space U_t denoted by $\int$ (the "integral" map). We put

$$\int x^{(i)} = x^{(i+1)} \quad (0 \leqslant i < p^n - 1), \qquad \int x^{(p^n-1)} = 0,$$

and then extend the map linearly to the whole of U_t. The motivation of our notation is the following. It is obvious that $\partial \int u = u$ if $\mathrm{pr}_{\langle x(p^n-1)\rangle} u = 0$; that is, $\int$ is an "almost" inverse operation to taking a derivative $\partial\colon U_t \to U_t$.

Let $\bar{\psi}$ be a cohomology class in $H^k(L, U_t)$. We prove the existence of a representative $\psi \in Z^k(L, U_t)$ in $\bar{\psi}$ such that the following normalization condition holds:

$$\psi(e_{-1}, l_1, \ldots, l_{k-1}) = \lambda(l_1, \ldots, l_{k-1}) x^{(p^n-1)}, \tag{8}$$

where $\lambda \in C^{k-1}(\mathfrak{L}_0, P)$. Let ψ' be a representative in $\bar{\psi}$ such that the above condition is violated. We must find a coboundary $d\omega \in B^k(L, U_t)$ such that $\psi = \psi' - d\omega$ satisfies (8). To do this we will construct elements $\omega(l_1, \ldots, l_{k-1})$, $l_1, \ldots, l_{k-1} \in L$ being homogeneous elements, by induction on the number $q = |l_1| + \cdots + |l_{k-1}|$. In this way we can construct a cochain $\omega \in C^{k-1}(L, U_t)$. We put $i(e_{-1})\omega = 0$. This, in particular, gives a basis for the above induction.

Now we suppose that for $q - 1$ all the elements $\omega(l_1, \ldots, l_{k-1})$ have been constructed. Let $l_1, \ldots, l_{k-1} \in \mathfrak{L}_0$ be linearly independent elements with $|l_1| + \cdots + |l_{k-1}| = q$. Then

$$d\omega(e_{-1}, l_1, \ldots, l_{k-1}) = e_{-1}(\omega(l_1, \ldots, l_{k-1})) + a,$$

where $a = \Sigma_i (-1)^i \omega([e_{-1}, l_i], l_1, \ldots, \hat{l}_i, \ldots, l_{k-1})$ is a well-determined (by the induction hypothesis) element in U_t. It remains to put

$$\omega(l_1, \ldots, l_{k-1}) = \int (\psi(e_{-1}, l_1, \ldots, l_{k-1}) - a).$$

The induction step is proved. Thus condition (8) can be satisfied.

Now we prove that

$$\lambda \in Z^{k-1}(\mathfrak{L}_0, 1_t).$$

Let $\lambda = \sum_{r\geq 0} \lambda_r$, where $\lambda_r \in C_r^{k-1}(\mathfrak{L}_0, 1_t)$. By induction on r we prove that $d\lambda_r = 0$. Since the cocycle space $Z^{k-1}(\mathfrak{L}_0, 1_t)$ is finite dimensional, this will give $d\lambda = 0$.

There is nothing to prove if $r < 0$. Suppose for $r - 1$ our statement is true. Then the condition of being a cocycle $d\psi(e_{-1}, l_1, \ldots, l_k) = 0$, where $|l_1| + \cdots + |l_k| = r$, can be rewritten as follows:

$$\sum_{i=1}^{k} \sum_{|l_i|=0} (-1)^{i+1} (l_i)_{U_t}(\psi(e_{-1}, l_1, \ldots, \hat{l}_i, \ldots, l_k))$$
$$+ \sum_{i=1}^{k} \sum_{|l_i|=0} (-1)^{i} \psi([l_i, e_{-1}], l_1, \ldots, \hat{l}_i, \ldots, l_k)$$
$$+ (i(e_{-1})\psi)'(l_1, \ldots, l_k) = e_{-1}(\psi(l_1, \ldots, l_k)).$$

