GENERALIZED CASIMIR ELEMENTS

1986 Math. USSR Izv. 27 391

(http://iopscience.iop.org/0025-5726/27/2/A09)

Изв. Акад. Наук СССР
Сер. Матем. Том 49(1985), № 5

Math. USSR Izvestiya
Vol. 27(1986), No. 2

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UDC 512.554.342

A. S. DZHUMADIL'DAEV

ABSTRACT. A method is given for constructing a central element in the universal enveloping algebra $\mathfrak{A}(L)$ of a Lie algebra L, generalizing the method for constructing a Casimir element and not requiring the existence of a nondegenerate invariant form on L. Generalized Casimir elements are constructed for certain Lie algebras of Cartan types over a field of positive characteristic.

Bibliography: 15 titles.

In the calculation of the center $\mathfrak{Z}(L)$ of the universal enveloping algebra $\mathfrak{A}(L)$ of a Lie algebra L an important role is played by methods of constructing central elements in $\mathfrak{Z}(L)$. The following method of constructing a central element in the case where L possesses a nondegenerate invariant form (,) is well known. If $V(L) = \{e_i | i \in I\}$ is a basis in L and $V(L) = \{e'_i \in L | (e'_i, e_j) = \delta_{i,j},\ i,j \in I\}$ is a dual basis, then the Casimir element $\sum_i e_i e'_i$ is central. In this paper we give a generalization of this construction that does not require the existence of a nondegenerate invariant form on L. Interesting examples of generalized Casimir elements arise in the case of simple Lie algebras of positive characteristic. Recall that almost all nonclassical simple Lie algebras known up to now are Lie algebras of Cartan types [1]. For many nonclassical simple Lie algebras every invariant form is degenerate. Nevertheless, among them occur algebras possessing nontrivial central elements. Such, for example, is the Zassenhaus algebra (among the nonclassical Lie algebras, $W_1(m)$ is the only one for which the center $\mathfrak{Z}(W_1(m))$ has been described completely; see [2]–[7]). We will give a simple method for constructing nontrivial central elements in terms of generalized Casimir elements both for the Zassenhaus algebra and for some other Lie algebras of Cartan types.

§1. A generalized Casimir element

We endow $\mathfrak{A}(L)$ with the structure of an adjoint L-module. Suppose M is an L-module relative to the representation $x \to (x)_M$, M' is a dual L-module, $\mathcal{B}: M \times M' \to P$ is the natural pairing, $V(M) = \{v_i | i \in I\}$ is a basis in M, and $V(M') = \{v'_i | \mathcal{B}(v_i, v'_j) = \delta_{i,j},\ i,j \in I\}$, where $\delta_{i,j}$ is the Kronecker symbol, is a dual basis. Let $M^Q = \langle v \in M | x(v) = 0,\ x \in Q\rangle$ be the space of invariants relative to the subalgebra Q (here and below, in notation like $(x)_M v$ the symbol M is omitted). A Lie algebra L will be called *Casimir* (more precisely, *M-Casimir*) if there exist L-module homomorphisms $F: M \to \mathfrak{A}(L)$ and $F': M' \to \mathfrak{A}(L)$. Then these homomorphisms can be extended to an L-module homomorphism

$$F \otimes F': M \otimes M' \cong \operatorname{End} M \to \mathfrak{A}(L), \qquad v \otimes v' \mapsto F(v)F'(v'),$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 17B35.

where $(\operatorname{End} M)^L$ is mapped into $\mathfrak{Z}(L)$. The element

$$c = \sum_i F(v_i)F'(v_i') \in \mathfrak{Z}(L)$$

corresponding to the identity endomorphism in M (it does not depend on the choice of basis in M and M') is called a *generalized* (more precisely, *M-generalized*) *Casimir element.*

The L-module $\mathfrak{A}(L)$ contains the L-submodule L in a natural way. Therefore if the L-module $A(L)$ contains a submodule isomorphic to the coadjoint module L', then, identifying it with L', we can consider the L-generalized Casimir element

$$\sum_i e_i e_i' \in \mathfrak{Z}(L),$$

where $e_i' \in \mathfrak{A}(L)$ and $\mathcal{B}(e_i, e_j') = \delta_{i,j}$. In particular, if L possesses a nondegenerate invariant form, then the adjoint L-module is isomorphic to the coadjoint module and this element is a Casimir element. Assume that the characteristic p of the ground field P is positive, and let $R(L)$ be a minimal p-hull of the Lie algebra L [8]. There is a natural embedding of the L-module $R(L)$ into $\mathfrak{A}(L)$. Therefore if the L-module $\mathfrak{A}(L)$ contains an L-submodule isomorphic to $R(L')$, we can construct $R(L)$-generalized Casimir elements.

If $p > 0$, the algebra $\mathfrak{Z}(L)$ contains the subalgebra $\mathfrak{Z}_0(L) = \langle x^p - x^{[p]} | x \in R(L)\rangle$. We will call $z \in \mathfrak{Z}(L)$ a nontrivial central element if $z \notin \mathfrak{Z}_0(L)$. Let $\tilde{\mathfrak{Z}}(L)$ and $\tilde{\mathfrak{Z}}_0(L)$ be the fields of fractions of the rings $\mathfrak{Z}(L)$ and $\mathfrak{Z}_0(L)$.

