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COHOMOLOGY OF TRUNCATED COINDUCED REPRESENTATIONS OF LIE ALGEBRAS OF POSITIVE CHARACTERISTIC

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ABSTRACT. The author proves that for any n-dimensional Lie algebra of characteristic $p > 0$ and any k, $0 \leq k \leq n$, there exists a finite-dimensional module with nontrivial k-cohomology; the nontrivial cocycles of such modules become trivial under some finite-dimensional extension. He also obtains a criterion for the Lie algebra to be nilpotent in terms of irreducible modules with nontrivial cohomology. The proof of these facts is based on a generalization of Shapiro's lemma. The truncated induced and coinduced representations are shown to be the same thing.

Bibliography: 22 titles.

In this paper, unless we specify otherwise, all Lie algebras and their modules are assumed to be finite dimensional, and except in §1 we shall always be working over fields of characteristic $p > 0$.

Let L be an n-dimensional Lie algebra over a field P of characteristic $p > 0$, and let M be an L-module. We say that M is *k-special* if its k-cohomology is nontrivial: $H^k(L, M) \neq 0$. We call M an *annihilable* module, and say that M is annihilated in the L-module N, if there exists a monomorphism of finite-dimensional L-modules $M \to N$ such that the corresponding cohomology homomorphism $H^k(L, M) \to H^k(L, N)$ is zero for all $k > 0$. We say that M is *strongly annihilable* if there exists a monomorphism of finite-dimensional L-modules $M \to N$ such that $H^k(L, N) = 0$ for all $k > 0$.

The following facts are true for any Lie algebra L of characteristic $p > 0$:

(i) *For any $0 \leq k \leq n$ there exists at least one k-special L-module* (Corollary 2 of Theorem 1.3).

(ii) *The number of nonisomorphic irreducible special L-modules, which we denote $\kappa(L)$, is finite* (see [7]).

(iii) *Every L-module is annihilable, but there exist L-modules which are not strongly annihilable* (Theorem 3.1).

If the field is algebraically closed, then we have the following characterization of nilpotent Lie algebras: *L is nilpotent if and only if $\kappa(L) = 1$.*

In the proof of these facts an important role is played by the concepts of a generalized coinduced module and generalized subalgebra, and by the corresponding generalized Shapiro lemma. Despite their simplicity, these generalizations are of independent interest.

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The assertion (i) was stated earlier as a conjecture by Seligman [9]. Jacobson [2] proved it for $k = 1$. The assertion (iii) was proved for $k = 2$ by Iwasawa [19]. The papers [14]–[16] are devoted to the question of annihilability of 3-cocyles—more precisely, the question of interpreting the space $H^3(L, M)$. When $k = 3$, (iii) shows that *the entire space* $H^3(L, M)$, $p > 0$, *can be realized as a space of obstructions for L-kernels with center* M. This fact is also demonstrated in [21]. The group analogue of this result is proved in [20]. In [17] annihilability is proved for modules of solvable Lie algebras of characteristic zero.

A few words on notation. Let $\langle \mathfrak{X} \rangle$ denote the linear span of the set $\mathfrak{X}$ over the field P. The following cohomological notation is standard (for details, see [13]):

$C^*(L, M) = \bigoplus_k C^k(L, M)$ is the space of cochains, $C^0(L, M) = M$, and $C^k(L, M) = 0$ if $k < 0$.

d is the coboundary operator, i.e., $d: C^k(L, M) \to C^{k+1}(L, M)$ is defined as follows:

$$d\psi(X_1, \dots, X_{k+1}) = \sum_{i<j} (-1)^{i+j} \psi([X_i X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) + \sum_i (-1)^{i+1} (X_i)_M \psi(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})$$

(the $\widehat{\ }$ means that the corresponding element is omitted).

$Z^*(L, M) = \langle \psi \in C^*(L, M) | d\psi = 0 \rangle$ is the space of cocycles.

$B^*(L, M) = \langle d\psi | \psi \in C^*(L, M) \rangle$ is the space of coboundaries.

$H^*(L, M) = Z^*(L, M)/B^*(L, M)$ is the cohomology space.

Recall that $H^0(L, M)$ is the space of invariants M^L. Given a space A, we let $V(A)$ denote a basis.

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§1. Generalized subalgebras and the generalized Shapiro lemma

1. Generalized (co)induced modules. We make the universal enveloping algebra $\mathfrak{U}(L)$ of the Lie algebra L into left and right regular $\mathfrak{U}(L)$-modules. We say that a Lie algebra $\widetilde{A}$ is a *generalized subalgebra* of L if $\mathfrak{U}(L)$ has a subalgebra isomorphic to $\mathfrak{U}(\widetilde{A})$ (which we shall identify with $\mathfrak{U}(\widetilde{A})$) and $\mathfrak{U}(L)$ is free as a $\mathfrak{U}(\widetilde{A})$-module (as a regular bimodule). By the Poincaré-Birkhoff-Witt theorem, a subalgebra in the usual sense is obviously a generalized subalgebra.

Let $\mathscr{A}$ and $\mathscr{B}$ be associative algebras with unit, with $\mathscr{B}$ a subalgebra of $\mathscr{A}$, and let M be a $\mathscr{B}$-module (unless we specify otherwise, modules over associative algebras will be assumed to be left modules: $(bb')m = b(b'm))$. We make $\mathscr{A}$ into a two-sided $\mathscr{B}$-module. We introduce an $\mathscr{A}$-module structure on the tensor product $A \otimes_{\mathscr{B}} M$ over $\mathscr{B}$ according to the rule $a(a' \otimes m) = aa' \otimes m$. We introduce an $\mathscr{A}$-module structure on the space of $\mathscr{B}$-homomorphisms

$$\mathrm{Hom}_{\mathscr{B}}(\mathscr{A}, M) = \langle f: \mathscr{A} \to M | f(ba) = bf(a), a \in \mathscr{A}, b \in \mathscr{B} \rangle$$

by means of the formula $(a \circ f)(a') = f(a'a)$, $a, a' \in \mathscr{A}$.