Since $(e_0)_{U_t}(x^{(p^n-1)}) = -x^{(p^n-1)} + tx^{(p^n-1)}$ and $[e_0, e_{-1}] = -e_{-1}$, this condition and the normalizing condition (8) give

$$\sum_{i=1}^{k} \sum_{|l_i|=0} (-1)^{i+1} ((l_i)_{\langle 1_t \rangle} \lambda_r(l_1, \ldots, \hat{l}_i, \ldots, l_k)) x^{(p^n-1)}$$
$$+ \lambda'_r(l_1, \ldots, l_k) x^{(p^n-1)} = e_{-1}(\psi(l_1, \ldots, l_k)).$$

Using the expression for the coboundary d in the cochain complex $C^*(\mathfrak{L}_0, 1_t)$, we can rewrite this as

$$(d\lambda(l_1, \ldots, l_k)) x^{(p^n-1)} = e_{-1}(\psi(l_1, \ldots, l_k)).$$

It is obvious that this is possible only if

$$d\lambda(l_1, \ldots, l_k) = 0, \quad e_{-1}(\psi(l_1, \ldots, l_k)) = 0.$$

The induction step is proved. Hence $\lambda \in Z^{k-1}(\mathfrak{L}_0, 1_t)$.

Using the pairing formula for the cocycle α, it is easy to verify that $\alpha \cup \mathfrak{A}'\lambda = \alpha \cup \lambda$ if $\lambda \in C^{k-1}(\mathfrak{L}_0, 1_t)$. Hence each cocycle $\psi \in Z^k(L, U_t)$ can be represented in the form $\psi = \alpha \cup \mathfrak{A}'\lambda + \varphi$, where $\lambda \in Z^{k-1}(\mathfrak{L}_0, 1_t)$ and $\varphi \in Z^k(L, U_t)$. We remark that $i(e_{-1})(\psi - \alpha \cup \lambda) = 0$; that is, $i(e_{-1})\varphi = 0$. Then, by (2),

$$\theta(e_{-1})\varphi = d(i(e_{-1})\varphi) + i(e_{-1})(d\varphi) = 0.$$

In other words, $\varphi \in Z^k(L, L_{-1}, U_t)$. Thus, in every class of cocycles belonging to the complement of the relative cohomology space $H^k(L, L_{-1}, U_t)$ in $H^k(L, U_t)$ there is a cocycle ψ representable in the form $\psi = \alpha \cup \mathfrak{A}'\lambda$, $\lambda \in Z^{k-1}(\mathfrak{L}_0, 1_t)$. It is obvious that if the class of λ is nontrivial, the same is true of ψ.

To finish we have to prove that if $\psi = d\omega$ is a coboundary then the same is true of λ. So, we assume that

$$\alpha \cup \mathfrak{A}'\lambda = d\omega. \tag{9}$$

We proceed by induction over the number of arguments $k = 0, 1, 2, \ldots$. There is nothing to do if $k = 0$. Assuming the statement true for $k - 1$, we prove it for k. According to

Proposition 1 the cohomology of $C^*(L, M)$ is isomorphic to that of $C^*(L, M)^{L_0}$, the subcomplex of invariants under the action of the torus L_0. Thus, in addition to (9) we may assume that

$$i(e_0)\omega = 0, \quad \theta(e_0)\omega = 0.$$

Then from (9), using (2), we find that

$$\alpha \cup i(e_0)\mathcal{C}'\lambda = -d(i(e_0)\omega),$$

(because $i(e_0)(\alpha \cup \mathcal{C}'\lambda) = \alpha \cup (i(e_0)\mathcal{C}'\lambda)$). Since $i(e_0)\mathcal{C}'\lambda \in Z^{k-2}(L, L_{-1}, U_t)$ by the induction hypothesis, for some $\sigma \in C^{k-3}(L, L_{-1}, U_t)$ we have $i(e_0)\mathcal{C}'\lambda = d\sigma$. Then, by Lemma 4,

$$\mathcal{C}'\lambda = -d(\beta \cup \sigma) + \mathcal{C}'\tilde{\lambda}$$

for some $\tilde{\lambda} \in Z^{k-1}(\mathfrak{L}_0, L_0, 1_t)$. Thus (9) can be rewritten in the form

$$\alpha \cup A'\tilde{\lambda} = d\kappa, \tag{10}$$

where $\kappa = \omega - \alpha \cup (\beta \cup \sigma) \in C^{k-1}(L, U_t)$.