PROPOSITION 1. *If L is a finite-dimensional Casimir Lie algebra, then there exists a finite set of generalized Casimir elements generating $\tilde{\mathfrak{Z}}(L)$ over $\tilde{\mathfrak{Z}}_0(L)$.*

PROOF. Suppose $c = \sum_i F(v_i)F(v_i')$ is a generalized Casimir element. Note that for any $z \in \mathfrak{Z}(L)$ the element $zc = \sum_i (zF(v_i))F'(v_i')$ is also a generalized Casimir element. It is known that the rank $t = [\tilde{\mathfrak{Z}}(L): \tilde{\mathfrak{Z}}_0(L)] < \infty$ (see [9]). Suppose $z_1, \dots, z_t \in \mathfrak{Z}(L)$ generate the field $\tilde{\mathfrak{Z}}(L)$ over $\tilde{\mathfrak{Z}}_0(L)$. Then the new generators $cz_1, \dots, cz_t$ are generalized Casimir elements.

CONJECTURE. *Suppose L is a finite-dimensional Lie algebra over an algebraically closed field P of characteristic $p > 5$. If L is Casimir, then as generators of $\mathfrak{Z}(L)$ over $\mathfrak{Z}_0(L)$ we can take a finite set of generalized Casimir elements. Moreover, if $\mathfrak{Z}(L)$ contains a nontrivial central element, then L is Casimir.*

Let $\overline{\mathfrak{A}}(L) = \mathfrak{A}(R(L))/\mathfrak{Z}_0(L)$ be the restricted universal enveloping algebra of the Lie p-algebra $R(L)$; we will call it the *u-universal enveloping algebra* of the Lie algebra L. Let $\overline{\mathfrak{Z}}(L)$ be the center of $\overline{\mathfrak{A}}(L)$. If there exist L-module homomorphisms $\overline{F}: M \to \overline{\mathfrak{A}}(L)$ and $\overline{F}': M' \to \overline{\mathfrak{A}}(L)$, then, as above, the central element $c = \sum_i \overline{F}(v_i)F'(v_i') \in \overline{\mathfrak{Z}}(L)$ is called an M-generalized Casimir u-element (in the future the M and u will often be omitted).

Suppose the elements $l_1, \dots, l_n$ form a basis in $R(L)$. Then in $\overline{\mathfrak{A}}(L)$ we can choose the basis $\{l^\alpha = \prod_i l_i^{\alpha_i} | 0 \le \alpha_i < p,\ i = 1, \dots, n\}$. Let $e = (p-1, \dots, p-1)$. We define a linear mapping $\varphi: \overline{\mathfrak{A}}(L) \to P$ by the rule $\varphi: l^\alpha \mapsto 0$, $\alpha \ne e$, $\varphi: l^e \mapsto 1$. Assume that for all $l \in R(L)$ we have

$$\operatorname{tr} \operatorname{ad} l = 0. \qquad (*)$$

It was shown in [10] and [11] that the bilinear form

$$\Phi: \overline{\mathfrak{A}}(L) \times \overline{\mathfrak{A}}(L) \to P, \qquad \Phi(u, v) = \varphi(uv),$$

is symmetric, nondegenerate, and L-invariant. Thus we have

PROPOSITION 2. *If a finite-dimensional Lie algebra L satisfies $(*)$, then it is u-Casimir: the L-module $\overline{\mathfrak{A}}(L)$ contains an L-submodule isomorphic to L'. In particular, a Lie algebra that is equal to its commutant is u-Casimir.*

It is obvious that a generalized Casimir u-element of such Lie algebras has the form $\dim R(L)l^e$. In general, $l^e \in \overline{\mathfrak{Z}}(L)$ if $(*)$ holds. Later we will construct more interesting examples of generalized Casimir u-elements.

§2. A nondegenerate invariant form in the space of exterior differential forms

We introduce the following notation: $\langle X\rangle$ is the linear span of the set of vectors X, I is an index set of order $|I|$,

$$\Gamma_I(\boldsymbol{m})\ (=\Gamma_{|I|}(\boldsymbol{m})) = \{\alpha = (\dots,\alpha_i,\dots) | 0 \le \alpha_i < p^{m_i},\ i \in I\},$$

where $\boldsymbol{m} = (\dots, m_i, \dots)$, $m_i \in \mathbf{N}$, and

$$\varepsilon_i = \Big(0,\dots,0,\underset{i}{1},0,\dots,0\Big), \quad e = \sum_i (p^{m_i}-1)\varepsilon_i \in \Gamma_I(\boldsymbol{m}), \quad |\alpha| = \sum_i \alpha_i.$$

Throughout this section, L is a Lie algebra of Cartan type, U is a natural L-module, $l, l_1, \dots \in L$, and $u, v, \dots \in U$. The module U has the structure of a commutative associative algebra. A module M will be called an (L,U)-module if the L-module M has the additional structure of a module over the algebra U and satisfies the condition $l(ua) = l(u)a + ul(a)$, $a \in M$. Recall that in the divided power algebra

$$O_I(\boldsymbol{m})\ (=O_{|I|}(\boldsymbol{m})) = \left\langle x^{(\alpha)} = \prod_i x_i^{(\alpha_i)} \middle| \alpha \in \Gamma_I(\boldsymbol{m}) \right\rangle$$

multiplication is defined by the rule

$$x^{(\alpha)} \cdot x^{(\beta)} = \prod_i \binom{\alpha_i+\beta_i}{\alpha_i} x^{(\alpha+\beta)}.$$