In what follows, without special mention we shall make use of the fact that every module structure over a Lie algebra can be uniquely extended to a module structure over the universal enveloping algebra, and, conversely, every module over the universal enveloping algebra is also a module over the Lie algebra.

Let $\widetilde{A}$ be a generalized subalgebra of L, and let M be an $\widetilde{A}$-module. By the *generalized induced module*, denoted $\mathrm{Ind}_{\widetilde{A}}M$, we mean the L-module $\mathfrak{U}(L)\otimes_{\mathfrak{U}(\widetilde{A})} M$. We similarly define the *generalized coinduced module*, denoted $\mathrm{Coind}_{\widetilde{A}}M$, to be $\mathrm{Hom}_{\mathfrak{U}(\widetilde{A})}(\mathfrak{U}(L), M)$. We note that in the case of subalgebras in the usual sense these definitions coincide with the usual definitions of induced and coinduced modules.

2. The case of characteristic $p > 0$. Truncated (co)induced modules. Let A be a subalgebra of codimension s in a Lie algebra L over a field P of characteristic $p > 0$. We choose a basis $V(L) = \{e_1, \dots, e_n\}$ of L in such a way that the subset $V(A) = \{e_{s+1}, \dots, e_n\}$ is a basis of the subalgebra A. It is known that for any $e_i \in V(L)$ there exists a central element of the form $z_i = \sum_{t=0}^{m_i} \lambda_i(t)e_i^{p^t} \in \mathfrak{U}(L)$ (see [2]).

To any subalgebra A we can associate a generalized subalgebra $\widetilde{A}$ of the same dimension as L, which is the direct sum of A and an s-dimensional abelian subalgebra $\widetilde{Z}$. In fact, if we associate each basis element $\tilde{z}_i$ of $\widetilde{Z}$ to a central element z_i, $1 \le i \le s$ (we identify z_i with $\tilde{z}_i$), and if we leave alone the elements of the subalgebra A, we see that the induced map $\mathfrak{U}(\widetilde{A}) \to \mathfrak{U}(L)$ is a monomorphism and, by Lemma 4 in [2], Chapter V, §7, $\mathfrak{U}(L)$ is a free $\mathfrak{U}(\widetilde{A})$-module.

In what follows, given any A-module M, we shall suppose that it is made into an $\widetilde{A}$-module by means of the rule $\tilde{z}_i M = 0$, $1 \le i \le s$.

Let $I(L)$ be the ideal of the algebra $\mathfrak{U}(L)$ which is generated by the elements $z_1, \dots, z_n$, and let $\overline{\mathfrak{U}}(L) = \mathfrak{U}(L)/I(L)$. Since $I(A) = I(L) \cap \mathfrak{U}(A)$, we may take $\overline{\mathfrak{U}}(A)$ to be a subalgebra of $\overline{\mathfrak{U}}(L)$. Thus, we may introduce the L-modules

$$\overline{\mathrm{Ind}}_A M = \overline{\mathfrak{U}}(L) \otimes_{\overline{\mathfrak{U}}(A)} M, \qquad \overline{\mathrm{Coind}}_A M = \mathrm{Hom}_{\overline{\mathfrak{U}(A)}}(\overline{\mathfrak{U}}(L), M).$$

They are called the *truncated induced module* and the *truncated coinduced module*, respectively. Apparently the first published definition of three modules is due to Yu. B. Ermolaev.

It is obvious that truncated (co)induced modules are isomorphic to generalized (co)induced modules if $p > 0$. We prefer to work with the latter concept, since we have the following generalization of Shapiro's lemma.

3. Cohomology of a generalized coinduced module.

THEOREM. *Let $\widetilde{A}$ be a generalized subalgebra of the Lie algebra L, and let M be a module over $\widetilde{A}$. Then $H^k(L, \mathrm{Coind}_{\widetilde{A}}M) \cong H^k(\widetilde{A}, M)$.*

The proof is a repetition of the proof of Shapiro's lemma. According to Proposition 7.2 in [3], Chapter X, if we use one of the definitions of Lie algebra cohomology ($H^k(L, S) \cong \mathrm{Ext}^k_{\mathfrak{U}(L)}(P, S)$) we have

$$H^k(L, \mathrm{Coind}_{\widetilde{A}}M) \cong \mathrm{Ext}^k_{\mathfrak{U}(L)}(P, \mathrm{Coind}_{\widetilde{A}}M) \cong \mathrm{Ext}^k_{\mathfrak{U}(\widetilde{A})}(P, M) \cong H^k(\widetilde{A}, M).$$

COROLLARY 1 $(p > 0)$. *If A is a subalgebra of L and M is an A-module, then*

$$H^k(L, \mathrm{Coind}_{\widetilde{A}}M) \cong \bigotimes_{l=0}^{k} \bigwedge^{l}(L/A) \otimes H^{k-l}(A, M),$$

where $\bigwedge^l(L/A)$ is the lth exterior power of the space L/A.

PROOF. The following isomorphisms are compatible with the actions of the coboundary operators in $C^*(\widetilde{Z}, P)$ and $C^*(A, M)$ (the prime denotes the adjoint module):

$$C^*(\widetilde{A}, M) \cong \Lambda^*(\widetilde{Z} \oplus A)' \otimes M \cong \Lambda^*(\widetilde{Z})' \otimes (\Lambda^*(A)' \otimes M)$$
$$\cong C^*(\widetilde{Z}, P) \otimes C^*(A, M).$$

Thus, by the Künneth formula,

$$H^*(\widetilde{A}, M) \cong H^*(\widetilde{Z}, P) \otimes H^*(A, M) \cong \Lambda^*(L/A) \otimes H^*(A, M).$$

COROLLARY 2 ($p > 0$). *Every n-dimensional Lie algebra has a finite-dimensional module with nontrivial k-cohomology, for any $0 \le k \le n$.*

PROOF. We take A to be the zero subalgebra. Then the L-module $\widetilde{V} = \mathrm{Coind}_{\tilde{0}} P$ has the required property.

In the next section we construct cocycles whose classes form a basis for the space $H^*(L, \widetilde{V})$. We then prove a refinement of Corollary 2.