The following implication is true:

$$i(e_0)\tilde{\lambda} = 0, \quad \theta(e_0)\tilde{\lambda} = 0 \Rightarrow i(e_0)(\alpha \cup \mathcal{C}'\tilde{\lambda}) = 0, \quad \theta(e_0)(\alpha \cup \mathcal{C}'\tilde{\lambda}) = 0.$$

Hence $i(e_0)d\kappa = 0$ and $\theta(e_0)d\kappa = 0$. By Proposition 1 we may restrict ourselves to the case where $\theta(e_0)\kappa = 0$. Then $d(i(e_0)\kappa) = 0$. In other words, $\kappa = \beta \cup i(e_0)\kappa + \tilde{\omega}$ for some $\tilde{\omega} \in C^{k-1}(L, U_t)$ such that $i(e_0)\tilde{\omega} = 0$, $\theta(e_0)\tilde{\omega} = 0$ and $d\tilde{\omega} = d\kappa$. Thus (10), and hence (9), can be rewritten in the form

$$\alpha \cup \mathcal{C}'\tilde{\lambda} = d\tilde{\omega}, \tag{11}$$

where $i(e_0)\tilde{\omega} = 0$, $\theta(e_0)\tilde{\omega} = 0$ and $\tilde{\lambda} \in Z^{k-1}(\mathfrak{L}_0, L_0, 1_t)$, the difference of λ and $\tilde{\lambda}$ being a coboundary. Now we remark that in proving the normalizing condition (8) we did not use the fact that ψ is a cocycle. Hence we may apply the same procedure for the cochain $\tilde{\omega}$.

Hence we may assume that the following normalizing condition holds:

$$i(e_{-1})\tilde{\omega} = \mu \cup x^{(p^n-1)}, \qquad \mu \in C^{k-2}(\mathfrak{L}_0, 1_t). \tag{12}$$

We recall the formula for pairing the cochain μ and the 0-cochain $x^{(p^n-1)}$:

$$\left(\mu \cup x^{(p^n-1)}\right)(l_1, \ldots, l_{k-2}) = \mu(l_1, \ldots, l_{k-2})x^{(p^n-1)}.$$

Moreover, we can have our previous normalizing condition:

$$i(e_0)\tilde{\omega} = 0, \quad \theta(e_0)\tilde{\omega} = 0. \tag{13}$$

Then from (11) and (12), using (2), we find that

$$\mathcal{C}'\tilde{\lambda} \cup x^{(p^n-1)} + d\left(\mu \cup x^{(p^n-1)}\right) = \theta(e_{-1})\tilde{\omega} \tag{14}$$

(because $i(e_{-1})(\alpha \cup \mathcal{C}'\tilde{\lambda}) = \mathcal{C}'\tilde{\lambda} \cup x^{(p^n-1)}$).

Now we derive from (14) the fact that $\tilde{\lambda} + d\mu = 0$. Clearly this will complete our induction over k and then the whole proof of the lemma. We will verify that

$$(\tilde{\lambda} + d\mu)(l_1, \ldots, l_{k-1}) = 0$$

for an arbitrary choice of linearly independent homogeneous elements $l_1, \ldots, l_{k-1} \in \mathfrak{L}_0$. We will proceed by induction over $q = |l_1| + \cdots + |l_{k-1}|$. Our assertion is true if at least one of the elements $l_1, \ldots, l_{k-1}$ is in the torus L_0. This was essentially verified above (we recall (13), and also the fact that $\tilde{\lambda} \in Z^{k-1}(\mathfrak{L}_0, L_0, 1_t))$. This, in particular, forms the basis for the induction. Now we assume that the assertion is true for $q-1$, and prove it for q. As we noted above, $l_1, \ldots, l_{k-1}$ may be taken in $\mathfrak{L}_1$.