Let

$$\Omega_n^k(\boldsymbol{m}) = \langle x^{(\alpha)}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} | i_1 < \dots < i_k,\ \alpha \in \Gamma_n(\boldsymbol{m})\rangle,$$

the space of exterior differential k-forms, and let $d\colon \Omega_n^k(\boldsymbol{m}) \to \Omega_n^{k+1}(\boldsymbol{m})$ be the exterior differentiation operator in $\Omega_n^*(\boldsymbol{m}) = \bigoplus_k \Omega_n^k(\boldsymbol{m})$. By analogy with the characteristic zero case, the cochain complex $(\Omega_n^*(\boldsymbol{m}), d)$ will be called the *de Rham complex*. Let $Z^k(\Omega) = \langle \omega \in \Omega_n^k(\boldsymbol{m}) | d\omega = 0\rangle$ be the space of closed k-forms (k-cocycles), $B^k(\Omega) = \langle d\omega | \omega \in \Omega_n^{k-1}(\boldsymbol{m})\rangle$ the space of exact k-forms (k-coboundaries), and $H^k(\Omega) = Z^k(\Omega)/B^k(\Omega)$ the k-homology space of the cochain complex $\Omega_n^*(\boldsymbol{m})$ (de Rham k-cohomology). Recall that in the characteristic zero case the Hodge inner product in the space $\Omega^*(\mathfrak{X})$ of differential forms on the variety $\mathfrak{X}$ is given by the bilinear form $\mathcal{A}$ defined by $\mathcal{A}(\omega,\omega') = \int_{\mathfrak{X}}(\omega \wedge *\omega')$. In this section we will construct a modular analogue of this form.

The complex $\Omega_n^*(\boldsymbol{m})$ has the structure of an $(W_n(\boldsymbol{m}), O_n(\boldsymbol{m}))$-module, where

$$W_n(\boldsymbol{m}) = \langle x^{(\alpha)}\partial_i | \alpha \in \Gamma_I(\boldsymbol{m}),\ I = \{1,\dots,n\}\rangle$$

is a general Lie algebra of Cartan type (see [1] for details). Therefore we can impose on $\Omega_n^*(\boldsymbol{m})$ the additional structures of $W_n(\boldsymbol{m})$-modules by means of the rule

$$(l)_t\omega = l(\omega) + t(\operatorname{Div} l)\omega, \qquad t \in P.$$

We denote these modules by $(\Omega_n^*(\boldsymbol{m}))_t$. Exterior multiplication in the Grassmann algebra $\Omega_n^*(\boldsymbol{m})$ is coordinated with the actions of the Lie algebra and the coboundary mapping by means of the following conditions:

$$l(\omega \wedge \omega') = l(\omega) \wedge \omega' + \omega \wedge l(\omega'), \tag{1}$$

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega', \qquad \omega \in \Omega_n^k(\boldsymbol{m}). \tag{2}$$

Consider the mappings

$$\pi\colon \quad O_I(\boldsymbol{m}) \to P, \qquad \pi\colon \quad \sum_{\alpha \in \Gamma_I(\boldsymbol{m})} \lambda_\alpha x^{(\alpha)} \to \lambda_e,$$

$$\sigma\colon \quad \Omega_n^n(\boldsymbol{m}) \to P, \qquad \sigma\colon \quad u dx_1 \wedge \cdots \wedge dx_n \to \pi(u).$$

Note that the space $H^n(\Omega)$ is one-dimensional and is generated by the class of the form $x^{(e)} dx_1 \wedge \cdots \wedge dx_n$. Since the action of $W_n(\boldsymbol{m})$ on $H^*(\Omega)$ is trivial, we have the relation

$$\sigma(l\omega) = 0, \qquad \omega \in \Omega_n^n(m),\ l \in W_n(\boldsymbol{m})$$

(this can be verified directly: $l(u dx_1 \wedge \cdots \wedge dx_n) = (l(u) + (\operatorname{Div} l)u)\, dx_1 \wedge \cdots \wedge dx_n$). It then follows from (1) and (2) that the form $\mathcal{A}\colon \Omega_n^*(\boldsymbol{m}) \times \Omega_n^*(\boldsymbol{m}) \to P$ defined by the rule

$$\mathcal{A}(\omega, \omega') = \sigma(\omega \wedge \omega')$$

is nondegenerate and invariant under $W_n(\boldsymbol{m})$, $O_n(\boldsymbol{m})$, and d:

$$\begin{aligned} &\mathcal{A}(l\omega, \omega') + \mathcal{A}(\omega, l\omega') = 0, \qquad l \in W_n(\boldsymbol{m}), \\ &\mathcal{A}(u\omega, \omega') = \mathcal{A}(\omega, u\omega'), \qquad u \in O_n(\boldsymbol{m}), \end{aligned} \tag{3}$$

$$\mathcal{A}(d\omega, \omega') + (-1)^k \mathcal{A}(\omega, d\omega') = 0. \tag{4}$$

It follows from (3) that the form

$$\mathcal{A}_t\colon \quad (\Omega_n^*(\boldsymbol{m})_t \times (\Omega_n^*(\boldsymbol{m}))_{-t} \to P, \qquad \mathcal{A}_t(\omega, \omega') = \sigma(\omega \wedge \omega'),$$

is also invariant under $W_n(\boldsymbol{m})$.

It is easy to show that the linear mapping

$$\sum_i u_i \partial_i \to \sum_i (-1)^i u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \in \Omega_n^{n-1}(\boldsymbol{m})$$

defines an isomorphism of the $W_n(\boldsymbol{m})$-modules $W_n(\boldsymbol{m})$ and $(\Omega_n^{n-1}(\boldsymbol{m}))_{-1}$ (the roof $\widehat{\ }$ signifies that the corresponding element is omitted). The elements

$$D_{i,j}(u) = \partial_i(u)\partial_j - \partial_j(u)\partial_i \in S_n(\boldsymbol{m}) = \langle l \in W_{n+1}(\boldsymbol{m}) \mid \operatorname{Div} l = 0 \rangle$$

are sent into the differential forms $(-1)^{i+j} d(u dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})$. The simple Lie algebra $S_n(\boldsymbol{m})$ is generated by the elements $D_{i,j}(u)$, $i < j$. Put $B_{n+1}^*(\boldsymbol{m}) = B^*(\Omega_{n+1}(\boldsymbol{m}))$. Then the $S_n(\boldsymbol{m})$-modules $S_n(\boldsymbol{m})$ and $B_{n+1}^n(\boldsymbol{m})$ are isomorphic.

