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CENTRAL EXTENSIONS OF THE ZASSENHAUS ALGEBRA AND THEIR IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. It is shown that the Zassenhaus algebra $W_1(m)$ over a field of characteristic $p > 3$ has, up to equivalence, a unique nontrivial central extension $\tilde{W}_1(m)$ (the modular Virasoro algebra). For the Virasoro algebra we construct a generalized Casimir element. All the irreducible $\tilde{W}_1(m)$-modules are described. It is shown that there is no simple graded Lie algebra with zero component $L_0 \cong \tilde{W}_1(m)$.

Bibliography: 16 titles.

In contradistinction to the case of zero characteristic, the problem of classifying simple Lie algebras over a field of positive characteristic remains unsolved. All the nonclassical simple Lie algebras known up to now are obtained from irreducible L_0-modules by Cartan extensions, where L_0 is a simple algebra or a trivial central extension of a simple Lie algebra. A natural question arises: is it possible to obtain a new simple Lie algebra by this method, if L_0 is a nontrivial central extension of a simple Lie algebra? This question, in turn, engenders two problems. The first is to find all central extensions of simple Lie algebras. We note that in the case of zero characteristic, every central extension of a finite-dimensional simple Lie algebra splits. The second problem is to describe the irreducible representations of nontrivial central extensions of simple Lie algebras. Incidentally, this problem is interesting in its own right; according to Proposition 4, for the study of Cartan extensions it suffices to describe the irreducible representations of the algebra L_0 whose dimensions do not exceed the dimension of this algebra.

In this paper we solve these problems in the case where L_0 is a nontrivial central extension of the Zassenhaus algebra $W_1(m)$.

We recall that a Lie algebra $\tilde{L}$ with central element z is called a central extension of the algebra L if the quotient algebra $\tilde{L}/\langle z\rangle$ is isomorphic to L. The 2-cohomology space $H^2(L, P)$ can be regarded as the space of nonequivalent central extensions of L. In the standard cochain complex $C^*(L, L^*)$ we distinguish a cochain subcomplex $\bar{C}^*(L)$ whose homology is isomorphic (with the grading shifted by one) to the cohomology of the Lie algebra L with coefficients in the trivial module. In particular, $H^1(L, L^*)$ contains a subspace isomorphic to $H^2(L, P)$. This fact is used in computing the central extensions of

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the Zassenhaus algebra (see §1). It has been proved that for $p > 3$, as in the case of infinite-dimensional Lie algebras of zero characteristic, the Zassenhaus algebra $W_1(m)$ has a unique (up to equivalence) nontrivial central extension $\tilde{W}_1(m)$. In analogy with characteristic zero, we call this algebra a (modular) Virasoro algebra. We recall that in the Lie algebra $W_1(m) = \langle e_i \,|\, -1 \leqslant i \leqslant p^m - 2\rangle$ the multiplication is given by

$$[e_i, e_j] = N_{i,j}e_{i+j}, \qquad N_{i,j} = \binom{i+j+1}{j} - \binom{i+j+1}{i}.$$

Then the multiplication in the Virasoro algebra $\tilde{W}_1(m) = \langle e_i, z \,|\, -1 \leqslant i \leqslant p^m - 2\rangle$ may be given by the rule

$$\{e_i, e_j\} = [e_i, e_j] + (-1)^i \delta_{i+j,p^m} z,$$

where $\delta_{i,j}$ is the Kronecker symbol. As in the Zassenhaus algebra, the rule

$$L_i = \langle e_i\rangle, \quad -1 \leqslant i \leqslant p^m - 2,\ e_{p^m-1} = 0, \qquad L_{p^m} = \langle z = e_{p^m}\rangle, \qquad \mathscr{L}_i = \langle e_j | j \geqslant i\rangle$$

gives a grading $L = \tilde{W}_1(m) = \oplus_{-1}^{p^m} L_i$, and a filtration $\mathscr{L}_{-1} \supset \mathscr{L}_0 \supset \cdots \supset \mathscr{L}_{p^m} \supset 0$. We introduce the subalgebra $\bar{\mathscr{L}}_1 = \langle e_i | 1 \leqslant i \leqslant p^m,\ i \neq 2\rangle$.

In §2 we construct a nontrivial central element z_1, the generalized Casimir element of the nilpotent Lie algebra $\bar{\mathscr{L}}_1$. We note that the well-known method of constructing the Casimir element only applies for semisimple algebras, i.e. Lie algebras with nondegenerate invariant forms. To construct a generalized Casimir element, it suffices to determine whether the L-module $U(L)$ contains as a submodule the coadjoint L-module L^*. The properties of the generalized Casimir element z_1 play an important role in the study of the irreducible representations of the Virasoro algebra (§3). Let M be an irreducible $\tilde{W}_1(m)$-module with respect to the representation $x \to (x)_M$. For a subalgebra L' containing z all the representations are divided into two types: we refer it to the first type if $(z)_M = 0$, and to the second type otherwise. The classes of irreducible $\tilde{W}_1(m)$-modules of the first type coincide with the classes of irreducible $W_1(m)$-modules. We give a description of the irreducible $W_1(m)$-modules of nonextremal height ($p \geqslant 2$). The analogous result for $p > 3$ is stated in [6]. Let $\overline{W}_1(m)$ be the direct sum of a 3-dimensional simple Lie algebra of type A_1 and a nilpotent subalgebra $\mathscr{L}_2$. It turns out that the classes of irreducible modules of the second types over $\tilde{W}_1(m)$ and over $\overline{W}_1(m)$ are in one-to-one correspondence. A similar fact in the case of the Lie p-algebra $W_1(1)$ was established in [12]. There the existence of a central element z_1 was proved. We give an explicit construction for z_1, and this enables us to give the correspondence more exactly. In §4 we prove that there does not exist a simple Lie algebra with zero component isomorphic to $\tilde{W}_1(m)$.

In this paper we use the following notation. All vector spaces are considered over a field P of characteristic $p > 0$ (in §3, P is algebraically closed); $\langle X\rangle$ is the linear span of a set X of vectors; $U(L)$ is the universal enveloping algebra of the Lie algebra L, $U(L)^{L'} = \langle f \in U(L) \,|\, [x, f] = 0,\ x \in L'\rangle$ is the space of invariants in $U(L)$ with respect to the subalgebra L', and $Z(L) = U(L)^L$ is the center of $U(L)$; in the case of the Virasoro algebra $L = \tilde{W}_1(m)$ this notation is abbreviated to $U, U^{L'}$, Z. For subalgebras L' containing the element z, we denote by $\hat{U}(L')$ the localization of $U(L')$ by the ideal $\langle z\rangle$; that is, $\hat{U}(L)$ is the algebra of fractions of the form fz^α, $f \in U(L')$, $-\infty < \alpha < \infty$. Let $\operatorname{Ad} x$ and $\widetilde{\operatorname{Ad}} x$ be the adjoint derivations of the universal enveloping algebras corresponding to the adjoint derivations of the algebras $W_1(m)$ and $\tilde{W}_1(m)$. The multiplication in $U(L)$ is denoted in the same way as the multiplication in the Lie algebra L. Let $\varepsilon_i = (0,\dots,0,1_i,0,\dots,0)$, and for a vector $\alpha = (\dots,\alpha_i,\dots)$, $i \in I$, we put $e^\alpha = \prod_i e_i^{\alpha_i}$ (I is a set of indices; the lengths of these vectors are to be determined from the context).

§1. Cohomology of Lie algebras with trivial coefficients

Let L^* be a coadjoint module over the Lie algebra L (in arbitrary characteristic), and let $(\ ,\ )$: $L^* \times L \to P$ be the natural pairing. In what follows $x, x', x_1, x_2, \dots \in L$. Let $C^k(L, L^*)$ be the space of skew-symmetric multilinear maps $L \times \cdots \times L \to L^*$, $k > 0$, $C^0(L, L^*) = L^*$, and let $C^*(L, L^*) = \bigoplus_k C^k(L, L^*)$ be the standard cochain complex of the Lie algebra L with coefficients in the coadjoint module with the coboundary operator d. We introduce the subspaces

$$\begin{aligned}
&\bar{C}^0(L) = L^*,\\
&\bar{C}^k(L) = \big\langle \psi \in C^k(L, L^*) \,|\, (\psi(x_1,\dots,x_{k-1},x_k), x_{k+1})\\
&\qquad\qquad + (\psi(x_1,\dots,x_{k-1},x_{k+1}), x_k) = 0 \big\rangle, \qquad k > 0,\\
&\tilde{C}^k(L) = \big\langle \psi \in C^k(L, L^*) \,|\, d\psi \in \bar{C}^{k+1}(L) \big\rangle,
\end{aligned}$$

and the linear map

$$\mathscr{A}\colon C^{k+1}(L, P) \to C^k(L, L^*), \qquad (\mathscr{A}\psi(x_1,\dots,x_k), x_{k+1}) = \psi(x_1,\dots,x_{k+1}).$$

Since ψ is skew-symmetric, it is clear that $\mathscr{A}\psi \in \bar{C}^k(L)$; that is, indeed the map $\mathscr{A}$: $C^{k+1}(L, P) \to \bar{C}^k(L)$ is defined. Conversely, the following mapping is also well defined:

$$\mathscr{B}\colon \bar{C}^k(L, L^*) \to C^{k+1}(L, P), \qquad \mathscr{B}\bar{\psi}(x_1,\dots,x_{k+1}) = (\bar{\psi}(x_1,\dots,x_k), x_{k+1}),$$

where the following conditions hold:

$$\mathscr{A}\mathscr{B}\bar{\psi} = \bar{\psi}, \quad \bar{\psi} \in \bar{C}^k(L), \qquad \mathscr{B}\mathscr{A}\psi = \psi, \quad \psi \in C^{k+1}(L, P).$$

A simple verification shows that the following result holds.