So, we consider restrictions of the cochains $\tilde{\lambda}$, μ and $\tilde{\omega}$ to $\mathfrak{L}_1$. It is easily seen that

$$d\left(\mu \cup x^{(p^n-1)}\right) = d\mu \cup x^{(p^n-1)}.$$

Thus (14) is equivalent to the system of equations

$$(\tilde{\lambda} + d\mu)(e_{i_1}, \ldots, e_{i_{k-1}})x^{(p^n-1)} = e_{-1}\left(\tilde{\omega}(e_{i_1}, \ldots, e_{i_{k-1}})\right) + \sum_{j=1}^{k-1} (-1)^j \tilde{\omega}\left(e_{i_j-1}, e_{i_1}, \ldots, \hat{e}_{i_j}, \ldots, e_{i_{k-1}}\right), \quad (15)$$

where $e_{i_j} \in L_{i_j}$, $i_j > 0$, $j = 1, \ldots, k-1$. Let $A_{i_1, \ldots, i_{k-1}} \in P$ denote the coefficient at the basis monomial $x^{(p^n-q+i_1+\cdots+i_{k-1})}$ in the representation of $\omega(e_{i_1}, \ldots, e_{i_{k-1}})$ in the form of a linear combination of basic vectors of U_t. Now we derive conditions imposed on these coefficients by equations (15). It is easy to verify that we get the following equations, each corresponding to a certain choice of $(i_1, \ldots, i_{k-1})$:

$$(\tilde{\lambda} + d\mu)(e_{i_1}, \ldots, e_{i_{k-1}}) + \sum_{j=1}^{k-1} (-1)^j A_{i_j-1, i_1, \ldots, \hat{i}_j, \ldots, i_{k-1}},$$

if $i_1 + \cdots + i_{k-1} = q$. By the induction hypothesis we have

$$0 = A_{i_1, \ldots, i_{k-1}} + \sum_{j=1}^{k-1} (-1)^j A_{i_j-1, \ldots, \hat{i}_j, \ldots, i_{k-1}},$$

if $i_1 + \cdots + i_{k-1} < q$. Now we say that the collections $(i_1 - 1, i_2, \ldots, i_{k-1})$, $(i_1, i_2 - 1, \ldots, i_{k-1}), \ldots, (i_1, i_2, \ldots, i_{k-1} - 1)$ are *associated with* the collection $(i_1, \ldots, i_{k-1})$. If $(i_1, \ldots, i_{k-1})$ and $(i_1', \ldots, i_{k-1}')$ are associated with each other and the same is true of $(i_1', \ldots, i_{k-1}')$ and $(i_1'', \ldots, i_{k-1}'')$, then we say that $(i_1, \ldots, i_{k-1})$ and $(i_1'', \ldots, i_{k-1}'')$ are also associated. Let $(i_1, \ldots, i_{k-1})$ be an arbitrary but fixed collection such that $i_1 + \cdots + i_{k-1} = q$ and $0 < i_1, \ldots, 0 < i_{k-1}$. We pick up all equations corresponding to the collections $(i_1', \ldots, i_{k-1}')$ associated with $(i_1, \ldots, i_{k-1})$ and such that $i_1' + \cdots + i_{k-1}' = q' < q$, and add them. Then all the right-hand side terms are annihilated, and, in the left-hand side, only one term remains, namely, $(\tilde{\lambda} + d\mu)(e_{i_1}, \ldots, e_{i_{k-1}})$. Thus the induction argument is complete, proving the whole of the lemma.

Theorem 5 follows immediately from Lemmas 4 and 5.

It should be remarked that our proof lets us effectively construct bases of the cohomology space $H^*(L, U_t)$ once we are given a basis of $H^*(\mathfrak{L}_1, 1_t)^{L_0}$. Indeed, if the class of ψ is basic in $H^*(\mathfrak{L}_1, 1_t)^{L_0}$, then the classes $\mathcal{C}'\psi$, $\alpha \cup \mathcal{C}'\psi$, $\beta \cup \mathcal{C}'\psi$ and $\alpha \cup (\beta \cup \mathcal{C}'\psi)$ are basic in $H^*(L, U_t)$ and span it. Now we give an alternative statement of Theorem 5.