On the other hand, the existence of a nondegenerate invariant form

$$\mathcal{A}_{-1}\colon (\Omega_n^{n-1}(\boldsymbol{m}))_{-1} \times (\Omega_n^{n-1}(\boldsymbol{m}))_1 \to P$$

shows that the $W_n(\boldsymbol{m})$-modules $(\Omega_n^{n-1}(\boldsymbol{m}))_{-1}$ and $(\Omega_n^1(\boldsymbol{m}))_1$ are dual. Thus the $W_n(\boldsymbol{m})$-module $(\Omega_n^1(\boldsymbol{m}))_1$ is isomorphic to the coadjoint module. Also,

$$\mathcal{A}(d\omega, \omega') = 0, \qquad \omega \in \Omega_{n+1}^{n-1}(\boldsymbol{m}),\ \omega' \in Z^1(\Omega_{n+1}(\boldsymbol{m}))$$

(according to (4)). Since $B_{n+1}^2(\boldsymbol{m}) \cong \Omega_{n+1}^1(\boldsymbol{m})/Z^1(\Omega_{n+1}(\boldsymbol{m}))$, this means that the $S_n(\boldsymbol{m})$-modules $B_{n+1}^n(\boldsymbol{m})$ and $B_{n+1}^2(\boldsymbol{m})$ are dual. Consequently, the $S_n(\boldsymbol{m})$-module $B_{n+1}^2(\boldsymbol{m})$ is isomorphic to the coadjoint module.

For the Hamiltonian Lie algebra

$$H_n(\boldsymbol{m}) = \langle x^{(\alpha)} | \alpha \in \Gamma_I(\boldsymbol{m}),\ \alpha \neq 0, e,\ I = \{\pm 1, \ldots, \pm n\} \rangle,$$
$$\{x^{(\alpha)}, x^{(\beta)}\} = -\sum_i \operatorname{sgn} i \partial_i(x^{(\alpha)}) \partial_{-i}(x^{(\beta)}),$$

the natural $H_n(\boldsymbol{m})$-module is isomorphic to the adjoint module: $L = H_n(\boldsymbol{m})$, $U = O_{2n}(\boldsymbol{m})$, and $l(u) = \{l, u\}$. Since $x^{(e)} \notin [L, L]$, the form $\mathcal{B}: L \times L \to P$ defined by

$$\mathcal{B}(l, l_1) = \pi(l l_1)$$

is nondegenerate and invariant, i.e., $H_n(\boldsymbol{m})' \cong H_n(\boldsymbol{m})$.(1)

In the contact Lie algebra

$$K_{n+1}(\boldsymbol{m}) = \langle x^{(\alpha)} | \alpha \in \Gamma_I(\boldsymbol{m}),\ I = \{0, \pm 1, \ldots, \pm n\} \rangle,$$

multiplication is defined by the rule

$$[x^{(\alpha)}, x^{(\beta)}] = \partial_0(x^{(\alpha)}) \Delta(x^{(\beta)}) - \partial_0(x^{(\beta)}) \Delta(x^{(\alpha)}) - \sum_{i \neq 0} \operatorname{sgn} i \partial_i(x^{(\alpha)}) \partial_{-i}(x^{(\beta)}),$$

$$\Delta(x^{(\alpha)}) = \left(2 - \sum_{i \neq 0} \alpha_i\right) x^{(\alpha)}.$$

When $n \equiv -2 \pmod p$, the algebra $K_{n+1}(\boldsymbol{m})$ has an ideal of codimension 1: $\overline{K}_{n+1}(\boldsymbol{m}) = \langle x^{(\alpha)} | \alpha \neq e,\ \alpha \in \Gamma_I(\boldsymbol{m}) \rangle$. Let U_t denote the $K_{n+1}(\boldsymbol{m})$-module defined in the space $U = O_{2n+1}(\boldsymbol{m})$ by the rule

$$(l)_t u = [l, u] + (t - 2)\partial_0(l) u, \qquad t \in P.$$

Note that the adjoint L-module is isomorphic to U_2, and the natural L-module is equal to U_0. It is easy to see that

$$\pi((x^{(\alpha)})_t x^{(\beta)}) = -\delta_{\alpha+\beta, e+\varepsilon_0}(t + 2(n+1))(-1)^{|\alpha|}, \qquad \alpha, \beta \in \Gamma_I(\boldsymbol{m}),$$
$$[l, uv] = [l, u]v + u[l, v] - 2\partial_0(l) uv, \qquad u, v \in U_0,\ l \in L.$$

Consequently for any $l, u \in L$ and $v \in U_t$, we have

$$\pi([l, u]v) + \pi(u((l)_t v)) = \pi([l, uv] + t\partial_0(l) uv) = (t + 2(n+2))\lambda(l, u, v),$$

where $\lambda(l, u, v) \in P$. In other words, the form

$$\mathcal{B}: L \times U_{-2(n+2)} \to P, \qquad \mathcal{B}(l, u) = \pi(lu),$$

is invariant and nondegenerate, i.e., $K_{n+1}(\boldsymbol{m})' \cong (O_{2n+1}(\boldsymbol{m}))_{-2(n+2)}$.

Let us summarize what we have obtained.