PROPOSITION 1. $\bar{C}^k(L) \subseteq \tilde{C}^k(L)$, $0 \leqslant k$.

Thus, $C^*(L, L^*)$ contains the cochain subcomplex $\bar{C}^*(L) = \bigoplus_k \bar{C}^k(L)$. Let

$$\bar{Z}^*(L) = \big\langle \psi \in \bar{C}^*(L) \,|\, d\psi = 0 \big\rangle, \qquad \bar{B}^*(L) = \big\langle d\omega \,|\, \omega \in \bar{C}^*(L) \big\rangle,$$
$$\bar{H}^*(L) = \bar{Z}^*(L)/\bar{B}^*(L)$$

be the spaces of cocycles, coboundaries, and cohomology of this complex.

PROPOSITION 2. *The following diagram is commutative*:

$$\begin{array}{ccccccc}
\cdots & \to & C^k(L,P) & \to & C^{k+1}(L,P) & \to & \cdots\\
 & & \mathscr{A}\downarrow \ \ \uparrow\mathscr{B} & & \mathscr{A}\downarrow \ \ \uparrow\mathscr{B} & & \\
\cdots & \to & \bar{C}^{k-1}(L) & \to & \bar{C}^k(L) & \to & \cdots.
\end{array}$$

PROOF. Let $\psi \in C^k(L, P)$. We note that for any permutation $\sigma \in S_k$ we have

$$(\mathscr{A}\psi(x_1,\dots,x_{k-1}), x_k) = \operatorname{sgn}\sigma\big(\mathscr{A}\psi(x_{\sigma(1)},\dots,x_{\sigma(k-1)}), x_{\sigma(k)}\big).$$

Therefore

$$\begin{aligned}
((x_i \circ \mathscr{A}\psi)(x_1,\dots,\hat{x}_i,\dots,x_k), x_{k+1}) &= -(\mathscr{A}\psi(x_1,\dots,\hat{x}_i,\dots,x_k), [x_i, x_{k+1}])\\
&= (-1)^k \psi([x_i, x_{k+1}], x_1,\dots,\hat{x}_i,\dots,x_k)
\end{aligned}$$

(the caret means that the corresponding element is omitted), and

$$\begin{aligned}(d\mathscr{A}\psi(x_1,\dots,x_k),x_{k+1}) &= \sum_{1<j\leqslant k}(-1)^{i+j}\psi([x_i,x_j],\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1}) \\ &\quad+\sum_{i=1}^{k}(-1)^{i+k+1}\psi([x_i,x_{k+1}],x_1,\dots,\hat{x}_i,\dots,x_k) \\ &= d\psi(x_1,\dots,x_k,x_{k+1}) = (\mathscr{A}d\psi(x_1,\dots,x_k),x_{k+1}).\end{aligned}$$

Analogously, $\mathscr{B}d\overline{\psi} = d\mathscr{B}\overline{\psi}$, $\overline{\psi}\in\overline{C}^{k-1}(L)$, $k>0$.

Thus, we have proved the following result.

THEOREM 1. $H^{k+1}(L,P)\cong H^k(L)$, $k\geqslant 0$.

Let var_k: $H^{k+1}(L,P)\to H^k(L,L^*)$ be the cohomology map induced by the map $\mathscr{A}$.

COROLLARY 1. $\operatorname{Ker}\operatorname{var}_k\cong\tilde{C}^{k-1}(L)/\overline{C}^{k-1}(L)$, $k\geqslant 0$.

Since $\tilde{C}^0(L)=\overline{C}^0(L)=L^*$, we obtain that $\operatorname{Ker}\operatorname{var}_k=0$ if $k=1$. In other words, the following is true:

COROLLARY 2. *The space $H^2(L,P)$ of central extensions is isomorphic to the subspace of $H^1(L,L^*)$ consisting of the cochains which preserve the pairing (,):*

$$H^2(L,P)\cong\overline{H}^1(L)=\langle\psi\in H^1(L,L^*)\,|\,(\psi(x),y)+(\psi(y),x)=0,\forall x,y\in L\rangle.$$

In particular if L has a nondegenerate invariant form then $L\cong L^*$ and $\overline{H}^1(L)$ appears as a direct summand in the space $H^1(L,L^*)$ of outer derivations, namely as the subspace of derivations preserving the form. If in addition all the derivations of L are inner, then every central extension of L is trivial, which is, by the way, well known in the zero characteristic case.

Let $L=W_1(m)$. We give the space $O_1(m)$ a module structure by the rule $u\partial$: $v\mapsto u\partial(v)+2\partial(u)v$. The resulting module is denoted by U_2. We note that the coadjoint L-module is isomorphic to the L-module U_2. Indeed, the form F: $U_2\times L\to P$ defined by the rule

$$F(u,v\partial)=\lambda_{p^m-1}(uv),$$

where $w=\sum_{i=0}^{p^m-1}\lambda_i(w)x^{(i)}\in O_1(m)$, is invariant:

$$\begin{aligned}F((u\partial)(v),w\partial)+F(v,[u\partial,w\partial]) &= \lambda_{p^m-1}(u\partial(v)w+2\partial(u)vw+vu\partial(w)-vw\partial(u)) \\ &= \lambda_{p^m-1}(\partial(uvw))=0.\end{aligned}$$

By Corollary 2 of Theorem 5 of [14] we have the isomorphism

$$H^1(L,U_2)\cong\begin{cases}\langle\partial^3\colon u\partial\to\partial^3(u)\rangle, & \text{if } p>3,\\ \langle\partial^{p^k}\colon u\partial\mapsto\partial^{p^k}(u)\,|\,0<k<m\rangle, & \text{if } p=3,\\ \langle u\partial\mapsto x^{(p^m-1)}u\rangle\oplus\langle\partial^{p^k-1}\colon u\partial\mapsto\partial^{p^k-1}(u)\,|\,0<k<m\rangle, & \text{if } p=2.\end{cases}$$

It can easily be verified that the given cocycles preserve the form F. For example,

$$\begin{aligned}F(\partial^3(u),v\partial)+F(u,\partial^3(v\partial)) &= \lambda_{p^m-1}(\partial(\partial^2(u)v-\partial(u)\partial(v)+u\partial^2(v))) \\ &= 0,\qquad r\geqslant 3.\end{aligned}$$

Thus, by Theorem 1 the map $H^1(L, U_2) = \overline{H}^1(L) \to H^2(L, P)$, $\bar{\psi} \mapsto \mathscr{B}\bar{\psi}$ gives an isomorphism. The following theorem is proved.

THEOREM 2. *The Lie algebra $W_1(m)$ over a field of characteristic $p \geqslant 2$ has* 1, $m-1$, *or* m *nonequivalent central extensions according as $p > 3$, $p = 3$, or $p = 2$. In the corresponding algebras with central elements $z, v_1, v_2, u_1, \ldots$ the multiplications can be defined by the rules*

$$\begin{aligned}
\{e_i, e_j\} &= [e_i, e_j] + (-1)^i \delta_{i+j, p^m} z, && p > 3,\\
\{e_i, e_j\} &= [e_i, e_j] + \sum_{k=1}^{m-1} (-1)^i \delta_{i+j, p^m+p^k-p} v_k, && p = 3,\\
\{e_i, e_j\} &= [e_i, e_j] + \delta_{i,-1}, \delta_{j,-1} u_m + \sum_{k=1}^{m} \delta_{i+j, p^m+p^k-4} u_k, && p = 2.
\end{aligned} \tag{1}$$

For the Lie p-algebra $W_1(1)$ this fact was obtained earlier in [12], and also in [7] for $P = \mathbf{Z}/p\mathbf{Z}$.* The papers [8] and [9] present erroneous results on $H^2(W_1(m), P)$ (in [8], p. 39, the equality on the ninth line from the top is false). By analogy with characteristic zero, the Lie algebra $\tilde{W}_1(m) = \langle e_i, z \mid -1 \leqslant i \leqslant p^m - 2\rangle$ with the multiplication given by (1) is called a (modular) Virasoro algebra. For the irreducible representations of the Virasoro algebra of zero characteristic and their connections with other problems, see for example [10]. In the description of the irreducible representations of a modular Virasoro algebra an important role is played by the formula for a central element z_1 given in the next section.