§2. Existence of a special module of the form $M \otimes N$

1. Statement of the basic result.

THEOREM ($p > 0$). *For any n-dimensional Lie algebra L and for any finite-dimensional L-module M there exists a finite-dimensional L-module N such that $H^k(L, M \otimes N) \ne 0$ for all $n \ge k \ge 0$.*

Before proving the theorem, we introduce some notation. Let ε_i be the vector with 1 in the ith coordinate and zeros everywhere else (the number of coordinates will be clear from the context), let $\mathbf{m} = \sum_i m_i \varepsilon_i$, where the positive integer m_i is determined from z_i (see §1), and let $\Gamma_n = \Gamma_n(\infty) = \{\alpha = \sum_i \alpha_i \varepsilon_i | 0 \le \alpha_i, i = 1, \dots, n\}$ and $\Gamma_n(\mathbf{m}) = \{\alpha \in \Gamma_n | 0 \le \alpha_i < p^{m_i}, i = 1, \dots, n\}$. In addition, instead of $\prod_i e_i^{\alpha_i}$ and $\prod_i x_i^{(\alpha_i)}$, where $\alpha \in \Gamma_n$, we shall write simply e^α and $x^{(\alpha)}$. Recall that multiplication in the divided power algebra $O_n(\infty) = \langle x^{(\alpha)} | \alpha \in \Gamma_n(\infty)\rangle$ is defined by

$$x^{(\alpha)} x^{(\beta)} = \prod_i \binom{\alpha_i + \beta_i}{\alpha_i} x^{(\alpha+\beta)}$$

and that this algebra contains the finite-dimensional subalgebra $O_n(\mathbf{m}) = \langle x^{(\alpha)} | \alpha \in \Gamma_n(\mathbf{m})\rangle$ of dimension p^m, where $m = \sum_i m_i$. Further, recall that a derivation $D \in \mathrm{Der}\, O_n(\infty)$ is said to be *special* if $D(x^{(\alpha)}) = \sum_i x^{(\alpha - \varepsilon_i)} D(x_i)$ for any $\alpha \in \Gamma_n(\infty)$. The Lie algebra of special derivations of $\widetilde{U} = O_n(\mathbf{m})$ is called a *general Cartan Lie algebra* and is denoted $W_n(\mathbf{m})$. For more details about these algebras, see [1].

The space $V = \mathrm{Coind}_0 P$ has an algebra structure relative to the bilinear map which is adjoint to the comultiplication, and every element of L acts as a derivation on V [11]. Moreover, under the isomorphism of associative algebras

$$\pi: V \to U = O_n(\infty), \qquad \pi(f) = \sum_{\alpha \in \Gamma_n} x^{(\alpha)} f(e^\alpha) \tag{1}$$

the derivation $X^\pi \in \mathrm{Der}\, U$, corresponding to $X \in L$ is special [10]. Recall that $X^n(u) = \pi(X\pi^{-1}(u))$, $u \in U$. Let $\mathrm{pr}: O_n(\infty) \to O_n(\mathbf{m})$ be the natural projection. It is easy to verify that

$$\tilde{\pi} = \mathrm{pr}\, \pi: \widetilde{V} = \mathrm{Coind}_{\tilde{0}} P \to \widetilde{U} = O_n(\mathbf{m}), \qquad \tilde{\pi}(f) = \sum_{\alpha \in \Gamma_n(\mathbf{m})} x^{(\alpha)} f(e^\alpha),$$

gives an isomorphism of algebras, under which

$$e_i^{\tilde{\pi}} = \operatorname{pr} e_i^{\pi} - \sum_{t=0}^{m_i-1} \lambda_i(t) x_i^{(p^{m_i}-1)} \partial_i^{p^t}, \qquad i = 1, \dots, n. \tag{2}$$

In particular, the L-module $\tilde{U}$ is filtered:

$$e_i^{\tilde{\pi}}(x^{(\alpha)}) - x^{(\alpha-\varepsilon_i)} \in \tilde{U}_{|\alpha|} = \langle x^{(\beta)} \mid |\beta| \geq |\alpha|, \beta \in \Gamma_n(\mathbf{m}) \rangle, \qquad |\alpha| = \sum_{i=1}^{n} \alpha_i. \tag{3}$$

2. Admissible systems of central elements. If M is an L-module, we let $(X)_M$ denote the endomorphism corresponding to the element $X \in \mathfrak{U}(L)$. For example, $(e_i)_U = e_i^{\pi}$ and $(e_i)_{\tilde{U}} = e_i^{\tilde{\pi}}$. We let id denote the identity endomorphism of a space (which space will be clear from the context). We shall omit the M in the notation $(X)_M v$ in cases when this will not lead to confusion. We shall say that a set of central elements $\{z_1, \dots, z_n\}$ is an *admissible system* for M if $(z_i)_M = 0$, $1 \leq i \leq n$. Recall that a polynomial of the form $z_i(a) = \sum_{t \geq 0} \lambda_i(t) a^t \in P[a]$ is called a *p-polynomial.*

LEMMA. *For any finite dimensional L-module M there exist p-polynomials $z_i(a) \in P[a]$, $1 \leq i \leq n$, such that the elements $z_i = z_i(e_i) \in \mathfrak{U}(L)$, $1 \leq i \leq n$, are central and $(z_i)_M = 0$, $1 \leq i \leq n$.*

The proof is the same as the proof of Lemma 5 in [2], Chapter VI, §5. Because M is finite dimensional, for any z_i there exists a polynomial $g_i(a) \in P[a]$ such that $g_i((z_i)_M) = 0$. Since any polynomial is a divisor of some p-polynomial, the assertion follows.