PROPOSITION 3. *For any $0 \leq i \leq n$ and $t \in P$ we have a $W_n(\boldsymbol{m})$-module isomorphism $(\Omega_n^i(\boldsymbol{m}))_t' \cong (\Omega_n^{n-i}(\boldsymbol{m}))_{-t}$. For a simple Lie algebra L of Cartan type over a field P of characteristic $p \geq 3$ the coadjoint L-module L' and the corresponding pairing $\mathcal{B}: L \times L' \to P$ are defined by the following rules:*

$$W_n(\boldsymbol{m})' \cong (\Omega_n^1(\boldsymbol{m}))_1, \qquad \mathcal{B}\left(\sum_i u_i \partial_i, \sum_j v_j dx_j\right) = \sum_i \pi(u_i v_i),$$

$$S_n(\boldsymbol{m})' \cong B_{n+1}^2(\boldsymbol{m}), \qquad \mathcal{B}\left(D_{i,j}(u), d\left(\sum_s v_s dx_s\right)\right) = \pi(u(\partial_i(v_j) - \partial_j(v_i)));$$

(1) *Added in proof.* After the present paper was submitted for publication there appeared that of Ya. S. Krylyuk [15], in which were also described the coadjoint representations of Lie p-algebras of Cartan types, except for the contact algebra.

in the cases of Hamiltonian and contact algebras the adjoint and coadjoint modules can be defined over the same space and the pairing $\mathcal{B}$ given by $\mathcal{B}(l,u)=\pi(lu)$:

$$H_n(\boldsymbol{m})'\cong H_n(\boldsymbol{m});$$

$$K_{n+1}(\boldsymbol{m})'\cong (O_{2n+1}(\boldsymbol{m}))_{-2(n+2)},\qquad n\not\equiv -2 \pmod p;$$
$$\overline{K}_{n+1}(\boldsymbol{m})'\cong O_{2n+1}(\boldsymbol{m})/P,\qquad n\equiv -2 \pmod p.$$

COROLLARY. *The simple Lie algebras of Cartan types over a field of characteristic $p\geq 3$ with nondegenerate invariant forms are exhausted by the following list: $W_1(\boldsymbol{m})$ $(p=3)$, $S_2(\boldsymbol{m})$, $H_n(\boldsymbol{m})$, $K_{n+1}(\boldsymbol{m})$ $(n\equiv -3 \pmod p)$. Any nondegenerate invariant form of these algebras is symmetric and differs from the following forms $(\ ,\ )\colon L\times L\to P$ by nonzero factors:*

$$(e_i,e_j)=(-1)^i\delta_{i+j,p^m-p},$$

$$L=W_1(\boldsymbol{m})=\langle e_i=x^{(i+1)}\partial \mid -1\leq i\leq p^m-2\rangle,\qquad p=3;$$

$$(D_{i,j}(u),D_{i,k}(v))=\operatorname{sgn}\begin{pmatrix}0&1&2\\ i&j&k\end{pmatrix}\pi(\partial_i(u)v),$$

$$(D_{i,j}(u),D_{i,j}(v))=0,\qquad L=S_2(\boldsymbol{m}),$$

where i,j,k are distinct elements of the set $\{0,1,2\}$ and $\operatorname{sgn}\left(\begin{smallmatrix}0&1&2\\ i&j&k\end{smallmatrix}\right)$ *is the parity of the permutation $\left(\begin{smallmatrix}0&1&2\\ i&j&k\end{smallmatrix}\right)$; for contact and Hamiltonian algebras the form $(\ ,\ )$ is defined by $(l,l_1)=\pi(ll_1)$.*

PROOF. A Lie algebra of Cartan type has a grading $L=\bigoplus_{i=q}^{r}L_i$. Let $V(L_q)=\{\mathfrak{d}_i \mid i\in I\}$ be a standard basis of the subalgebra L_q, and let $\mathcal{L}_-=\bigoplus_{i=q}^{-1}L_i$ (see [1] for details). If $l\in L_i$, we will write $|l|=i$. We will say that l is *integrable with respect to* $\mathfrak{d}_i$ and write $l'=\int_i l$ if $[\mathfrak{d}_i,l']=l$. We will call l *strongly nonintegrable* if $l\notin[\mathfrak{d},L]$ for all $\mathfrak{d}\in\mathcal{L}_-$. Suppose T is a standard torus and $V(L)$ a basis in L. Then we can choose $\int_i l$ for $l\in V(L)$ so that $|\int_i l|>|l|$. In particular, the sets of strongly nonintegrable basis elements and basis elements of the subspace L_r are the same. Suppose $(\ ,\ )\colon L\times L\to P$ is a nondegenerate invariant form on L. For any $\mathfrak{d}_i\in V(L_q)$ we choose $\mathfrak{d}_i^*\in V(L)$ such that $(\mathfrak{d}_i,\mathfrak{d}_i^*)\neq 0$. Then $\mathfrak{d}_i^*$ is strongly nonintegrable, since in the case $\mathfrak{d}_i^*=[\mathfrak{d},l]$, $l\in L$, $\mathfrak{d}\in\mathcal{L}_-$, we would have $(\mathfrak{d}_i,[\mathfrak{d},l])=([\mathfrak{d}_i,\mathfrak{d}],l)=0$. Thus $\mathfrak{d}_i^*\in L_r$. Moreover, for any $h\in T\subset L_0$ we have

$$([h,\mathfrak{d}_i],\mathfrak{d}_i^*)+(\mathfrak{d}_i,[h,\mathfrak{d}_i^*])=0.$$

As can be seen from the results of [1], the situation where the basis vectors $\mathfrak{d}_i\in L_q$ and $\mathfrak{d}_i^*\in L_r$ satisfy these two conditions can arise only in the following cases:

L	$W_1(\boldsymbol{m}),\ p=3$	$S_2(\boldsymbol{m})$	$H_n(\boldsymbol{m})$	$K_{n+1}(\boldsymbol{m}),\ n\equiv -3 \pmod p$
$(\mathfrak{d}_i,\mathfrak{d}_i^*)$	$(\partial, x^{(p^m-1)}\partial)$	$(\partial_i, D_{j,k}(x^{(e)}))$	$(x_i, x^{(e-\varepsilon_i)})$	$(1,x^{(e)})$
				$i\neq j\neq k,\ i\neq k.$

It follows from Proposition 2 that in these cases nondegenerate invariant forms do exist.