§2. A generalized Casimir element of the Virasoro algebra

We call the Lie algebra L a *Casimir* algebra if the L-module $U(L)$ contains as a submodule the coadjoint L-module L^*. Let $V(L) = \{e_i \mid i \in I\}$ be a basis for the Casimir Lie algebra L and let $V(L^*) = \{e_i^* \in U(L) \mid (e_i^*, e_j) = \delta_{i,j},\ i, j \in I\}$ be the dual basis. Then

$$c = \sum_{i \in I} e_i e_i^* \in Z(L).$$

This element is called a generalized Casimir element of the Lie algebra L. If L has a nonsingular invariant form, then this construction coincides with the well-known one. Despite the simplicity of the generalization, our construction makes sense for the production of central elements for the universal enveloping nonclassical Lie algebras. For example, the Virasoro algebra for $p \geqslant 5$ has no nonsingular invariant form, but, as is shown in this section, $U(\tilde{W}_1(m))$ has a nontrivial central element.

By the Poincaré-Birkhoff-Witt theorem, in the universal enveloping Virasoro algebra U one can choose a basis

$$\left\{ e^\alpha = \prod_{i=-1}^{p^m} e_i^{\alpha_i} \,\middle|\, 0 \leqslant \alpha_i,\ i \neq p^m - 1,\ \alpha_{p^m-1} = 0 \right\}.$$

Then in $\hat{U}$ one can choose a basis

$$\{ e^\alpha \mid 0 \leqslant \alpha_i,\ -1 \leqslant i \leqslant p^m - 2,\ -\infty < \alpha_{p^m} < \infty \}.$$

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* *Added in translation*. After the paper was published, the author learned that a similar result for $p > 3$ was also obtained in [16].

We decompose U into the direct sum of proper subspaces with respect to $\mathrm{Ad}\, e_0$:

$$U = \bigoplus_{l \in \mathbf{Z}/p\mathbf{Z}} U^{(l)}, \qquad U^{(l)} = \left\langle e^{\alpha} \,\middle|\, \sum_i i\alpha_i = l \right\rangle.$$

We introduce the subspaces $U(i) = \langle e^{\alpha} | \alpha_j = 0,\ i < j \leqslant p^m \rangle$. In particular we have an isomorphism of vector spaces $U(p^m - 2) \cong W_1(m)$. Let π_i: $U \to U(i)$ be the natural projection. We define the linear maps

$$d/di\colon U \to U, \qquad (d/di)(e^{\alpha}) = \alpha_i e^{\alpha - \varepsilon_i}$$

and

$$\int_i \colon \langle e^{\alpha} | \alpha_i \not\equiv -1 \ (\mathrm{mod}\, p) \rangle \to U, \qquad \int_i e^{\alpha} = e^{\alpha + \varepsilon_i}/(\alpha_i + 1).$$

Let Γ_m be the set of vectors $\alpha = (\alpha_1, \dots, \alpha_{p^m})$ for which $\alpha_2 = \alpha_{p^m - 1} = 0$. We put $e_{p^m - 1} = 0$ and $e_{p^m} = z$. Then $U(\bar{\mathscr{L}}_1) = \langle e^{\alpha} \,|\, \alpha \in \Gamma_m \rangle$. Obviously, the multiplication in U can be defined by the rule

$$\{f, g\} = [f, g] + \Phi(f, g) z,$$

where

$$\Phi(f, g) = \sum_{i=2}^{p^m - 2} (-1)^i \frac{d}{di} f \frac{d}{d\bar{i}}, \qquad \bar{i} = p^m - i.$$

Let $\overline{W}_1(m)$ be the direct sum of the 3-dimensional simple Lie algebra

$$A_1 = \left\langle \bar{e}_{-1}, \bar{e}_0, \bar{e}_1 \,\middle|\, [\bar{e}_{-1}, \bar{e}_1] = \bar{e}_0,\ [\bar{e}_0, \bar{e}_{\pm 1}] = \pm \bar{e}_{\pm 1} \right\rangle$$

and the $(p^m - 2)$-dimensional nilpotent subalgebra

$$\left\langle \bar{e}_i \,\middle|\, 2 \leqslant i \leqslant p^m,\ i \neq p^m - 1 \right\rangle,$$

isomorphic to the subalgebra $\mathscr{L}_2(\tilde{W}_1(m))$ (the isomorphism is given by the rule $e_i \mapsto \bar{e}_i$, $2 \leqslant i \leqslant p^m$).

THEOREM 3. *Let $p \geqslant 3$. The Virasoro algebra $\tilde{W}_1(m)$ and its subalgebra $\bar{\mathscr{L}}_1$ are Casimir. The following recurrence relation is well defined*:

$$f_0 = e_1, \quad f_{l+1} = \sum_{i=2}^{p^m - 2} (-1)^{i+1} \int_i \pi_{\bar{i}} [e_i, f_l] \in U(p^m - 2),$$
$$0 \leqslant l < t = (p^m - 3)/2, \quad (2)$$

and

$$z_1 = \sum_{l=0}^{t} f_l z^{t-l} \in U(\bar{\mathscr{L}}_1)^{\mathscr{L}_1}.$$

In particular z_1 is a generalized Casimir element of the Lie algebra $\bar{\mathscr{L}}_1$. The rule

$$\bar{e}_{-1} \mapsto z^{-t}\left(\left(\widetilde{\mathrm{Ad}}\, e_{-1}\right)^2 z_1 - z_1 e_{p^m - 2} z^{-1}\right),$$

$$\bar{e}_0 \mapsto z^{-t}\left(\widetilde{\mathrm{Ad}}\, e_{-1}\right) z_1, \quad \bar{e}_1 \mapsto z^{-t} z_1, \quad \bar{e}_i \mapsto e_i, \qquad 2 \leqslant i \leqslant p^m,$$

gives an algebra isomorphism $\hat{U}(\tilde{W}_1(m)) = \hat{U}(\tilde{W}_1(m))$.

REMARK. Let $Z_0^{(i)} = \langle x^p - x^{[p]} | x \in \mathscr{L}_i \rangle$ be the p-center of $\mathscr{L}_i$. It can be shown that

$$U(\mathscr{L}_i)^{\mathscr{L}_1} = Z_0^{(i)}, \quad 2 \leqslant i \leqslant p^m, \qquad U(\bar{\mathscr{L}}_1)^{\mathscr{L}_1} = Z(\bar{\mathscr{L}}_1) = Z_0^{(1)}[z_1].$$

In other words every element of $U(\bar{\mathscr{L}}_1)^{\mathscr{L}_1}$ is a polynomial in z_1 with coefficients in $Z_0^{(1)}$.

We remark that

$$\{z_1, e_i\} = \sum_l \left([f_l, e_i] + (-1)^{i+1} \frac{d}{d\bar{i}} f_{l+1} \right) z^{t-l}.$$

Hence $z_1 \in U(\bar{\mathscr{L}}_1)^{\mathscr{L}_1}$ if and only if the following "deformation equations" are satisfied:

$$[e_1, f_l] = 0, \qquad 0 \leqslant l, \tag{3}$$

$$[e_i, f_{l-1}] + (-1)^i (d/d\bar{i}) f_l = 0, \qquad 0 \leqslant l,\ 2 \leqslant i \leqslant p^m - 2. \tag{4}$$

(We recall that $\partial/d\bar{1} = 0$.) We suppose that the elements $f_0, f_1, \ldots, f_k$ have been constructed. For the existence of f_{k+1} the following condition is necessary:

$$\pi_i f_{k+1} = (-1)^{i+1} \int_{\bar{i}} \pi_{\bar{i}} [e_i, f_k].$$

In other words, f_{k+1} must be given by (2). We must prove the existence of the "integral" $\int_{\bar{i}} \pi_{\bar{i}}[e_i, f_k]$ and the validity of (3) and (4) for f_{k+1}. For the proof of these facts we need the following remark:

$$f_l \in \left\langle e^\alpha = e_i \prod_{\substack{j \geqslant (p^m+1)/2 \\ i+j \geqslant p^m+1}} e_j^{\alpha_j} \middle| i \geqslant 2l+1,\ e_i \in [\mathscr{L}_1, \mathscr{L}_1] \right\rangle. \tag{5}$$

For $l = 0$, this inclusion is trivial. We can assume that it holds for $l = k$. Then the assertion for $l = k + 1$ follows from (2). We remark that in the Virasoro algebra $(\widetilde{\operatorname{ad}} e_s)^2 e_i = 0$, if $s > (p^m + 1)/2$.

The proof of Theorem 3 is based on the following lemma.

