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Simple Lie algebras with a subalgebra of codimension one

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We study Lie algebras over a field P of positive characteristic p. The general Lie algebra of Cartan type

$$W_1(m) = \langle e_i \mid -1 \leqslant i \leqslant r = p^m - 2\rangle, \quad [e_i, e_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) e_{i+j}$$

is simple for $p > 2$. When $p = 2$, it has an ideal of codimension 1: $\overline{W}_1(m) = \langle e_i \mid -1 \leqslant i \leqslant r = 2^m - 3\rangle$ (see [1]). By a *Zassenhaus algebra* we mean the simple Lie algebra $W_1(m)$ $(p > 2)$, $\overline{W}_1(m)$ $(p = 2)$.

Theorem. *Every simple Lie algebra with a subalgebra of codimension* 1 *over a perfect field of characgeristic* $p > 0$ *is isomorphic to a three-dimensional Lie algebra of type* A_1 *or to a Zassenhaus algebra.*

With a filtered Lie algebra $\mathscr{L} = \mathscr{L}_{-1} \supset \mathscr{L}_0 \supset \dots \supset \mathscr{L}_r$ with multiplication $\{\ ,\ \}$ we can associate the graded algebra $L = \bigoplus_{i=-1}^{r-1} L_i$, $L_i \cong \mathscr{L}_i/\mathscr{L}_{i+1}$, in which the multiplication is given by the rule $[\bar{x}, \bar{y}] = \overline{\{x, y\}}/\mathscr{L}_{i+j+1}$ for $\bar{x} \in L_i$, $\bar{y} \in L_j$. In such cases we say that $\mathscr{L}$ is an $\{L_i\}$*-deformation* of L. After identification of L_i with $\mathscr{L}_i/\mathscr{L}_{i+1}$ the multiplication in $\mathscr{L}$ can be defined by the rule $\{x, y\} = [x, y] + \sum_{q \geqslant 1} \psi_q(x, y)$, where $\psi_q : L \times L \to L$ is a bilinear skew-symmetric map. Then the multiplication $\{\ ,\ \}_t = [\ ,\ ] + \sum_{q \geqslant 1} t^q \psi_q$ defines a deformation of L, and $\{\ ,\ \}_{t=1} = \{\ ,\ \}$. Every simple Lie algebra with a subalgebra of codimension 1 except A_1 is isomorphic to some $\{L_i\}$-deformation of a Zassenhaus algebra (Kostrikin). It follows from the results of [2] and [3] that such algebras are isomorphic to one of the following algebras, depending on parameters $\varepsilon_1, \dots, \varepsilon_{m-1} \in P$:

$$\mathscr{L} = W_1(m, \varepsilon), \quad p \geqslant 2, \qquad \{e_i, e_j\} = [e_i, e_j] + \sum_{k=1}^{m-1} \varepsilon_k(\delta_{i,-1}\delta_{j,p^k-1} - \delta_{j,-1}\delta_{i,p^k-1}) e_{p^m-2},$$

$$\mathscr{L} = \overline{W}_1(m, \varepsilon), \quad p = 2, \qquad \{e_i, e_j\} = [e_i, e_j] + \sum_{k=1}^{m-1} \varepsilon_k(\delta_{i,-1}\delta_{j,2^k-2} - \delta_{j,-1}\delta_{i,2^k-2}) e_{2^m-3},$$

where $\delta_{i,j}$ is the Kronecker delta. It remains to construct an isomorphism $\mathscr{L} \cong L$. To do this we find an element $E \in \mathscr{L}$ such that $(\operatorname{ad} E)^{r+2} = 0$, $(\operatorname{ad} E)^{r+1} \neq 0$. Then the map $e_i \mapsto (\operatorname{ad} E)^{r-i}(e_r)$, $-1 \leqslant i \leqslant r$, gives the required isomorphism.