3. Nontrivial cocycles of the module $\tilde{U}$. We set

$$\mathrm{Sq}_i e = x_i^{(p^{m_i}-1)} e^{\tilde{\pi}}(x_i), \qquad 1 \leq i \leq n.$$

By (2), the cochain $\mathrm{Sq}_i \in C^1(L, \tilde{U})$ is a cocycle:

$$d\mathrm{Sq}_i(e, e') = e^{\tilde{\pi}}(x_i^{(p^{m_i}-1)})(e')^{\tilde{\pi}}(x_i) - (e')^{\tilde{\pi}}(x_i^{(p^{m_i}-1)}) e^{\tilde{\pi}}(x_i) = 0.$$

Here is an invariant definition of the cocycle Sq_i: one can introduce an L-module structure on $U \cong V$ (but with formulas for the map from V to U which are not as nice as (1)) such that it naturally contains $\tilde{U}$ as an L-submodule; since the class of the element $x_i^{(p^{m_i})}$ in $U/\tilde{U}$ is L-invariant, the cochain $dx_i^{(p^{m_i})}$—which is the image of $x_i^{(p^{m_i})}$ under the Bockstein homomorphism $Z^0(L, U/\tilde{U}) \to Z^1(L, \tilde{U})$—is a cocycle; this is the cocycle Sq_i.

We let $\wedge$ denote the multiplication in $C^*(L, \tilde{U})$ which is induced by the multiplication in $\tilde{U}$: if $\psi \in C^k(L, \tilde{U})$ and $\varphi \in C^l(L, \tilde{U})$, then the cochain $\psi \wedge \varphi \in C^{k+l}(L, \tilde{U})$ is defined by

$$(\psi \wedge \varphi)(X_1, \dots, X_{k+l}) = \sum_{\tau} (\operatorname{sgn} \tau) \psi(X_{\tau(1)}, \dots, X_{\tau(k)}) \varphi(X_{\tau(k+1)}, \dots, X_{\tau(k+l)})$$

(the summation is taken over all permutations $\tau \in S_{k+l}$ for which $\tau(1) < \cdots < \tau(k)$ and $\tau(k+1) < \cdots < \tau(k+l)$). Given an L-module M, we extend this multiplication to a pairing

$$C^*(L, \tilde{U}) \times C^*(L, \tilde{U} \otimes M) \to C^*(L, \tilde{U} \otimes M),$$

where the action of $\tilde{U}$ on $\tilde{U} \otimes M$ is given as follows: $u(v \otimes m) = uv \otimes m$.

LEMMA. *Suppose that* $\{z_1, \dots, z_n\}$ *is an admissible system of central elements for the L-module* M, *and the submodule of L-invariants* M^L *is not zero. Then* $H^k(L, \widetilde{U} \otimes M) \neq 0$ *for any* $0 \leq k \leq n$.

PROOF. We prove that the classes of the cocycles $\mathrm{Sq}_{i_1} \wedge \cdots \wedge \mathrm{Sq}_{i_k} \otimes m$, where $0 \neq m \in M^L$ and $1 \leq i_1 < \cdots < i_k \leq n$, are linearly independent in $H^k(L, \widetilde{U} \otimes M)$. Suppose that for some $a_{i_1, \dots, i_k} \in P$, $1 \leq i_1 < \cdots < i_k \leq n$, we have

$$\sum a_{i_1, \dots, i_k} \mathrm{Sq}_{i_1} \wedge \cdots \wedge \mathrm{Sq}_{i_k} \otimes m = d\Delta \in B^k(L, \widetilde{U} \otimes M). \tag{4}$$

Given an ordered k-tuple $\bar{i} = (i_1, \dots, i_k)$, we let $(i_{k+1}, \dots, i_n)$ denote the ordered $(n-k)$-tuple which is complementary to $\bar{i}$ in the set $\{1, \dots, n\}$. We multiply both sides of (4) by the cocycle $\mathrm{Sq}_{i_{k+1}} \wedge \cdots \wedge \mathrm{Sq}_{i_n} \in Z^{n-k}(L, \widetilde{U})$. We have

$$\pm a_{i_1, \dots, i_k} \mathrm{Sq}_1 \wedge \cdots \wedge \mathrm{Sq}_n = d(\Delta \wedge \mathrm{Sq}_{i_{k+1}} \wedge \cdots \wedge \mathrm{Sq}_{i_n}).$$

Thus, it is sufficient to prove the lemma in the case $k = n$.

So suppose that $k = n$ and $\mathrm{Sq}_1 \wedge \cdots \wedge \mathrm{Sq}_n \otimes m = d\omega \in B^n(L, \widetilde{U} \otimes M)$. Since

$$\mathrm{Sq}_1 \wedge \cdots \wedge \mathrm{Sq}_n(e_1, \dots, e_n) = x^{(\theta)}, \qquad \theta = \sum_{i=1}^{n} (p^{m_i} - 1)\varepsilon_i$$

and

$$d\omega(e_1, \dots, e_n) = \sum_{i=1}^{n} \delta_i(\omega(e_1, \dots, \hat{e}_i, \dots, e_n)), \qquad \delta_i = (e_i)_{\widetilde{U} \otimes M} - \mathrm{tr}(\mathrm{ad}\, e_i)\mathrm{id}$$

(the caret means that the corresponding element is omitted), it follows that

$$x^{(\theta)} = \sum_{i=1}^{n} \delta_i(u_i) \tag{5}$$

for some $u_i \in \widetilde{U} \otimes M$. We shall show that (5) is impossible.

We note that $(z_i)_{\widetilde{U}} = 0$, $1 \leq i \leq n$, since $(z_i)_{\widetilde{V}} = 0$:

$$(z_i \circ f)(X) = f(X z_i) = f(z_i X) = z_i(f(X)) = 0, \qquad Z \in \mathfrak{U}(L), \; f \in \widetilde{V}.$$

Hence, $(z_i)_{\widetilde{U} \otimes M} = (z_i)_{\widetilde{U}} \otimes \mathrm{id} + \mathrm{id} \otimes (z_i)_M = 0$ and

$$\delta_i^{p^{m_i}} = \lambda_i(m_i)^{-1} \left((z_i)_{\widetilde{U} \otimes M} - \sum_{t=0}^{m_i - 1} \lambda_i(t) \delta_i^{p^t} \right) \in \langle \delta_i^{p^t} \mid 0 \leq t < m_i \rangle$$

(we may suppose that $\lambda_i(m_i) \neq 0$). Thus, for any $\alpha \in \Gamma_n(\mathbf{m})$

$$\delta_i \delta^\alpha \in \langle \delta^\beta \mid \beta \in \Gamma_n(\mathbf{m}), \, \beta \neq 0 \rangle. \tag{6}$$

On the other hand, by (3), the set of elements $\delta^\alpha(x^{(\theta)} \otimes v)$, $v \in V(M)$, $\alpha \in \Gamma_n(\mathbf{m})$, forms a basis of the space $\widetilde{U} \otimes M$. If we write u_i as a linear combination of basis elements and use (6), we obtain a contradiction with (5): $\delta^0 x^{(\theta)}$ written linearly in terms of the other elements of the basis. The lemma is proved.