LEMMA 1. *Let*

$$f \in \left\langle e^\alpha | (\operatorname{ad} e_s)^2 e_i = 0, \text{ if } \alpha_s \geqslant 2,\ \alpha \in \Gamma_m \right\rangle.$$

Then

$$\frac{d}{dj}[e_i, f] = \left[e_i, \frac{d}{dj} f \right] + N_{i,j-i} \frac{d}{d(j-i)} f, \qquad -1 \leqslant i,\ j \leqslant p^m - 2.$$

PROOF. If $(\operatorname{ad} e_s)^2 e_i = 0$, $\alpha_s \geqslant 2$, then it is obvious that

$$[e_i, e^\alpha] = N_{i,j-i} \alpha_{j-i} e^{\alpha - \varepsilon_{j-i} + \varepsilon_j} + N_{i,j} \alpha_j e^{\alpha - \varepsilon_j + \varepsilon_{i+j}} + X_i,$$

where $X_i \in \langle e^\beta | \beta \in \Gamma_m,\ \beta_j = \alpha_j \rangle$. Analogously,

$$\alpha_i [e_i, e^{\alpha - \varepsilon_j}] = \alpha_i \left(N_{i,j-i} \alpha_{j-i} e^{\alpha - \varepsilon_{j-i} + \varepsilon_j} + N_{i,j} (\alpha_j - 1) e^{\alpha - 2\varepsilon_j + \varepsilon_{i+j}} + Y_i \right),$$

where $Y_i \in \langle e^\beta | \beta \in \Gamma_m,\ \beta_j = \alpha_j - 1 \rangle$, and $dX_i/dj = \alpha_j Y_i$.

The following congruence is established by induction on $i + j$:

$$N_{i,j} \equiv (-1)^i N_{p^m - i - j, i} + (-1)^j N_{j, p^m - i - j} \pmod{p}. \tag{6}$$

It follows from the next lemma that (2) is well defined:

LEMMA 2. $(d/d\bar{i})^{p-1} [e_i, f_k] = 0$, $2 \leqslant i \leqslant p^m - 2$.

PROOF. We put $a_i = (d/d\bar{i})^{p-1}[e_i, f_k]$. By (5), when computing da_i/dj we can use Lemma 1:

$$\begin{aligned}\frac{d}{dj}a_i &= \left(\frac{d}{d\bar{i}}\right)^{p-1}\frac{d}{d\bar{j}}[e_i, f_k] = \left(\frac{d}{d\bar{i}}\right)^{p-1}\left(\left[e_i, \frac{d}{d\bar{j}}f_k\right] + N_{i,j-i}\frac{d}{d(\overline{j-i})}f_k\right)\\
&= (\text{by conditions (3), (4)})\\
&= (-1)^j\left(\frac{d}{d\bar{i}}\right)^{p-1}\left[e_i\left[e_j, f_{k-1}\right]\right] + (-1)^{j-i}N_{i,j-i}\left(\frac{d}{d\bar{i}}\right)^{p-1}\left[e_{\overline{j-i}}, f_{k-1}\right]\\
&= (-1)^j\left(N_{i,p^m-j} + (-1)^iN_{i,j-i}\right)\left(\frac{d}{d\bar{i}}\right)^{p-1}\left[e_{\overline{j-i}}, f_{jk-1}\right]\\
&\quad + (-1)^j\left(\frac{d}{d\bar{i}}\right)^{p-1}\left[e_j, \left[e_i, f_{k-1}\right]\right]\\
&= (\text{by formula (6) and conditions (3), (4)})\\
&= (-1)^{i+j+1}\left(\frac{d}{d\bar{i}}\right)^{p-1}\left[e_j, \frac{d}{d\bar{i}}f_k\right] + N_{p^m-j,j-i}(-1)^{j-i}\left(\frac{d}{d\bar{i}}\right)^{p-1}\frac{d}{d(\overline{j-i})}f_k\\
&= (\text{by Lemma 1}) = (-1)^{i+j}\left(\frac{d}{d\bar{i}}\right)^p\left[e_j, f_k\right].\end{aligned}$$

Since $(d/d\bar{i})^p = 0$, this means that $a_i \in \langle e^\alpha | \alpha_j \equiv 0 \pmod p), \forall j\rangle$. Hence $a_i \in U^{(0)}$. Since $a_i \in U^{(1)}$, this is possible only for $a_i = 0$.

LEMMA 3.

$$(-1)^i(d/d\bar{j})[e_i, f_k] = (-1)^j(d/d\bar{i})\left[e_j, f_k\right], \qquad 2 \leqslant i,\ j \leqslant p^m - 2.$$

PROOF. By (3)–(6) and Lemma 1, we have

$$\begin{aligned}(-1)^i\frac{d}{d\bar{j}}[e_i, f_k] &= (-1)^i\left(\left[e_i, \frac{d}{d\bar{j}}f_k\right] + N_{i,p^m-i-j}\frac{d}{d(\overline{i+j})}f_k\right)\\
&= (-1)^{i+j+1}\left[e_i, \left[e_j, f_{k-1}\right]\right] + (-1)^{j+1}N_{i,p^m-i-j}\left[e_{i+j}, f_{k-1}\right]\\
&= (-1)^{i+j+1}\left[e_j, \left[e_i, f_{k-1}\right]\right] - \left((-1)^{i+j}N_{i,j} + (-1)^jN_{i,p^m-i-j}\right)\left[e_{i+j}, f_{k-1}\right]\\
&= (-1)^{i+j+1}\left[e_j, \left[e_i, f_{k-1}\right]\right] + (-1)^{i+1}N_{j,p^m-i-j}\left[e_{i+j}, f_{k-1}\right]\\
&= (-1)^j\left[e_j, \frac{d}{d\bar{i}}f_k\right] + (-1)^jN_{j,p^m-i-j}\frac{d}{d(\overline{i+j})}f_k = (-1)^j\frac{d}{d\bar{i}}\left[e_j, f_k\right].\end{aligned}$$

PROOF OF THEOREM 3. We note that

$$\int \frac{d}{d\bar{j}} = \frac{d}{d\bar{j}}\int_{\bar{i}}, \quad i \neq j, \quad \text{and} \quad \int_{\bar{i}}\frac{d}{d\bar{i}}e^\alpha = e^\alpha, \quad \alpha_i > 0.$$

By Lemma 3

$$(-1)^i\pi_{\bar{i}}(d/d\bar{j})[e_i, f_k] = (-1)^j\pi_{\bar{i}}(d/d\bar{i})\left[e_j, f_k\right], \qquad 2 \leqslant i < j \leqslant p^m - 2.$$

Moreover, $d\pi_j/di = \pi_j d/di$ and

$$\frac{d}{d\bar{j}}f_{k+1} = \sum_{\bar{s} \geqslant \bar{j}}(-1)^{s+1}\int_{\bar{s}}\pi_{\bar{s}}\frac{d}{d\bar{j}}[e_s, f_k].$$

Hence for any $\bar{i} > \bar{j}$ we have

$$(\mathrm{id} - \pi_{\bar{i}-1})\frac{d}{d\bar{j}}f_{k+1} = \sum_{\bar{s} \geqslant \bar{j}} (-1)^{s+1} \int_{\bar{s}} \pi_{\bar{s}} \frac{d}{d\bar{j}}[e_s, f_k].$$

(Let id be the identity map.) Then

$$\pi_{\bar{i}}(\mathrm{id} - \pi_{\bar{i}-1})\frac{d}{d\bar{j}}f_{k+1} = (-1)^{i+1}\pi_{\bar{i}}\int_{\bar{i}} \pi_{\bar{i}} \frac{d}{d\bar{j}}[e_i, f_k],$$

since $\pi_{\bar{i}} f_{\bar{s}} = 0$ for $\bar{s} > \bar{i}$. Thus,

$$\pi_{\bar{i}}(\mathrm{id} - \pi_{\bar{i}-1})\frac{d}{d\bar{j}}f_{k+1} = (-1)^{j+1} \int_{\bar{i}} \pi_{\bar{i}} \frac{d}{d\bar{i}}[e_j, f_k]$$

for any $2 \leqslant i \leqslant p^m - 2$. In other words, condition (4), $l = k + 1$, is satisfied.

To verify condition (3), $l = k + 1$, we put $b = [e_1, f_{k+1}]$. According to (3)–(5), by Lemma 1, for any $2 \leqslant j \leqslant p^m - 2$ we have

$$\begin{aligned} db/dj &= [e_1, df_{k+1}/dj] + N_{1,j-1} df_{k+1}/d(j-1) \\ &= (-1)^j [e_1, [e_{\bar{j}}, f_k]] - (-1)^j N_{1,j-1}[e_{\overline{j-1}}, f_k] \\ &\equiv (\text{by condition } (3)) = (-1)^j (N_{1,p^m-j} - N_{1,j-1})[e_{\overline{j-1}}, f_k]. \end{aligned}$$

Since

$$N_{1,p^m-j} = (p^m - j + 2)(p^m - j - 1)/2 \equiv (j+1)(j-2)/2 = N_{1,j-1} \pmod p,$$

this means that

$$b \in \left\langle e^{\beta} | \beta_1 = 0,\ \beta_j \equiv 0 \pmod p),\ 2 \leqslant j,\ \beta \in \Gamma_m \right\rangle.$$

Hence $b \in U^{(0)}$. On the other hand, $b \in [U^{(1)}, U^{(1)}] \subseteq U^{(2)}$. For $p > 2$ this is possible only for $b = 0$. Thus, f_{k+1} satisfies (3) and (4).