THEOREM 5'. *Let L be the Zassenhaus algebra $W_1(n)$, and let $t \in P$. The cohomology space $H^*(L, U_t)$ can be represented as the tensor product of its subspaces*

$$H^*(L, U_t) \cong H^*(L_{-1}, U) \otimes H^*(L_0) \otimes \left(H^*(\mathfrak{L}_1) \otimes \langle 1_t \rangle\right)^{L_0}.$$

We remark that $H^1(L) = L/[L, L] = 0$ whereas the dimension of the space

$$H^1(\mathfrak{L}_1) \cong \mathfrak{L}_1/[\mathfrak{L}_1, \mathfrak{L}_1] \cong \langle e_1, e_2, e_{p^k-1} | 0 < k < n \rangle$$

is $n + 1$. Since, by Theorem 5, the 1-cohomology space has a decomposition

$$H^1(L, U_t) \cong \left(H^1(L_{-1}, U) \oplus H^1(L_0)\right) \otimes \langle 1_t \rangle^{L_0} \oplus \left(H^1(\mathfrak{L}_1) \otimes \langle 1_t \rangle\right)^{L_0},$$

we immediately derive the following result.

COROLLARY 2. *Let $L = W_1(n)$ and $t \in P$. Then $H^0(L, U_t)$ is trivial if $t \neq 0$. The 1-cohomology space $H^1(L, U_t)$ is trivial with the following exceptions:*

$$H^1(L, U) \cong H^1(L_{-1}, U) \oplus H^1(L_0) \quad \textit{has dimension } 2 \textit{ if } t = 0;$$

$$H^1(L, U_1) \cong \langle e_1 \rangle \quad \textit{has dimension } 1 \textit{ if } t = 1;$$

$$H^1(L, U_2) \cong \langle e_1 \rangle \quad \textit{has dimension } 1 \textit{ if } t = 2;$$

$$H^1(L, U_{-1}) \cong \left\langle e_{p^k - 1} \,|\, 0 < k < n \right\rangle \quad \textit{has dimension } n - 1 \textit{ if } t = -1.$$

It should be remarked that U_{-1} is an L-module isomorphic to the adjoint L-module. Thus, in the latter case we deal with a well-known result saying that the outer derivation space $H^1(L, L)$ is generated by derivations in the set $\{\partial^{p^k} \,|\, 0 < k < n\}$, and, in particular, it is $(n - 1)$-dimensional. Now if $t = 0$, then we see that the classes of the cocycles α and β are linearly independent, proving what was promised above.

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BIBLIOGRAPHY

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chaps. 1–3, Actualités Sci. Indust., nos. 1285, 1349, Hermann, Paris, 1971, 1972; English transl., Hermann, Paris, and Addison-Wesley, Reading, Mass., 1975.

2. J. Dixmier, *Cohomologie des algèbres de Lie nilpotentes*, Acta Sci. Math. (Szeged) **16** (1955), 246–250.

3. G. Hochschild and J.-P. Serre, *Cohomology of Lie algebras*, Ann. of Math. (2) **57** (1953), 591–603.

4. A. I. Kostrikin and I. R. Shafarevich, *Graded Lie algebras of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 251–322; English transl. in Math. USSR Izv. **3** (1969).

5. A. N. Rudakov and I. R. Shafarevich, *Irreducible representations of a simple three-dimensional Lie algebra over a field of finite characteristic*, Mat. Zametki **2** (1967), 439–454; English transl. in Math. Notes **2** (1967).

6. A. A. Mil'ner, *Irreducible representations of the Zassenhaus algebra*, Uspekhi Mat. Nauk **30** (1975), no. 6, 178. (Russian)

7. Richard Block, *New simple Lie algebras of prime characteristic*, Trans. Amer. Math. Soc. **89** (1958), 421–449.

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