REMARK. It is clear from this proof that the classes of the cocycles $\mathrm{Sq}_{i_1} \wedge \cdots \wedge \mathrm{Sq}_{i_k}$, $1 \le i_1 < \cdots < i_k \le n$, are linearly independent in $H^k(L, \widetilde{U})$. By Corollary 1 of Theorem 1.3, they form a basis of $H^k(L, \widetilde{U})$.

4. Proof of Theorem 2.1. According to Lemma 2.2, we may suppose that $(z_i)_M = 0$, $1 \le i \le n$. Then $(z_i)_{M \otimes M'} = 0$, where M' is the adjoint L-module. In addition, $(M \otimes M')^L \neq 0$, since $M \otimes M' \cong \operatorname{End} M$ and the identity endomorphism of the module M is L-invariant. Thus, if we take $N = \widetilde{U} \otimes M'$, by Lemma 2.3 we have

$$H^k(L, M \otimes N) \cong H^k(L, \widetilde{U} \otimes (M \otimes M')) \neq 0,$$

as was to be shown.

§3. Annihilability of modules

1. Statement of the basic result.

THEOREM ($p > 0$). *For any finite-dimensional Lie algebra, all of its finite-dimensional modules are annihilable, but there exist finite-dimensional modules which are not strongly annihilable.*

More precisely, we prove that the cohomology homomorphisms

$$H^k(L, M) \to H^k(L, \widetilde{V} \otimes M), \qquad k > 0,$$

corresponding to the natural monomorphism $M \to \widetilde{V} \otimes M$, are zero for any admissible system of central elements. We also prove that the trivial module is not strongly annihilable. We first prove the latter assertion.

2. A property of modules with nontrivial cohomology.

LEMMA. *If $H^k(L, M) \neq 0$, then $H^l(L, M) \neq 0$ for some $l \neq k$.*

PROOF. For any finite-dimensional L-module M, the Euler characteristic

$$\chi(L, M) = \sum_k (-1)^k c_k,$$

where c_k is the dimension of $C^k(L, M)$, is equal to zero. In fact,

$$C^k(L, M) \cong \Lambda^k L' \otimes M, \qquad c_k = \binom{n}{k} \dim M$$

(L' is the coadjoint L-module), and hence

$$\chi(L, M) = \left(\sum_k (-1)^k \binom{n}{k}\right) \dim M = 0.$$

Thus, the Euler-Poincaré formula takes the form

$$\sum_k (-1)^k \dim H^k(L, M) = 0.$$

Hence, the condition $H^k(L, M) \neq 0$ implies that $H^l(L, M) \neq 0$, $l \neq k$.

COROLLARY. *If the trivial L-module is contained as a submodule in the L-module M, then $H^l(L, M) \neq 0$ for some $l \neq 0$.*

3. The modules $\operatorname{Cnd} M$, $\operatorname{Cnd}_+ M$, $\widetilde{\operatorname{Cnd}} M$, $\widetilde{\operatorname{Cnd}}_+ M$, and their cohomological properties. Let pr be the natural projection from $\mathfrak{U}(L)$ to the trivial L-module P, and let $\mathfrak{U}_+(L) = L\mathfrak{U}(L)$ be the kernel. For the L-module M, the exact sequence of L-modules

$$0 \to \mathfrak{U}_+(L) \to \mathfrak{U}(L) \overset{\mathrm{pr}}{\to} P \to 0$$

gives another exact sequence of L-modules

$$0 \to \operatorname{Hom}(P, M) \cong M \overset{\mathrm{pr}^*}{\to} \operatorname{Hom}(\mathfrak{U}(L), M) \to \operatorname{Hom}(\mathfrak{U}_+(L), M) \to 0, \qquad (7)$$

where the module structures in the homomorphism spaces $\operatorname{Hom}(\mathfrak{U}(L), M)$ and $\operatorname{Hom}(\mathfrak{U}_+(L), M)$ are defined as follows:

$$(e\Box f)(X) = (e \circ f)(X) + (e)_M f(X), \qquad X \in \mathfrak{U}(L) \text{ or } X \in \mathfrak{U}_+(L).$$

Here $e \circ f \in \operatorname{Hom}(\mathfrak{U}(L), M)$ or $e \circ f \in \operatorname{Hom}(\mathfrak{U}_+(L), M)$ is defined just as in the coinduced module: $(e \circ f)(X) = f(Xe)$. We let $\operatorname{Cnd} M$ and $\operatorname{Cnd}_+ M$ denote the L-modules obtained from the spaces $\operatorname{Hom}(\mathfrak{U}(L), M)$ and $\operatorname{Hom}(\mathfrak{U}_+(L), M)$ using the rule $(e, f) \mapsto e\Box f$. We take central elements $z_1, \dots, z_n$ so that $(z_i)_M = 0$, $1 \leq i \leq n$ (this is possible by Lemma 2.2). Obviously,

$$(z_i\Box f)(X) = (z_i\Box f)(X), \qquad X \in \mathfrak{U}(L) \text{ or } X \in \mathfrak{U}_+(L).$$

We can now introduce the finite-dimensional submodules

$$\widetilde{\operatorname{Cnd}} M = \langle f \in \operatorname{Hom}(\mathfrak{U}(L), M) | z_i\Box f = 0,\ 1 \leq i \leq n\rangle \subset \operatorname{Cnd} M,$$
$$\widetilde{\operatorname{Cnd}}_+ M = \langle f \in \operatorname{Hom}(\mathfrak{U}_+(L), M) | z_i\Box f = 0,\ 1 \leq i \leq n\rangle \subset \operatorname{Cnd}_+ M.$$