The inclusion (5) implies that $t = (p^m - 3)/2$. Hence at the tth step our procedure terminates, and we obtain the element $z_1 \in U(\bar{\mathscr{L}}_1)^{\mathscr{L}_1}$. By the same computations as above, one can verify that the map

$$e_i^* \to dz_1/di \in U(\bar{\mathscr{L}}_1), \qquad 1 \leqslant i \leqslant p^m,\ i \neq 2,$$

gives an embedding of the coadjoint $\bar{\mathscr{L}}_1$-module in the $\bar{\mathscr{L}}_1$-module $U(\bar{\mathscr{L}}_1)$. Hence the element

$$-2z_1 = \sum_{i=1, i\neq 2}^{p^m-2} e_i \frac{d}{di} z_1$$

is a generalized Casimir element of the algebra $\bar{\mathscr{L}}_1$.

The action of the Virasoro algebra in the coadjoint module is given by the rule

$$e_{-1} \circ e_i^* = -e_{i+1}^*, \quad -1 \leqslant i \leqslant p^m - 3, \qquad e_{-1} \circ e_{p^m-2}^* = 0,$$

$$e_i \circ z^* = (-1)^{i+1} e_{p^m-i}^*, \quad 2 \leqslant i \leqslant p^m - 2, \qquad e_i \circ z^* = 0, \quad -1 \leqslant i \leqslant 1,$$

and the generalized Casimir element of the Virasoro algebra has the form

$$c = \sum_{i=-1}^{p^m-2} (-1)^i e_i \left(\widetilde{\mathrm{Ad}}\, e_{-1}\right)^{i+1}(z_1) + z \cdot z^*.$$

But the derivation of an explicit formula for z^* would require extensive computations, and we construct a central element in $U(\tilde{W}_1(m))$ in a somewhat different way (we shall require some points of these constructions in future). We put

$$E_1 = z_1 z^{-t}, \qquad E_0 = \widetilde{\mathrm{Ad}}\, e_{-1} E_1, \qquad E_{-1} = \left(\widetilde{\mathrm{Ad}}\, e_{-1}\right)^2 E_1 - E_1 e_{p^m-2} z^{-1}.$$

We remark that

$$E_0 - e_0 \in \hat{U}(\mathscr{L}_2), \qquad E_{-1} - e_{-1} - e_1 e_{p^m-2} z^{-1} \in \hat{U}(\mathscr{L}_2).$$

Hence

$$\begin{aligned} &\{E_0, E_1\} = \{e_0, E_1\} = E_1, \\ &\{e_i, E_0\} = \{e_i, \{e_{-1}, E_1\}\} = -\{e_{i-1}, E_1\} = 0, \qquad i \geqslant 2, \end{aligned}$$

and

$$\begin{aligned} &\{E_{-1}, E_1\} = \{e_{-1}, E_1\} = E_0, \qquad \{e_1, E_0\} = \{e_1, \{e_{-1}, E_1\}\} = -\{e_0, E_1\} = -E_1, \\ &\{E_{-1}, E_0\} = \left\{e_{-1} + e_1 e_{p^m-2} z^{-1}, E_0\right\} = \{e_{-1}, \{e_{-1}, E_1\}\} - \{e_1, E_0\} e_{p^m-2} z^{-1} = E_{-1}, \\ &\{e_i, E_{-1}\} = \{e_i, \{e_{-1}, E_0\}\} = -\{e_{i-1}, E_0\} + \{e_{-1}, \{e_i, E_0\}\} = 0, \qquad i \geqslant 3, \\ &\{e_2, E_{-1}\} = \left\{e_2, \{e_{-1}, E_0\} - E_1 e_{p^m-2} z^{-1}\right\} = -\{e_1, E_0\} - E_1 \{e_2, e_{p^m-2}\} z^{-1} = 0. \end{aligned}$$

Thus, $\hat{U}$ contains the subalgebra $A = \langle E_{-1}, E_0, E_1 \rangle$ isomorphic to the three-dimensional simple Lie algebra of type A_1, and the subalgebras A and $\hat{U}(\mathscr{L}_2)$ pairwise commute. Moreover, the elements e_{-1}, e_0, and e_1 can be represented as linear combinations of the elements E_{-1}, E_0, and E_1 with coefficients from $\hat{U}(\mathscr{L}_2)$. In other words, we can choose such a basis in $\hat{U}$:

$$\left\{ \prod_{i=-1}^{1} E_i^{\alpha_i} \prod_{j=2}^{p^m} e_j^{\alpha_j} \,\middle|\, -\infty < \alpha_{p^m} < \infty, 0 \leqslant \alpha_s, -1 \leqslant s \leqslant p^m - 2 \right\}.$$

Thus

$$\hat{U}(\tilde{W}_1(m)) \cong U(A)\hat{U}(\mathscr{L}_2) \cong \hat{U}(A_1 \oplus \mathscr{L}_2) \cong \hat{U}(\overline{W}_1(m)).$$

Moreover, the Casimir element of the Lie algebra A is central in $\hat{U}$:

$$\bar{z} = E_{-1}E_1 - E_0^2/2 - E_0/2 \in Z(\hat{U}).$$

Obviously, $c \in \langle z^{2t}\bar{z} \rangle \subseteq Z(\tilde{W}_1(m))$. It can be verified that this element is indeed a generalized Casimir element of the Virasoro algebra. Theorem 3 is proved.

EXAMPLE. Let $p = 5$ and $m = 1$. Then

$$\begin{aligned} &E_1 = e_1 + e_3^2 z^{-1}, \qquad E_0 = e_0 + 2e_2 e_3 z^{-1}, \qquad E_{-1} = e_{-1} + \left(e_1 e_3 + 2e_2^2\right) z^{-1} - e_3^3 z^{-2}, \\ &c = e_{-1}\left(e_1 z + e_3^2\right) - e_0\left(e_0 z + 2e_2 e_3 - z\right) + e_1\left(e_{-1} z + 2e_1 e_3 + 2e_2^2\right) \\ &\quad - e_2(2e_0 e_3 + e_1 e_2 + e_3) + e_3\left(2e_{-1}e_3 + 3e_0 e_2 + e_1^2 + e_2\right) + z\left(e_{-1}e_1 + 2e_0^2 + 2e_0\right). \end{aligned}$$

In the sequel we shall need the following properties of the element z_1:

LEMMA 4. *Let $p \geqslant 3$ and let $E_1 = \sum_{l \geqslant 0} f_l z^{-l}$, $E_0 = \sum_{l \geqslant 0} g_l z^{-l}$, and $E_{-1} = \sum_{l \geqslant 0} h_l z^{-l}$, where $f_0, \dots, f_t$, $g_0, \dots, g_t$, $h_0, \dots, h_{t+1} \in U(p^m - 2)$. Then the elements $f_0, f_1, \dots, f_t$ commute pairwise. The same is true for the elements $g_0, g_1, \dots, g_t$ and $h_0 + h_1 z^{-1}$, $h_2, \dots, h_{t+1}$.*

PROOF. We show that $\{f_i, f_j\} = 0$, $0 \leqslant i, j$. The remaining assertions of the lemma are proved analogously. It suffices to establish that $\Phi(f_i, f_j) = 0$. Then from the condition $\{z_1, f_j\} = 0$ we would obtain that

$$0 = \sum_{i \geqslant 0} \{f_i, f_j\} z^{-i} = \sum_{i \geqslant 0} [f_i, f_j] z^{-i},$$

whence $[f_i, f_j] = 0$. We argue by induction on $i + j = 0, 1, \ldots$. For $i + j = 0$, the assertion is trivial. We assume that it is proved for $i + j - 1$. By (3) and (4) we have

$$\begin{aligned}\Phi(f_i, f_j) &= \sum_{l=0}^{(p^m-1)/2} -\left(\frac{d}{dl} f_i\right)[e_l, f_{j-1}] + \sum_{l=(p^m+1)/2}^{p^m-2} [e_{p^m-l}, f_{i-1}] \frac{d}{dl} f_j \\ &= \sum_{l=2}^{(p^m-1)/2} -\left(\frac{d}{dl} f_i\right)[e_l, f_{j-1}] + [e_l, f_{i-1}] \frac{d}{dl} f_j.\end{aligned}$$

Let $2 \leqslant l \leqslant (p^m - 1)/2$. By (5) it is obvious that $(df_i/dl)e_l = f_i$ and

$$\begin{aligned}\left(\frac{d}{dl} f_i\right)[e_l, f_{j-1}] &= \left(\frac{d}{dl} f_i\right) e_l f_{j-1} - \left(\frac{d}{dl} f_i\right) f_{j-1} e_l \\ &= [f_i, f_{j-1}] - \left[\frac{d}{dl} f_i, f_{j-1}\right] e_l = (-1)^{l+1} [[e_{p^m-l}, f_{i-1}], f_{j-1}] e_l.\end{aligned}$$

Analogously, for $2 \leqslant l \leqslant (p^m - 1)/2$,

$$[e_l, f_{i-1}] \frac{d}{dl} f_j = e_l f_{i-1} \frac{d}{dl} f_j - f_{i-1} e_l \frac{d}{dl} f_j = (-1)^l e_l [[f_{i-1}, e_{p^m-l}], f_{j-1}].$$

Thus,

$$\Phi(f_i, f_j) = \sum_{l=2}^{(p^m-1)/2} (-1)^l [e_l, [[e_{p^m-l}, f_{i-1}], f_{j-1}]].$$

According to the inclusion (5),

$$[[e_{p^m-l}, f_{i-1}], f_{j-1}] \in \langle e^\alpha | \alpha_j = 0,\ j \leqslant p^m - l \rangle.$$

Hence $[e_l, [[e_{p^m-l}, f_{i-1}], f_{j-1}]] = 0$, as we were required to prove.