REMARK. Nondegenerate invariant forms of the Lie algebra $H_n(\boldsymbol{m})$ and of the Lie p-algebra $S_2(1,1,1)$ were constructed earlier in [12]. Analogous forms can be constructed in the characteristic zero case: the Lie algebras of smooth solenoidal vector fields on a three-dimensional sphere and of Hamiltonian vector fields on an even-dimensional sphere have nondegenerate invariant forms. Nondegenerate forms can be defined as in the modular case, if in the role of a strongly nonintegrable element $x^{(e)} \in U$ we take $\prod_i x_i^{-1}$.

§3. Central elements in the universal enveloping algebra of a Lie algebra of Cartan type

For $l \in L$ we denote by $\operatorname{Ad} l$ the adjoint derivation in $\mathfrak{A}(L)$ (and in $\overline{\mathfrak{A}}(L)$). For $\alpha \in \Gamma_I(\boldsymbol{m})$, $I = I(L)$, we consider the endomorphism $D^\alpha \colon \mathfrak{A}(L) \to \mathfrak{A}(L)$ defined by

$$D^\alpha = \prod_i (\operatorname{Ad} \partial_i)^{\alpha_i} \quad \text{if } L = W_n(\boldsymbol{m}), S_n(\boldsymbol{m}),$$

$$D^\alpha = (-1)^{\sum_{i>0} \alpha_i} \prod_i (\operatorname{Ad} x_i)^{\alpha_{-i}} \quad \text{if } L = H_n(\boldsymbol{m}),$$

$$D^\alpha = (-1)^{\sum_{i \geq 0} \alpha_i} \left(\frac{\operatorname{Ad} 1}{2}\right)^{\alpha_0} \prod_i (\operatorname{Ad} x_i)^{\alpha_{-i}} \quad \text{if } L = K_{n+1}(\boldsymbol{m}),$$

i.e., in our definitions of Lie algebras of Cartan type

$$D^\alpha = \begin{cases} \prod_i \partial_i^{\alpha_i} & \text{if } L = W_n(\boldsymbol{m}), S_n(\boldsymbol{m}), H_n(\boldsymbol{m}), \\ \partial_0^{\alpha_0} \prod_{i \neq 0} (\partial_i - \operatorname{sgn} i x_{-i} \partial_0)^{\alpha_i} & \text{if } L = K_{n+1}(\boldsymbol{m}). \end{cases}$$

The endomorphism $D^\alpha \colon \overline{\mathfrak{A}}(L) \to \overline{\mathfrak{A}}(L)$ is defined analogously.

THEOREM 1 ($p \geq 3$). *If L is a simple Lie algebra of Cartan type, the following elements lie in the center $\mathfrak{Z}(L)$*:

$$\begin{gathered} L = W_1(m) = \langle e_i = x^{(i+1)} \partial \mid -1 \leq i \leq p^m - 2 \rangle, \\ c_0 = (\operatorname{Ad} e_{-1})^{p^m - 1} (e_{p^m-2}^{(p+1)/2}), \end{gathered} \tag{5}$$

$$c_t = e_{-1}^{p^t} e_{p^m-2}^{p} - (\operatorname{Ad} e_{-1})^{p^m-1} ((\operatorname{Ad} e_{-1})^{p^t - 1} (e_{p^m-2}^{(p+1)/2}) e_{p^m-2}^{(p+1)/2}), \qquad 0 < t < m, \tag{6}$$

$$L = S_2(\boldsymbol{m}), \qquad c = D^e \left(\sum_{\gamma \in S_3} \operatorname{sgn} \gamma x^{(e)} \partial_{\gamma(0)} \partial_{\gamma(1)} (x^{(e)}) \partial_{\gamma(2)} \right) \tag{7}$$

(*the summation extends over all permutations* $\gamma = \begin{pmatrix} 0 & 1 & 2 \\ \gamma(0) & \gamma(1) & \gamma(2) \end{pmatrix} \in S_3$),

$$L = H_n(\boldsymbol{m}), \qquad c = D^e ((x^{(e)})^2), \tag{8}$$

$$L = K_{n+1}(\boldsymbol{m}), \quad n \not\equiv -2 \pmod p, \qquad c = D^e ((x^{(e)})^{s+1}), \tag{9}$$

where s satisfies the condition $s(n+2) \equiv -1 \pmod p$. These and the elements listed below lie in the center $\overline{\mathfrak{Z}}(L)$:

$$L = W_n(\boldsymbol{m}), \quad n \equiv -2 \pmod p, \qquad \bar{c} = D^e \left(\prod_i (x^{(e)} \partial_i) \right)^{p-1}, \tag{10}$$

$$L = S_n(\boldsymbol{m}), \qquad c = D^e \left(\prod_{i<j} (D_{i,j}(x^{(e)})) \right)^{p-1}, \tag{11}$$

$$\begin{gathered} L = H_n(\boldsymbol{m}), \quad c = D^e \left(\prod_i (x^{(e - \varepsilon_i)})^{p-1} \right), \\ L = \overline{K}_{n+1}(\boldsymbol{m}), \qquad n \equiv -2 \pmod p, \end{gathered} \tag{12}$$

$$\bar{c} = D^e\left(x^{(e)}(x^{(e-\varepsilon_0)})^{p-3}\prod_{i\neq 0}(x^{(e-\varepsilon_i)})^{p-1}\right) - x^{(e)}D^e\left((x^{(e-\varepsilon_0)})^{p-3}\prod_{i\neq 0}(x^{(e-\varepsilon_i)})^{p-1}\right). \tag{13}$$