We note that the homomorphism pr^* factors through $\widetilde{\operatorname{Cnd}} M$:

$$(\mathrm{pr}^* m)X = (\mathrm{pr}\, X)m, \qquad \mathrm{pr}^* m \in \widetilde{\operatorname{Cnd}} M \subset \operatorname{Cnd} M, \qquad m \in M.$$

Since $\widetilde{\operatorname{Cnd}}_+ M = \widetilde{\operatorname{Cnd}} M \cap \operatorname{Cnd}_+ M$, by (7) we have an exact sequence

$$0 \to M \overset{\mathrm{pr}^*}{\to} \widetilde{\operatorname{Cnd}} M \to \widetilde{\operatorname{Cnd}}_+ M \to 0.$$

Let $\eta: \widetilde{\operatorname{Cnd}}_+ M \to \operatorname{Cnd}_+ M$ be the natural imbedding, and let

$$\mathscr{F} = \mathrm{pr}^*: M \to \operatorname{Cnd} M \cong V \otimes M, \qquad \widetilde{\mathscr{F}} = \mathrm{pr}^*: M \to \widetilde{\operatorname{Cnd}} M \cong \widetilde{V} \otimes M,$$

which are L-module monomorphisms. By Proposition 1 of [4], Chapter 3, $H^k(L, \operatorname{Cnd} M) = 0$, $k > 0$. Hence the following homomorphism vanishes:

$$\mathscr{F}_*: H^k(L, M) \to H^k(L, \operatorname{Cnd} M), \qquad k > 0.$$

In other words, M is annihilated in the infinite-dimensional module $\operatorname{Cnd} M$. Our goal is to prove that $\operatorname{Cnd} M$ can be "truncated" in such a way that M "vanishes" in the finite-dimensional submodule $\widetilde{\operatorname{Cnd}} M$.

LEMMA. *For any finite-dimensional L-module M and any admissible system of central elements $\{z_1, \dots, z_n\}$ the cohomology homomorphism $\eta_*: H^*(L, \widetilde{\mathrm{Cnd}}_+M) \to H^*(L, \mathrm{Cnd}_+M)$ is zero.*

PROOF. Let $(z_i)_M = 0$, $1 \le i \le n$ (such a system of central elements exists by Lemma 2.2). We shall prove that for any $\psi \in Z^k(L, \mathrm{Cnd}_+M)$ the cohomology class of this cocycle has a representative φ such that $\varphi \in Z^k(L, \widetilde{\mathrm{Cnd}}_+M)$.

Suppose that for $0 < l \le n$ we have the conditions

$$z_i \square \psi = 0, \qquad l < i \le n. \tag{8}$$

Let ρ be a representation of the Lie algebra L in the space of cochains $C^*(L, \mathrm{Cnd}_+M)$, and let $i(e)$ be the exterior multiplication operator for $e \in L$:

$$i(e): C^k(L, \mathrm{Cnd}_+M) \to C^{k-1}(L, \mathrm{Cnd}_+M),$$
$$i(e)\psi(X_1, \dots, X_{k-1}) = \psi(e, X_1, \dots, X_{k-1}).$$

We introduce the cochain

$$\omega_l = \sum_{t\ge 0} \lambda_l(t)\rho(e_l)^{p^t-1} i(e_l)\psi.$$

In [5] it was shown that $\rho(z_l)\psi = d\omega_l$. In other words,

$$z_l \square \psi = d\omega_l. \tag{9}$$

We define a cochain $\Delta_l \in C^{k-1}(L, \mathrm{Cnd}_+M)$ by the rule

$$\Delta_l(\cdots)(e^\alpha z^\beta) = \begin{cases} \omega_l(\cdots)(e^\alpha z^{\beta-\varepsilon_l}), & \text{if } \beta_l > 0, \\ 0, & \text{if } \beta_l = 0 \end{cases}$$
$$(\alpha \in \Gamma_n(\mathbf{m}),\ \beta \in \Gamma_n(\infty),\ \alpha + \beta \ne 0,\ \omega_l(\cdots)(1) = 0).$$

Here and below we use the three dots to indicate $(k-1)$ arguments $X_1, \dots, X_{k-1} \in L$.

We shall show that

$$z_l \square \Delta_l = \omega_l, \qquad z_i \square \Delta_l = 0, \qquad l < i \le n. \tag{10}$$

From (7) it follows that

$$z_i \square \omega_l = \sum_{t \ge 0} \lambda_l(t) \rho(e_l)^{p^t-1} i(e_l)(z_i \square \psi) = 0. \tag{11}$$

We must prove that for any $\alpha \in \Gamma_n(\mathbf{m})$ and $\beta \in \Gamma_n(\infty)$, $\alpha + \beta \ne 0$, we have

$$\Delta_l(\cdots)(e^\alpha z^\beta z_l) = \omega_l(\cdots)(e^\alpha z^\beta),$$
$$\Delta_l(\cdots)(e^\alpha z^\beta z_i) = \omega_l(\cdots)(e^\alpha z^\beta), \qquad l < i \le n.$$

The first of these is obvious from the definition of Δ_l. The other equality is also obvious in the case $\beta_l = 0$, and if $\beta_l > 0$, then by (11)

$$\Delta_l(\cdots)(e^\alpha z^\beta z_i) = \omega_l(\cdots)(e^\alpha z^{\beta-\varepsilon_l} z_i) = (z_i \square \omega_l(\cdots))(e^\alpha z^{\beta - \varepsilon_l}) = 0.$$

According to (9) and (10), for $\psi_l = \psi - d\Delta_l$ we have

$$z_i \square \psi_l = 0, \qquad l \le i \le n.$$

If we repeat this procedure for $l = n, \dots, 1$, $\psi_n = \psi$, we find that $\varphi = \psi_0 \in Z^k(L, \widetilde{\mathrm{Cnd}}_+ M)$. The lemma is proved.