To describe the irreducible representations of the Virasoro algebra it is useful to give the explicit formulas

$$f_1 = \sum_{i=2}^{p^m-2} \left((-1)^i (i+2)(i-1) e_{i+1} e_{p^m-i}\right)/4,$$

$$g_1 = \sum_{i=2}^{p^m-2} \left((-1)^i i e_i e_{p^m-i}\right)/2, \qquad h_1 = e_1 e_{p^m-2} - \sum_{i=2}^{p^m-3} \left((-1)^i e_i e_{p^m-i-1}\right)/2.$$

§3. The irreducible representations of the Virasoro algebra

In this section we assume that P is algebraically closed.

Let M be an irreducible $\tilde{W}_1(m)$-module with respect to the representation $x \to (x)_M$. By Schur's lemma each of the endomorphisms $(e_i)_M$, $i \neq 0$, and $(e_0^p - e_0)_M$ has a unique eigenvalue θ_i, $i \neq 0$, or θ_0^p. We call the ordered set of these eigenvalues $(\theta_{-1}, \theta_0, \ldots, \theta_{p^m})$, $\theta_{p^m-1} = 0$, the invariant of the $\tilde{W}_1(m)$-module M. Analogously, for the irreducible$\overline{W}_1(m)$-module $\overline{M}$, we call the set $(\bar{\theta}_{-1}, \bar{\theta}_0, \ldots, \bar{\theta}_{p^m})$, where $\bar{\theta}_i$, $i \neq 0$, and $\bar{\theta}_0^p$ are the eigenvalues of the endomorphisms $(\bar{e}_i)_{\overline{M}}$, $i \neq 0$, and $(\bar{e}_0^p - \bar{e}_0)_{\overline{M}}$, the invariant of the

$\overline{W}_1(m)$-module $\overline{M}$. We say that M has height q if $\theta_i = 0$, $i \geqslant q$, and $\theta_{q-1} \neq 0$. As was shown in [15], Proposition 3, this is equivalent to the definition of height in the sense of A. N. Rudakov: $\mathscr{L}_q M = 0$ and $\mathscr{L}_{q-1}M \neq 0$. For every subalgebra L' in $\tilde{W}_1(m)$ or in $\overline{W}_1(m)$ which contains z, we call M or $\overline{M}$ and L'-module of the first type if $(z)_M = 0$ or $(z)_{\overline{M}} = 0$, and of the second type otherwise.

Every irreducible $W_1(m)$-module structure can be extended to an irreducible $\tilde{W}_1(m)$-module structure by assuming $(z)_M = 0$, and conversely, every irreducible $\tilde{W}_1(m)$-module of the first type is an irreducible $W_1(m)$-module. All irreducible $W_1(m)$-modules of nonextremal height $1 < q \leqslant p^m - 2$ are induced by irreducible $\mathscr{L}_0$-modules of height q; that is, by irreducible $\mathscr{L}_0/\mathscr{L}_q$-modules. We give a proof of this fact based on the ideas of [15]. In the latter, this is done for the Witt algebra $W_1(1)$. In the case of the Zassenhaus algebra $W_1(m)$ it remained to prove that every linear combination of the form

$$f = \sum_{\alpha=0}^{s} \lambda_\alpha e_{-1}^{\alpha}(v) = 0$$

is trivial (we recall that every irreducible $\mathscr{L}_0(W_1(m))$-submodule is induced by a one-dimensional module $\langle v \rangle$ over the weight subalgebra $\mathscr{A}$. We can assume that s is divisible by p. We represent it in the form $s = p^r s'$, where p and s' are coprime. If $q < p^m - p^{m-1}$, then $e_{q+p^r-1} \in \mathscr{A}$, and from the condition

$$0 = e_{q+p^r-1}(f) = \lambda_s s'\theta(e_{q-1}) e_{-1}^{p^r(s'-1)}(v) + g, \qquad g \in \left\langle e_{-1}^{k}(v) | k < p^r(s'-1) \right\rangle,$$

we obtain the contradiction $\lambda_s s'\theta(e_{q-1}) = 0$. Now let $q \geqslant p^m - p^{m-1}$. Then there is an $i \geqslant p^r$ for which $e_i \in \mathscr{A}$ but $e_{i-p^r} \notin \mathscr{A}$ otherwise we would have $\mathscr{A} = \mathscr{L}_0$; and this is possible only when q has the form $p^j - 1$ and $2^j - 2$ for $p = 2$). Thus, for the dual element $e_{i-p^r}^* \in \mathscr{A}$ we have

$$0 = e_{i-p^r}^* e_i(f) = \lambda_s s'\theta\left(\left[e_{i-p^r}^*, e_{i-p^r}\right]\right) e_{-1}^{p^r(s'-1)}(v) + g',$$

where $g' \in \langle e_{-1}^k(v) | k < p^r(s'-1)\rangle$. We obtain the contradiction $\lambda_s s' = 0$.

The considerations of [13] regarding the irreducible representations of the Witt algebra $W_1(1)$ of maximal height $q = p - 1$ go through with appropriate natural modifications also in the case of the Lie algebra $W_1(m)$. An irreducible $W_1(m)$-module of maximal height is also irreducible as an $\mathscr{L}_0(W_1(m))$-module. We only add that the Zassenhaus algebra is a Casimir algebra: the embedding of the coadjoint $W_1(m)$-module in the $W_1(m)$-module $U(W_1(m))$ is given by the rule

$$e_i \mapsto (-1)^{i+1}(\operatorname{Ad} e_{-1})^{i+1}\left(e_{p^m-2}^{(p-1)/2}\right), \qquad -1 \leqslant i \leqslant p^m - 2,$$

and hence the generalized Casimir element of the Zassenhaus algebra has the form

$$\sum_{i=-1}^{p^m-2} (-1)^{i+1} e_i (\operatorname{Ad} e_{-1})^{i+1}\left(e_{p^m-2}^{(p-1)/2}\right).$$

For $p > 3$ the irreducible $W_1(m)$-modules are also studied in [6].

We begin the description of the irreducible $\tilde{W}_1(m)$-modules of the second type. Let A and B be arbitrary Lie algebras, M_1 an A-module, and M_2 a B-module. On the tensor product $M_1 \otimes M_2$ we introduce a module structure over the direct sum $A \oplus B$ by the rule $(a + b)(v_1 \otimes v_2) = av_1 \otimes v_2 + v_1 \otimes bv_2$.

LEMMA 5. *Let M_1 and M_2 be irreducible A- and B-modules. Then the $A \oplus B$-module $M_1 \otimes M_2$ is also irreducible.*

PROOF. Let M be a nonzero $A \oplus B$-submodule of $M_1 \otimes M_2$. Then

$$M(v_2) = \langle v_1 \in M_1 | v_1 \otimes v_2 \in M \rangle \neq \langle 0 \rangle$$

for some $v_2 \in M_2$. Since $M(v_2)$ is invariant with respect to A, this implies that $M_1 = M(v_2)$. In addition, $M(bv_2) \supseteq M(v_2)$ for any $b \in B$. Hence $M = M_1 \otimes M_2$.

In the case $B = \mathscr{L}_2(\tilde{W}_1(m))$, the reverse is also true. To prove this, we give a description of the irreducible $\mathscr{L}_2$-modules of the second type. The algebra $\mathscr{L}_2$ is nilpotent, its subalgebra $\mathscr{A} = \mathscr{L}_{(p^m+1)/2} = \langle e_i | (p^m + 1)/2 \leqslant i \leqslant p^m \rangle$ is abelian, and, moreover, $[e_i, e_j] \in \mathscr{A}$, $i + j \geqslant p^m$, and

$$\theta([e_i, e_j]) = \theta(0) = 0, \quad i + j > p^m, \qquad \theta([e_i, e_{p^m - i}]) \neq 0, \quad 2 \leqslant i \leqslant (p^m - 1)/2.$$

According to the results of [15] this means that for any irreducible $\mathscr{L}_2$-module M_2 of the second type the subalgebra $\mathscr{A}$ is a weight subalgebra, and M_2 is induced by the one-dimensional $\mathscr{A}$-module

$$M_2 = \operatorname{Ind}_{\mathscr{L}_2}(\mathscr{L}_{(p^m+1)/2}, \langle v \rangle) = \left\langle \prod_{i=2}^{(p^m-1)/2} e_i^{\alpha_i}(v) \,\middle|\, 0 \leqslant \alpha_i < p \right\rangle.$$

In particular, independently of the θ_i, $2 \leqslant i \leqslant p^m - 2$, the dimension of M_2 is equal to $p^{(p^m-3)/2}$. Let $\overline{M}$ be an irreducible $\overline{W}_1(m)$-module and M_1 the space of eigenvectors with respect to the weight subalgebra $\mathscr{L}_{(p^m+1)/2}$. Then M_1 is an A_1-module and, as was shown above,

$$\overline{M} = \left\langle \sum_{i=2}^{(p^m-1)/2} e_i^{\alpha_i}(v) \,\middle|\, v \in M_1 \right\rangle \cong M_1 \otimes M_2.$$

It is obvious that M_1, as an A_1-module, is irreducible, and this isomorphism is a module isomorphism over $A_1 \oplus \mathscr{L}_2$. In particular, the dimension of every irreducible $\overline{W}_1(m)$-module has the form $ip^{(p^m-3)/2}$, $0 < i \leqslant p$.