PROOF. The coadjoint L-module L' is a homomorphic image or submodule of the L-module induced from the $\mathcal{L}_0$-module L'_q, where q is the depth of the grading in L (Proposition 3). Note that $\mathcal{L}_1 L'_q = 0$ and $\mathfrak{d}_i \circ (D^\alpha X)' = -(D^{\alpha-\varepsilon_i}X)'$, where $\mathfrak{d}_i \in V(\mathcal{L}_-)$, $\alpha_i > 0$, $X \in V(L_r)$, and $D^\alpha X \neq 0$. The following mappings are L_0-module homomorphisms $f'\colon L'_q \to \mathfrak{A}(L)^{\mathcal{L}_1}$ or $\overline{f}'\colon L'_q \to \overline{\mathfrak{A}}(L)^{\mathcal{L}_1}$:

$$L = W_1(\boldsymbol{m}), \qquad f'(e'_{-1}) = e_{p^m-2}^{(p-1)/2},$$

$$L = S_2(\boldsymbol{m}), \qquad f'(\partial'_i) = D_{j,k}(x^{(e)}), \quad i \neq j \neq k,\ i \neq k,\ \operatorname{sgn}\begin{pmatrix} 0 & 1 & 2 \\ i & j & k\end{pmatrix} = 1,$$

$$L = H_n(\boldsymbol{m}), \qquad f'(x'_i) = x^{(e-\varepsilon_i)}),$$

$$L = K_{n+1}(\boldsymbol{m}), \quad n \not\equiv -2 \pmod p, \qquad f'(1') = (x^{(e)})^s,$$

$$L = W_n(\boldsymbol{m}), \quad n \equiv -2 \pmod p, \qquad \overline{f}'(\partial'_i) = \prod_{j\neq i}(x^{(e)}\partial_j)^{p-1-\delta_{i,j}},$$

$$L = \overline{K}_{n+1}(\boldsymbol{m}), \quad n \equiv -2 \pmod p, \qquad \overline{f}'(1') = (x^{(e-\varepsilon_0)})^{p-3}\prod_{i\neq 0}(x^{(e-\varepsilon_i)})^{p-1}.$$

Consequently, the rules

$$L = W_1(\boldsymbol{m}), \qquad F'(e_i) = (-1)^{i+1}(\operatorname{Ad} e_{-1})^{i+1}(e_{p^m-2}^{(p-1)/2}),$$

$$L = H_n(\boldsymbol{m}), \qquad F'(x^{(\alpha)}) = (-1)^{|\alpha|}D^\alpha(x^{(e)}) = (-1)^{|\alpha|}x^{(e-\alpha)},$$

$$L = K_{n+1}(\boldsymbol{m}), \quad n \not\equiv -2 \pmod p, \qquad F'(x^{(\alpha)}) = (-1)^{|\alpha|}D^\alpha((x^{(e)})^s),$$

$$L = W_n(\boldsymbol{m}), \quad n \equiv -2 \pmod p, \qquad \overline{F}'(x^{(\alpha)}\partial_i) = (-1)^{|\alpha|}D^\alpha\left(\prod_{j\neq i}(x^{(e)}\partial_j)^{p-1-\delta_{i,j}}\right),$$

$$L = \overline{K}_{n+1}(\boldsymbol{m}), \quad n \equiv -2 \pmod p,$$

$$\overline{F}'(x^{(\alpha)}) = (-1)^{|\alpha|}D^\alpha\left((x^{(e-\varepsilon_0)})^{p-3}\prod_{i\neq 0}(x^{(e-\varepsilon_i)^{p-1}})\right), \qquad \alpha \neq e,$$

define L-module homomorphisms $F'\colon L' \to \mathfrak{A}(L)$ or $\overline{F}'\colon L' \to \overline{\mathfrak{A}}(L)$. Since

$$D^e(XY) = \sum_\alpha (-1)^{|\alpha|}D^\alpha(X)D^{e-\alpha}(Y), \qquad X, Y \in \mathfrak{A}(L) \text{ or } X, Y \in \overline{\mathfrak{A}}(L),$$

this means that the elements (5), (8), (9), (10), and (13) are central.

In case (7) this argument shows that $c \in \mathfrak{A}(W_3(\boldsymbol{m}))^{S_2(\boldsymbol{m})}$. We will prove that $c \in \mathfrak{A}(S_2(\boldsymbol{m}))$. We decompose c into a sum of elements c_1 and c_2, where

$$c_1 = \sum\nolimits_1 \sum_{\beta\in\Gamma_3(\boldsymbol{m}),\beta_i=0} (-1)^{|\beta|}x^{(\beta)}\partial_i D_{j,k}(x^{(e-\beta)}),$$

$$c_2 = \sum\nolimits_1 \sum_{\beta\in\Gamma_3(\boldsymbol{m}),\beta_i>0} (-1)^{|\beta|}x^{(\beta)}\partial_i D_{j,k}(x^{(e-\beta)})$$

(the symbol $\sum_1$ signifies that the summation extends over all $i, j, k \in \{0, 1, 2\}$ such that $\operatorname{sgn}\begin{pmatrix} 0 & 1 & 2 \\ i & j & k \end{pmatrix} = 1$). It is clear that $c_1 \in \mathfrak{A}(S_2(\boldsymbol{m}))$. Using the replacement $\beta = \alpha + \varepsilon_i$, where $0 \le \alpha_i < p^{m_i} - 1$, as is easily seen, we can represent c_2 in the form

$$c_2 = \sum_{\alpha} (-1)^{|\alpha|} (-D_{i,j}(x^{\alpha+\varepsilon_i+\varepsilon_j}) D_{j,k}(x^{(e-\alpha-\varepsilon_j)}) + D_{i,k}(x^{\alpha+\varepsilon_i+\varepsilon_k}) D_{i,j}(x^{(e-\alpha-\varepsilon_k)})).$$

Thus $c_2 \in \mathfrak{A}(S_2(\boldsymbol{m}))$, and hence $c \in \mathfrak{Z}(S_2(\boldsymbol{m}))$.