4. Proof of Theorem 3.1. We consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{\widetilde{\mathscr{F}}} & \widetilde{\mathrm{Cnd}}\, M & \to & \widetilde{\mathrm{Cnd}}_+ M & \to & 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \eta & & \\ 0 & \to & M & \xrightarrow{\mathscr{F}} & \mathrm{Cnd}\, M & \to & \mathrm{Cnd}_+ M & \to & 0. \end{array}$$

This diagram gives rise to another commutative diagram

$$\begin{array}{ccccccc} \cdots \to & H^{k-1}(L, \widetilde{\mathrm{Cnd}}_+ M) & \xrightarrow{\tilde{\delta}} & H^k(L, M) & \to & H^k(L, \widetilde{\mathrm{Cnd}}\, M) & \to \cdots \\ & \downarrow \eta_* & & \downarrow \mathrm{id}_* & & \downarrow & \\ \cdots \to & H^{k-1}(L, \mathrm{Cnd}_+ M) & \xrightarrow{\delta} & H^k(L, M) & \to & H^k(L, \mathrm{Cnd}\, M) & \to \cdots, \end{array}$$

whose rows are long exact cohomology sequences (δ and $\tilde{\delta}$ are the Bockstein homomorphisms). Let $(z_i)_M = 0$, $1 \le i \le n$. Then η_* is an epimorphism, by Lemma 3.3. As already noted, $\mathscr{F}_* = 0$ for $k > 0$. Hence, the Bockstein homomorphism δ is an epimorphism for $k > 0$. Thus, the composition of two epimorphisms

$$\delta \eta_* \colon H^{k-1}(L, \widetilde{\mathrm{Cnd}}_+ M) \to H^k(L, M)$$

is also an epimorphism. Consequently, from the commutativity of the diagram we see that the Bockstein homomorphism $\tilde{\delta}$ is an epimorphism. In other words, because of the exactness of the top row,

$$\widetilde{\mathscr{F}}_* \colon H^k(L, M) \to H^k(L, \widetilde{\mathrm{Cnd}}\, M), \qquad k > 0,$$

is the zero homomorphism.

From the corollary to Lemma 3.2 it follows that the trivial module is not strongly annihilable.

§4. Cohomological criterion for nilpotence of a Lie algebra in characteristic $p > 0$

Let $R(L)$ be the minimal p-span of the Lie algebra L [6]. We give it a p-structure as follows: $e_i^{[p]} = 0$ if $\mathrm{ad}\, e_i = 0$. Recall that an L-module M is said to be a *p-module* over L (even if L does not have a p-structure) if M is a p-module over $R(L)$. Let $\kappa(L)$ be the number of nonisomorphic finite-dimensional irreducible modules with nontrivial cohomology. As shown in [7] and in §2, we have $0 < \kappa(L) < \infty$.

We now give a characterization of the class of Lie algebras for which $\kappa(L) = 1$.

THEOREM. *Let P be an algebraically closed field of characteristic $p > 0$. The following conditions are equivalent:*

(i) *L is nilpotent.*

(ii) *every irreducible p-module over L is trivial.*

(iii) *every irreducible special L-module is trivial.*

(iv) *every irreducible 1-special L-module is trivial.*

(v) *for any L-module M the condition $H^1(L, M) \neq 0$ implies that $H^0(L, M) \neq 0$.*

Recall that all algebras and modules are assumed to be finite dimensional.

PROOF. The implication (i) $\Rightarrow$ (ii) is proved in [22]. In this connection, see also [6]. The implication (ii) $\Rightarrow$ (iii) follows from results in [6]. We note that (i) $\Rightarrow$ (iii) is also obtained in [18]. The fact that (iii) $\Rightarrow$ (iv) is obvious.

We shall prove (iv) $\Rightarrow$ (v) by induction on $t = \dim M$. If $t = 1$ and $H^1(L, M) \neq 0$, then M is an irreducible 1-special module, and so, by (iv), M is the trivial L-module. Hence, in this case $H^0(L, M) \neq 0$. Suppose that the implication holds for $t-1$; and is false for t, i.e., we have $H^1(L, M) \neq 0$ and $H^0(L, M) = 0$ for some L-module M of dimension t. Then M contains an L-submodule N different from M and 0, since otherwise, by (iv), M would have to be the trivial module, contradicting the condition $H^0(L, M) = M^L = 0$. Furthermore, $N^L = 0$, and so, by the induction assumption, $H^1(L, N) = 0$. Thus, the first few terms in the cohomology sequence corresponding to the short exact sequence $0 \to N \to M \to M/N \to 0$, have the form

$$\begin{aligned} 0 &\to H^1(L, N) = 0 \to H^1(L, M) = 0 \to H^0(L, M/N) \\ &\to H^1(L, N) = 0 \to H^1(L, M) \neq 0 \xrightarrow{\xi} H^1(L, M/N) \to \cdots . \end{aligned}$$

Hence, on the one hand, we have $H^0(L, M/N) = 0$, and on the other hand, because ξ is a monomorphism, we have $H^1(L, M/N) = 0$. These equalities contradict the induction assumption. This gives the induction step, and proves the implication (iv) $\Rightarrow$ (v).

We prove that (v) $\Rightarrow$ (i) by induction on $n = \dim L$. For $n = 1$ the implication is obvious. Suppose that it holds for $n - 1$. Suppose that L is an n-dimensional Lie algebra satisfying (v). We shall prove that every proper subalgebra A of L also satisfies the condition. Let M be a 1-special L-module. By Theorem 1.3 (more precisely, Corollary 1), $\mathrm{Coind}_{\tilde{A}}M$ is also a 1-special L-module. Hence, by (v), we have $(\mathrm{Coind}_{\tilde{A}}M)^L \neq 0$, and consequently, by Corollary 1 of Theorem 1.3, $M^A \neq 0$. In other words, the condition $H^1(A, M) \neq 0$ implies that $M^A \neq 0$. By the induction assumption, this means that every proper subalgebra A of L is nilpotent. We now show that the Lie algebra L itself is nilpotent. Suppose that this were not the case. Then L would contain a two-dimensional nonabelian subalgebra $\overline{L} = \langle X, Y \| [X, Y] = Y\rangle$. Since $\overline{L}$ is not nilpotent, this is possible only if $L = \overline{L}$. Then the one-dimensional L-module $\langle v\rangle$ for which $Xv = v$ and $Yv = 0$ has the following property: $H^1(L, \langle v\rangle) \neq 0$, but $H^0(L, \langle v\rangle) = 0$. In fact, the class of the cocycle $\psi \in Z^1(L, \langle v\rangle)$, given by $\psi(X) = 0$ and $\psi(Y) = v$, is nontrivial. Thus, we have constructed a module $\langle v\rangle$ which contradicts (v). Hence, L is nilpotent, and (v) $\Rightarrow$ (i).