Every structure of an irreducible $W_1(m)$-module for which $(z)_M \neq 0$ can be uniquely extended to the structure of an irreducible module over $\hat{U}$; and conversely, to every irreducible $\hat{U}$-module there corresponds an irreducible $\tilde{W}_1(m)$-module of the second type. By Theorem 3

$$\hat{U}(\tilde{W}_1(m)) \cong \hat{U}(\overline{W}_1(m)) \cong U(A_1) \otimes \hat{U}(\mathscr{L}_2).$$

Thus there is a one-to-one correspondence between the irreducible modules of the second type over $\tilde{W}_1(m)$ and over $\overline{W}_1(m)$. We shall find the relations between the invariants of the corresponding irreducible modules. We recall that the elements E_{-1}, E_0, and E_1 (see the proof of Theorem 3) are polynomials (although noncommutative) in the elements e_j, $-1 \leqslant j < p^m$, and z^{-1}. To emphasize this, for E_i we sometimes write $E_i(e)$. We show that

$$\theta(E_1(e)) = E_1(\theta(e)). \tag{7}$$

In other words, to compute the eigenvalue of the automorphism E_1 (this is unique), it suffices to make the substitution $e_i \mapsto \theta_i = \theta(e_i)$, $1 \leqslant i \leqslant p^m$, in the polynomial $E_1(e)$.

Let $f_l = \sum_\alpha \mu_\alpha e^\alpha$, $\mu_\alpha \in P$. If $e^\alpha = \prod_i e_i^{\alpha_i}$, $\mu_\alpha \neq 0$, then by the inclusion (5) we have $\{e_i^{\alpha_i}, e_j^{\alpha_j}\} = 0$. Therefore M contains a vector v which is an eigenvector for all the $e_i^{\alpha_i}$,

$1 \leqslant i \leqslant p^m$, and

$$\theta(e^\alpha)v = e^\alpha(v) = \prod_i \theta(e_i)^{\alpha_i} v = \theta(e)^\alpha v.$$

According to (5) and the Jacobson formula,

$$f_l^p = \sum_\alpha \mu_\alpha^p (e^\alpha)^p,$$

since in the Lie algebra $\tilde{W}_1(m)$ the commutators $\{\cdots\{e_{i_1}, e_{i_2}\}, \ldots, e_{i_p}\}$ are zero if none of the numbers $i_1, \ldots, i_p$ are of the form $p^k - 1, 0 < k < m$. Hence by Lemma 4

$$E_1^p = \sum_{l \geqslant 0} f_l^p z^{-lp} = \sum_{l \geqslant 0} \sum_\alpha \mu_\alpha^p (e^\alpha)^p z^{-lp}.$$

We note that the elements of the universal enveloping Virasoro algebra appearing in the sum E_1^p, namely $(e^\alpha)^p z^{-lp}$, are central (see (5)). By Schur's lemma, the corresponding module endomorphisms are scalar. Hence

$$\theta(E_1)^p = \sum_{l \geqslant 0} \sum_\alpha \mu_\alpha^p \theta(e)^{\alpha p} \theta(z)^{-lp}.$$

Having taken the pth roots of these equalities, we obtain (7).

The following relations are established using Lemma 4 and the inclusion (5) by analogous considerations:

$$\theta(E_0(e)) = E_0(\theta(e)) - ((E_0 - e_0)(\theta(e)))^{1/p}, \tag{8}$$

$$\theta(E_{-1}(e)) = \theta\left(e_{-1} + \left(e_1 e_{p^m-2} - \sum_{i=2}^{p^m-3} (-1)^i (e_i e_{p^m-i-1})/2\right) z^{-1}\right) + E'(\theta(e)), \tag{9}$$

where

$$E' = E_{-1} - h_0 - h_1 z^{-1} = \sum_{l \geqslant 2} h_l z^{-l}.$$

EXAMPLE. $p = 5$, $m = 1$. Then

$$\theta(E_1) = \theta_1 + \theta_3^2 \theta_5^{-1}, \qquad \theta(E_0) = \theta_0 + 2\theta_2\theta_3\theta_5^{-1} - 2\theta_2^{1/5}\theta_3^{1/5}\theta_5^{-1/5},$$

$$\theta(E_{-1}) = \theta\left(e_{-1} + \left(e_1 e_3 + 2e_2^2\right) z^{-1}\right) - \theta_3^3 \theta_5^{-2}.$$

We summarize our investigations.

THEOREM 4. *Let P be an algebraically closed field of characteristic $p \geqslant 3$. There is a one-to-one correspondence between the following classes of irreducible modules*:

(i) *over* $\tilde{W}_1(m)$, $(z)_M \neq 0$;

(ii) *over* $\overline{W}_1(m)$, $(z)_{\overline{M}} \neq 0$; *and*

(iii) *the set of pairs* (M_1, M_2), *where M_1 is an irreducible module over the three-dimensional simple Lie algebra of type A_1, and M_2 is an irreducible $\mathscr{L}_2(W_1(m))$-module for which* $(z)_{M_2} \neq 0$.

Here to a pair (M_1, M_2) there corresponds the irreducible $\overline{W}_1(m)$-module $\overline{M} = M_1 \otimes M_2$. The invariants of the irreducible $\tilde{W}_1(m)$-module M, $(z)_M \neq 0$, and of the corresponding $\overline{W}_1(m)$-module $\overline{M}$, $(z)_{\overline{M}} \neq 0$, are connected by the relations $\bar{\theta}_i = \theta_i$, $2 \leqslant i \leqslant p^m$, and $\bar{\theta}_i = \theta(E_i(e))$, $i = 0, \pm 1$, where the $\theta(E_i(e))$ are given by (7)–(9). Every irreducible $\mathscr{L}_2(\tilde{W}_1(m))$-module of the second type is induced by a one-dimensional $\mathscr{L}_{(p^m+1)/2}$-module. The dimension of an irreducible $\tilde{W}_1(m)$-module of the second type has the form $ip^{(p^m-3)/2}$, where $0 < i \leqslant p$.

The analog of the correspondence (i) $\leftrightarrow$ (iii) in the case of the Lie p-algebra $\tilde{W}_1(1)$ is established in [12]. The irreducible A_1-modules are described in [3]. We recall some of the results of this paper. Every irreducible p-module over $A_1 \cong \langle e_0, e_{\pm 1} | [e_0, e_{\pm 1}] = \pm e_{\pm 1}, [e_{-1}, e_{+1}] = e_0 \rangle$ is induced by a one-dimensional $\langle e_0, e_1 \rangle$-module $\langle v \rangle$ for which $e_0 v = \gamma(e_0)v$, $\gamma(e_0) \in \mathbf{Z}/p\mathbf{Z}$. They are all obtained by reduction modulo p from the standard A_1-modules of dimension at most p, where the maximal dimension is attained only in the case where $\gamma(e_0) = -1/2$. Using these facts, from Theorem 4 we obtain the following corollaries:

COROLLARY 1 ($p \geqslant 3$). *The least dimension of an irreducible $\tilde{W}_1(m)$-module M of the second type is $p^{(p^m-3)/2}$, and M has this least dimension if and only if*
$$E_1(\theta(e)) = 0, \qquad E_0(\theta(e))^p = (E_0 - e_0)(\theta(e)),$$
$$\theta(h_0 + h_1 z^{-1}) + E'(\theta(e)) = 0, \qquad \gamma(E_0) = 0,$$
where $E' = \sum_{l \geqslant 2} h_l z^{-l}$ and $E_{-1} = \sum_{l \geqslant 0} h_l z^{-l}$.

COROLLARY 2 ($p \geqslant 3$). *The greatest dimension of an irreducible $\tilde{W}_1(m)$-module M is $p^{(p^m-1)/2}$, and M has the greatest dimension if and only if M belongs to one of the following classes of modules: of height $q = p^m - 1$ (that is, $\theta_{p^m-2} \neq 0$, $(z)_M = 0$), or of height $q = p^m + 1$ (that is $(z)_M \neq 0$) for which*
$$E_1(\theta(e))\big(E_0(\theta(e))^p - (E_0 - e_0)(\theta)(e)\big)\big(\theta(h_0 + h_1 z^{-1}) + E'(\theta(e))\big) \neq 0$$
or
$$E_1(\theta(e)) = 0, \qquad E_0(\theta(e))^p = (E_0 - e_0)(\theta(e)),$$
$$\theta(h_0 + h_1 z^{-1}) + E'(\theta(e)) = 0, \qquad \gamma(E_0) = -1/2.$$

In the study of Cartan extensions we need from these descriptions of irreducible $\tilde{W}_1(m)$-modules only one fact: that the dimension of every irreducible $\tilde{W}_1(m)$-module of the second type is greater than the dimension of the Lie algebra $\tilde{W}_1(m)$, $(p, m) \neq (5, 1)$.