Let $E = e_{-1} + \sum_{i=1}^{m} \lambda_i e_{p^i-2} \in W_1(m, \varepsilon)$. We claim that

$$(\operatorname{ad} E)^{p^k} = \sum_{i=0}^{k} (\mu_i \operatorname{ad} e_{-1})^{p^{k-i}} + \operatorname{ad} x(k), \quad x(k) \in \mathscr{L}_0 \cong \bigoplus_{i=0}^{r} \langle e_i \rangle, \tag{1}$$

where $\mu_i \in P$ has the form $\mu_i = \lambda_i + f_i$, with $f_i = f_i(\lambda_1, \dots, \lambda_{i-1})$ independent of $\lambda_i, \dots, \lambda_m$. For $k = 0$ this is obvious. Suppose that (1) holds for $k-1$. If $x \in \mathscr{L}_0$, then $(\operatorname{ad} x)^p$ is an inner derivation. Therefore, by Jacobson's formula,

$$(\operatorname{ad} E)^{p^k} = \left(\sum_{i=0}^{k-1} (\mu_i \operatorname{ad} e_{-1})^{p^{k-i-1}} + \operatorname{ad} x\ (k-1)\right)^p =$$

$$= \sum_{i=0}^{k-1} (\mu_i \operatorname{ad} e_{-1})^{p^{k-i}} + (\lambda_k + f_k) \operatorname{ad} e_{-1} + \operatorname{ad} x(k),$$

where $x(k) \in \mathcal{L}_0$, $f_k \in P$ and $f_k \in P$ depends on $\mu_1, \ldots, \mu_{k-1}$, hence, by the inductive hypothesis, f_k depends only on $\lambda_1, \ldots, \lambda_{k-1}$.

Note that $(\mathrm{ad}\ e_{-1})^{p^m}(e_{p^m-2}) = \sum_{k=1}^{m-1} \varepsilon_k e_{p^m-p^k-2}$. Substituting this in (1) with $k = m$, we see that the condition $(\mathrm{ad}\ E)^{r+2}(e_{p^m-2}) = 0$ is equivalent to

$$\mu_i^{p^i} + \varepsilon_{m-i} = 0, \quad 1 \leqslant i \leqslant m-1, \quad \mu_m = 0, \quad \mu = 0, \tag{2}$$

where $[x(m), e_{p^m-2}] = \mu e_{p^m-3}$. Clearly, the system of equations (2) except the last one has a solution for $\lambda_1, \ldots, \lambda_m \in P$ if P is perfect. Since in $\mathcal{L} = W_1(m, \varepsilon)$ we can choose the basis $\{(\mathrm{ad}\ E)^i(e_r) \mid 0 \leqslant i \leqslant r+1\}$, we find that $(\mathrm{ad}\ E)^{p^m}(x) = \mu x$ for all $x \in L$. In particular for $x = E$ we conclude from this condition that $\mu = 0$. Thus, we have proved that $W_1(m, \varepsilon) \cong W_1(m)$ for $p \geqslant 3$.

When $L = \overline{W}_1(m, \varepsilon)$, we extend the algebra by adjoining a new basis vector e_{2^m-2} and defining $[e_i, e_{2^m-2}] = \delta_{i,-1} e_{2^m-3}$. Then $(\mathrm{ad}\ E)^{2^m-1}(e_{2^m-3}) = (\mathrm{ad}\ E)^{2^m}(e_{2^m-2}) = \mu e_{2^m-3}$ and all the above arguments carry over except for one: in (2), the equation $\mu_m = 0$ is insoluble, since $\lambda_m = 0$.

Remark. In the proof of Theorem 3.9 a) in [4] it is asserted that there is an ad-nilpotent element of the form $E' = e_{-1} + \lambda_0 e_0 + \sum_{i=1}^{m-1} \lambda_i e_{p^m-p^i-2} \in W_1(m, \varepsilon)$. This is false, in general, for $m > 2$, since

$$(\mathrm{ad}\ E')^{p^m} = \sum_{i=1}^{m} \lambda_0^{p^m-p^i}(\mathrm{ad}\ e_{-1})^{p^i} + \mathrm{ad}\ x, \quad x \in L, \quad \lambda_0 \in P$$

(the analogue of (1), above), and the $(m-1)$ equations $\lambda_0^{p^m-p^i} + \varepsilon_i = 0$, $1 \leqslant i \leqslant m-1$, in the single variable λ_0 (the analogue of (2)) are insoluble. More detailed calculations show that E' is ad-nilpotent only when $\lambda_i = 0$ for $1 \leqslant i \leqslant m-1$.

This error in the proof of 3.9 a) in [4] was noted earlier in [5], where the theorem is proved in the case of algebraically closed fields of characteristic $p \geqslant 3$, and an example is given to show that it is false in the case of a field that is not algebraically closed.

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