In cases (11) and (12) the embedding $\overline{f}': L'_q \to \overline{\mathfrak{A}}(L)^{\mathcal{L}_1}$ is defined in a somewhat more complicated fashion, and the centrality of these elements can be established more simply as follows. Note that $D^e(L) = 0$, and for $\overline{X} = \prod_{i<j} D_{i,j}(x^{(e)})^{p-1}$ when $L = S_n(\boldsymbol{m})$, $\overline{X} = \prod_i (x^{(e-\varepsilon_i)})^{p-1}$ when $L = H_n(\boldsymbol{m})$, we have $\overline{X} \in \overline{\mathfrak{A}}(L)^{\mathcal{L}_0}$. Consequently, for any $\partial_i \in V(\mathcal{L}_-)$ we have

$$[\partial_i, c] = \partial_i^{p^{m_i}} D^{e(\hat{i})}(\overline{X}) = 0, \qquad e(\hat{i}) = \sum_{j \ne i} (p^{m_j} - 1)\varepsilon_j \in \Gamma_I(\boldsymbol{m}).$$

Also, for any $Y \in L_r$

$$[Y, D^e(\overline{X})] = \sum_{\alpha} (-1)^{|\alpha|} D^{\alpha}([D^{e-\alpha} Y, \overline{X}]) = \sum_{\alpha \ne 0} (-1)^{|\alpha|} D^{\alpha}([D^{e-\alpha} Y, \overline{X}]) = 0,$$

since $D^e Y = 0$ and $D^{e-\alpha} Y \in \mathcal{L}_0$ if $\alpha \ne 0$. Thus $[L_r, \bar{c}] = 0$ and $[\mathcal{L}_-, \bar{c}] = 0$; hence $[L, \bar{c}] = 0$. Therefore $c = D^e(\overline{X}) \in \overline{\mathfrak{Z}}(L)$.

Suppose $L = W_1(m)$, $m > 1$. Consider the subspace $M = \langle e_{-1}^{p^t}, e_i \mid -1 \le i \le p^m - 2 \rangle$ in the minimal p-hull $R(L)$. Note that M has the structure of an L-module, and the L-module M' is defined by the rules

$$e_i \circ e'_{-1} = -\delta_{i,-1} e'_0 + \delta_{i,0} e'_{-1} + \delta_{i,p^t-1} (e_{-1}^{p^t})',$$
$$e_{-1} \circ e'_j = -e'_{j+1}, \quad -1 \le j < p^m - 2, \qquad L \circ (e_{-1}^{p^t})' = 0.$$

It is easy to verify that the element

$$z = (\operatorname{Ad} e_{-1})^{p^t - 1} (e_{p^m-2}^{(p+1)/2}) e_{p^m-2}^{(p-1)/2}$$

possesses the following property (see [4], Lemma 2):

$$[e_i, z] = -\delta_{i,-1}(-[e_{-1}, z]) + \delta_{i,0} z - \delta_{i,p^t-1} e_{p^m-2}^p, \qquad -1 \le i \le p^m - 2.$$

Thus the rule

$$F'((e_{-1}^{p^t})') = -e_{p^m-2}^p, \qquad F'(e'_i) = (-1)^{i+1} (\operatorname{Ad} e_{-1})^{i+1}(z), \qquad -1 \le i \le p^m - 2,$$

defines a homomorphism of the L-module M' into $\mathfrak{A}(L)$. It is obvious that M is an L-submodule of $\mathfrak{A}(L)$ and that c_t is an M-generalized Casimir element. This completes the proof of the theorem.

REMARK 1. The fact that $c_t \in \mathfrak{Z}(L)$, $0 \le t < m$, in the case $L = W_1(m)$ was established earlier in [4], and in cases (8) and (9) the inclusion $c \in \mathfrak{Z}(L)$ was noted in [13] and [14].

REMARK 2. Suppose L is a Lie algebra of Cartan type of depth q and $\operatorname{Hom}_{L_0}(M, N)$ is the space of L_0-homomorphisms $M \to N$, where M and N are L_0-modules. Let C be the space of L-generalized Casimir elements (we will assume that zero is also a generalized Casimir element). As can be seen from the proof of Theorem 1, there is a homomorphism

$$\operatorname{Hom}_{L_0}(L'_q, \mathfrak{A}(L)^{\mathcal{L}_1}) \to C.$$

Thus to describe the generalized Casimir elements it is first necessary to find the space $\mathfrak{A}(L)^{\mathcal{L}_1}$. In the proof of Theorem 1 we used the fact that $\mathfrak{A}(L)^{\mathcal{L}_1}$ contains the space of polynomials $P[L_r]$.

Analogously, if $\overline{C}$ is the space of L-generalized Casimir u-elements, then there exists a homomorphism

$$\mathrm{Hom}_{L_0}(L'_q, \overline{\mathfrak{A}}(L)^{\mathcal{L}_1}) \to \overline{C}.$$

Received 19/SEPT/83

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Translated by G. A. KANDALL