§4. Cartan extensions

Almost all the nonclassical simple Lie algebras of positive characteristic known up to now are the Lie algebras of Cartan types [1]. They are all graded: $I = \oplus_{i \geqslant -q} L_i$, $[L_i, L_j] \subseteq L_{i+j}$, where L_{-1} is an irreducible L_0-module and in the case where the grading has depth $q = 2$ (in the case of contact Lie algebras) there is a nonsingular skew-symmetric form F: $L_{-1} \times L_{-1} \to L_{-2}$; moreover, $\dim L_{-2} = 1$. For the Lie algebras of Cartan types, L_0 is a simple classical algebra or a trivial central extension of such. Now we suppose that we are given a Lie algebra L_0, an irreducible L_0-module L_{-1}, and, in the case $q = 2$, a skew-symmetric nonsingular form F: $L_{-1} \times L_{-1} \to L_{-2}$, $\dim L_{-2} = 1$. Is it possible to construct the spaces $L_1, L_2, \ldots$ in such a way that a new simple Lie algebra $\oplus_{i \geqslant -q} L_i$ would result (that is, one which does not fit the Kostrikin-Shafarevich conjecture)? To answer this question one must first compute the first Cartan extension $L_0^{(1)}$. We recall that
$$L_0^{(1)} = \big\langle \psi \in \operatorname{Hom}(L_{-1}, L_0) \,|\, \psi(x)y = \psi(y)x \big\rangle$$
for $q = 1$. Since there is a natural imbedding $L_0^{(1)} \supseteq L_1$, if $L_0^{(1)} = 0$, then we immediately obtain that $\oplus_{i \geqslant -q} L_i$ is not a simple Lie algebra. In the following cases it is impossible to obtain new simple Lie algebras by such methods: $L_0 = W_1(1)$, $q = 1$ (see [2]); $L_0 = W_1(m)$, $q = 1$, and L_{-1} is an irreducible L_0-module of nonextremal height (this follows

from the results of [4] and [6]); and $L_0 = W_1(m) \oplus P$, $q = 2$, $(p, m) \neq (5, 1)$ (see [5]). In [5], in the exceptional case $(p, m) = (5, 1)$ a new simple Lie algebra of depth 2 is constructed, for which $L_0 = W_1(1) \oplus P$. We study the case $L_0 = \tilde{W}_1(m)$. Our arguments are general. For example, they imply that the foregoing facts remain valid if $L_0 = W_1(m)$ and L_{-1} is an L_0-module of maximal height. Direct computations show that in the case $q = 1$ and $L_0 = W_1(m) \oplus P$, for $L_0^{(1)}$ (this is nontrivial only for $\dim L_{-1} \leqslant p^m$) we have the condition $[L_{-1}, L_0^{(1)}] \subseteq [L_0, L_0]$. So if $L_0/Z(L_0) \cong W_1(m)$, $\dim Z(L_0) \leqslant 1$, then a new simple Lie algebra can arise only in the case $L_0 = W_1(1) \oplus P$, $p = 5$, $q = 2$.

If L_0 is a nontrivial central extension of a simple algebra, then there does not exist a graded simple Lie algebra of depth 2. This follows from the following proposition.

PROPOSITION 3. *Let L_{-1} be an irreducible L_0-module, and let F: $L_{-1} \times L_{-1} \to L_{-2}$ be a skew-symmetric nonsingular form on it. If* $\dim L_{-2} = 1$ *and* $[L_0^{(1)}, L_{-2}] \neq 0$, *then* $[L_0, L_0] \neq L_0$.

PROOF. We assume that $[L_0, L_0] = L_0$. Then L_{-2} is a one-dimensional trivial L_0-module. Hence for any $x_{-1} \in L_{-1}$

$$F([x_{-2}, x_1], x_{-1}) = [x_{-2}, [x_1, x_{-1}]] = 0, \qquad x_1 \in L_0^{(1)}.$$

Since the form F is nondegenerate, $F[x_{-2}, x_1] = 0$. Contradiction.

In the computation of Cartan extensions, the following simple assertion plays an important role.

PROPOSITION 4. *Let L_{-1} be an arbitrary L_0-module and let $L_0^{(1)}$ be the first Cartan extension of depth* 1. *Assume that for some $x_{-1} \in L_{-1}$ and $x_1 \in L_0^{(1)}$ the endomorphism $([x_{-1}, x_1])_{L_{-1}}$ is nonsingular. Then the linear map $L_{-1} \to L_0$ given by the rule $v \mapsto [x_1, v]$ is injective. In particular*, $\dim L_{-1} \leqslant \dim L_0$.

PROOF. For any $u, v \in L_{-1}$, by definition $[[x_1, x_{-1}], v] = [[x_1, v], x_{-1}]$. Hence from the condition $[x_1, v] = 0$ it follows that $v = 0$.

COROLLARY 1. *Let L be a simple Lie algebra of Cartan type over an algebraically closed field P of characteristic $p \geqslant 5$, and let M be an irreducible L-module which is not a p-module. If* $\dim M > \dim L$, *then the first Cartan extension of depth* 1 *is trivial.*

PROOF. If $[L^{(1)}, M] = 0$, then the assertion is true. Since $[L^{(1)}, M]$ is an ideal in L, we can assume that $L = [L^{(1)}, M]$.

Let $V(L)$ be the standard basis in L, and let $|\ |$: $L \to \mathbf{Z}$ be a map giving the grading $L = \oplus_{i \geqslant -q} L_i$, $L_i = \langle v \in L \,|\, |v| = i \rangle$. Let T be the standard torus, and let $L_0^- \oplus T$ and $L_0^+ \oplus T$ be Borel subalgebras in L_0. Let $\mathscr{L}^- = L_0^- \oplus \oplus_{i<0} L_i$ and $\mathscr{L}^+ = L_0^+ \oplus \oplus_{i<0} L_i$. We assume that for any $x \in L$, the endomorphisms $(x)_M$ are nonsingular. Let $\mu(x)^p$ be an eigenvalue of the endomorphism $(x^p - x^{[p]})_M$. If ad x is nilpotent, then

$$\mu(x) = \sum_{i \geqslant 0} \theta\left(x^{[p^i]}\right)^{p^{-i}}.$$

Since in the minimal p-hull of a Lie algebra of Cartan type [1], [15], the p-map $x \mapsto x^{[p]}$ satisfies the condition $|x^{[p]}| > |x|$, $x \in \mathscr{L}^+$, or $|x^{[p]}| < |x|$, $x \in \mathscr{L}^-$, from this formula we obtain that $\mu(x) = 0$ for any $x \in \mathscr{L}^{\pm}$. For basis elements of the torus the p-map is given as $e_0^p = e_0$. Hence for $v \in \operatorname{Ker}(e_0)_M$ we have $\mu(e_0)^p v = (e_0^p - e_0)v = 0$. In other words, M is a p-module over L. The resulting contradiction shows that $([v, f])_M$ is nonsingular for some $v \in M$ and $f \in L^{(1)}$.

It is very likely that the following holds.

CONJECTURE. *Let L be a simple Lie algebra of Cartan type over an algebraically closed field of characteristic $p \geqslant 5$. If the height of an irreducible L-module M is greater than* 1, *or in the case of height at most* 1, *M contains an irreducible $\mathscr{L}_0$-submodule of dimension greater than* 1, *then* $\dim M > \dim L$. *Hence it suffices to study Cartan extensions of irreducible p-modules over L.*

COROLLARY 2. *Let $L_0 = W_1(m)$, and let L_{-1} be an irreducible L_0-module of height $p^m - 1$. Then $L_0^{(1)} = 0$, $p \geqslant 3$, $(p, m) \neq (3, 1)$.*

PROOF. As we established in §3, $\dim M > L_0$. Let $L_0 = \tilde{W}_1(m)$, $p \geqslant 3$. We put $|x| = i$ if $x \in \mathscr{L}_i$ but $x \notin \mathscr{L}_{i+1}$. Every ideal J in L_0 contains z. Indeed, there exists $x \in J$ for which $|x| \geqslant 2$ (otherwise the element $[e_3, x] \in J$ has this property); then $J \supseteq \langle [x, e_{p^m - |x|}] \rangle \ni z$. Let L_{-1} be an irreducible L_0-module. Since $[L_0^{(1)}, L_{-1}]$ is an ideal in L_0, we have $z \in [L_0^{(1)}, L_{-1}]$. If $(z)_{L_{-1}} \neq 0$, then by Theorem 4 and Proposition 4, $L_0^{(1)} = 0$ (in the case $(p, m) = (5, 1)$ this is established by direct verification). If $(z)_{L_{-1}} = 0$, then an easy induction on i shows that $[z, L_i] = 0$. Thus we have proved the following.

THEOREM 5. *For $p \geqslant 3$ there is no simple graded Lie algebra $\bigoplus_{i \geqslant -q} L_i$, $q \leqslant 2$ (in the case $q = 2$ it is assumed that* $\dim L_{-2} = 1$), *for which $L_0 \cong \tilde{W}_1(m)$.*

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