The proof of the theorem is complete.

§5. Connection between truncated induced and truncated coinduced modules

Given a subalgebra A of a Lie algebra L and an element $a \in A$, we let $\mathrm{ad}\, a|_A$ denote the restriction of ad a to A. We set $\mathrm{tr}_{L/A} a = \mathrm{tr}\,\mathrm{ad}\, a - \mathrm{tr}(\mathrm{ad}\, a|_A)$. To every A-module M we associate the adjoint A-module M' and the twisted A-module $\overline{M}$ with action of A defined as follows: $(a)_{\overline{M}} = (a)_m - \mathrm{tr}_{L/A} a$.

THEOREM ($p > 0$). *Let A be a subalgebra of the Lie algebra L, and let M be a module over A. Then*

$$\overline{\mathrm{Ind}}_A \overline{M} \cong \overline{\mathrm{Coind}}_A M.$$

PROOF. Let s be the codimension of A, and let $\theta = \sum_1^s (p^{m_i} - 1)\varepsilon_i$. We choose a basis of $\mathrm{Coind}_{\tilde{A}} M$ consisting of elements $f_{\alpha, v}$ such that

$$f_{\alpha, v}(e^\beta) = \begin{cases} v, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha, \beta \in \Gamma_s(\mathbf{m})$ and $v \in V(M)$. It is obvious that ($|\alpha| \overset{\text{def}}{=} \sum_1^s \alpha_i$)

$$a \in A, \ |\alpha| < |\theta|, \ \alpha \in \Gamma_s(\mathbf{m}) \Rightarrow e^\alpha a \in \langle e^\beta \mathfrak{U}(\tilde{A}) \,|\, |\beta| < |\theta|, \ \beta \in \Gamma_s(\mathbf{m})\rangle,$$
$$e^\theta a + (\mathrm{tr}_{L/A} a) e^\theta \in \langle e^\beta \mathfrak{U}(\tilde{A}) \,|\, |\beta| < |\theta|, \ \beta \in \Gamma_s(\mathbf{m})\rangle.$$

Hence, for any $a \in A$ and $v \in M$,

$$a \circ f_{\theta, v} = f_{\theta, av} - (\mathrm{tr}_{L/A} a) f_{\theta, v}.$$

In other words, $\mathrm{Coind}_{\tilde{A}} M$ contains the A-submodule $\langle f_{\theta, v} | v \in V(M)\rangle \cong M$. Since

$$e^\alpha \circ f_{\theta, v} - f_{\theta - \alpha, v} \in \langle f_{\alpha', v'} \,|\, |\alpha'| > |\theta - \alpha|, \ \alpha' \in \Gamma_s(\mathbf{m}), \ v' \in V(M)\rangle,$$

the set $\{e^\alpha \circ f_{\theta, v} | \alpha \in \Gamma_s(\mathbf{m}), \ v \in V(M)\}$ also forms a basis of the space $\mathrm{Coind}_{\tilde{A}} M$. Thus, the map

$$G: \mathrm{Ind}_{\tilde{A}} \overline{M} \to \mathrm{Coind}_{\tilde{A}} M, \qquad G(e^\alpha \otimes v) = e^\alpha \circ f_{\theta, v}, \qquad \alpha \in \Gamma_s(\mathbf{m}), \ v \in V(M),$$

gives an isomorphism of spaces. We verify that

$$G(Xe^\alpha \otimes v) = X \circ G(e^\alpha \otimes v), \qquad X \in L, \ \alpha \in \Gamma_s(\mathbf{m}), \ v \in V(M).$$

We represent Xe^α in the form

$$Xe^\alpha = \sum_{\beta \in \Gamma_s(\mathbf{m}), \, a \in V(\mathfrak{U}(\tilde{A}))} \lambda_{\alpha, \beta, a} e^\alpha a.$$

Then

$$\begin{aligned} G(Xe^\alpha \otimes v) &= \sum_{\beta, a} \lambda_{\alpha, \beta, a} G(e^\beta \otimes (a)_{\overline{M}} v) \\ &= \sum_{\beta, a} \lambda_{\alpha, \beta, a} e^\beta \circ f_{\theta, (a)_{\overline{M}} v} = \sum_{\beta, a} \lambda_{\alpha, \beta, a} e^\beta a \circ f_{\theta, v} = X \circ G(e^\alpha \otimes v). \end{aligned}$$

Thus, G gives an isomorphism of modules. The theorem is proved.

The theorem shows that, as in the case of finite groups, the concept of a truncated induced module and that of a truncated coinduced module coincide. In particular, every truncated induced module has an $(L, \tilde{U})$-module structure, i.e., along with its L-module structure $\mathrm{Ind}_{\tilde{A}} M$ can be given the structure of a module over the divided power algebra $\tilde{U} \cong \mathrm{Coind}_{\tilde{A}} P$ in such a way that

$$l(um) = l(u)m + ul(m), \qquad l \in L, \ u \in \tilde{U}, \ m \in M.$$

In the terminology of [12], in this situation one says that $\mathrm{Ind}_{\tilde{A}} M$ has a transitive imprimitivity system with base L/A. In the case of Cartan Lie algebras of characteristic $p > 0$ this fact was established in [8]. An $(L, \tilde{U})$-module structure can be introduced in a truncated (generalized) coinduced module in exactly the same way as in the case of coinduced modules [11].

In the preceding we have assumed that the central elements have p-form. By modifying somewhat the definition of the twisted module, one can prove Theorem 5 in the general case.

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