Central Extensions of Infinite-Dimensional Lie Algebras

A. S. Dzhumadil'daev UDC 517.986.5

It is reasonable to distinguish two kinds of Lie algebras of Cartan type. The first one is defined by means of algebras of formal power series. In order to define Lie algebras of Cartan type of the second kind, it is necessary to consider algebras of formal Laurent power series. For example, the one-sided Witt algebra (the Lie algebra of formal vector fields on the real line)

$$W_1^+ = \langle e_i \mid [e_i\,,\, e_j] = (j-i)e_{i+j}\,,\ i\,,\, j \geq -1\,,\ i\,,\, j \in \mathbb{Z}\rangle$$

is of the first kind, while the two-sided Witt algebra (the Lie algebra of formal vector fields on the circle)

$$W_1 = \langle e_i \mid [e_i\,,\, e_j] = (j-i)e_{i+j}\,,\ i\,,\, j \in \mathbb{Z}\rangle$$

is of the second kind. It is known that the Lie algebra W_1 possesses a nonsplittable central extension (the Virasoro algebra)

$$V_1 = \langle e_i\,,\, z \mid \{e_i\,,\, e_j\} = (j-i)e_{i+j} + \delta_{i+j\,,\,0}(i^3 - i)z\,,\ i\,,\, j \in \mathbb{Z}\rangle.$$

The Lie algebra W_1^+ has no such extensions. Actually, many facts are known on the cohomology of Lie algebras of the first kind (see, e.g., [1]); in particular, they possess no nontrivial central extensions, except the Hamiltonian algebra that has a one-dimensional central extension (the Poisson algebra).

Therefore, the natural problem on central extensions of Lie algebras of Cartan type is of interest only for Lie algebras of the second kind. It turns out that our hope to find new central extensions is justified: Lie algebras of formal divergentless vector fields and Hamiltonian vector fields possess nontrivial central extensions (Theorem 1). The method of calculating the second cohomology group is based on Proposition 1 of [2], which reduces the computation of $H^2(L\,,\,\mathbb{C})$ to the study of the first cohomology group of the coadjoint representation $H^1(L\,,\, L')$. Following this line of reasoning, the first groups $H^1(L\,,\, L')$ for Lie algebras of Cartan type of the second kind are described; in particular, the Lie algebras of outer derivations for the Hamiltonian algebra H_n and for the special Lie algebra S_n are determined (Theorem 2).

1. Statement of the main result. Let $U = \mathbb{C}[[x_i^{\pm 1} \mid i \in I]]$ be the algebra of formal Laurent power series in variables x_i with indices from the set I. We write

$$\varepsilon_i = (0\,,\, \dots\,,\, 0\,,\, \underset{i}{1}\,,\, 0\,,\, \dots\,,\, 0)\,, \quad \theta = -\sum_i \varepsilon_i\,, \quad x^\alpha = \prod_i x_i^{\alpha_i}\,, \quad \alpha = \sum_i \alpha_i \varepsilon_i\,, \quad \alpha_i \in \mathbb{Z}.$$

Let $\partial_i = d/dx_i$ be the derivation of the algebra U:

$$\partial_i(x^\alpha) = \alpha_i x^{\alpha - \varepsilon_i}.$$

The general Lie algebra of Cartan type W_n is defined as the Lie algebra of derivations of the algebra $\mathbb{C}[[x_i^{\pm 1} \mid i = 1\,,\, \dots\,,\, n]]$. The remaining three series of Lie algebras of Cartan type are defined as subalgebras in the general Lie algebra in terms of their actions on certain differential forms. Without going into the details of known definitions, we now give the definitions of these algebras in a convenient form. Note that in each particular case the meaning of I will be clear from the context.

The special Lie algebra

$$\widetilde{S}_n = \langle D \in W_{n+1} \mid \operatorname{Div} D = \sum_i \partial_i(u_i) = 0\,,\ I = \{0\,,\, 1\,,\, \dots\,,\, n\}\rangle\,, \qquad n \geq 2\,,$$

is not simple. Its commutant $S_n = [\widetilde{S}_n\,,\, \widetilde{S}_n]$ is a simple Lie algebra.

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We define the Hamiltonian algebra on the algebra of Laurent series

$$\widetilde{H}_n = \langle x^\alpha \mid \alpha \neq \theta\,,\ I = \{\pm 1\,, \dots\,, \pm n\}\rangle\,, \qquad n \geq 1\,,$$

with respect to the Poisson bracket

$$\{u\,,\, v\} = -\sum_i \operatorname{sgn} i\, \partial_i(u)\partial_{-i}(v).$$

This algebra has a one-dimensional center, and the corresponding quotient algebra $H_n = \widetilde{H}_n/\langle 1\rangle$ is simple.

Finally, we define the contact Lie algebra on the algebra of Laurent series with odd number of variables

$$K_{n+1} = \langle x^\alpha \mid I = \{0\,, \pm 1\,, \dots\,, \pm n\}\rangle\,, \qquad n \geq 1\,,$$

$$[u\,,\, v] = \partial_0(u)\Delta(v) - \partial_0(v)\Delta(u) - \sum_{i\neq 0} \operatorname{sgn} i\, \partial_i(u)\partial_{-i}(v)\,, \quad \text{where} \quad \Delta(u) = \Big(2 - \sum_{i \neq 0} x_i\partial_i\Big)u.$$

For $u \in U$, denote by $\pi(u)$ the coefficient of x^θ in the expansion of u with respect to the basis.

Theorem 1. *Let L be one of the simple Lie algebras of Cartan type listed above. Then $H^2(L\,,\mathbb{C}) = 0$ except in the following cases*:

L	basic cocycles	$\dim H^2(L\,,\mathbb{C})$
W_1	$(u\partial\,, v\partial) \mapsto \pi(u\partial^3(v))$	1
S_n	$(D_1\,, D_2) \mapsto \pi(x_i^{-1}x_j^{-1}(D_1(x_i)D_2(x_j) - D_1(x_j)D_2(x_i)))$	$(n+1)n/2$
H_n	$(u\,, v) \mapsto \pi(x_i^{-1}(u\partial_{-i}(v) - v\partial_{-i}(u)))$, $(u\,, v) \mapsto \pi(x^\theta\{u\,, v\})$	$2n+1$

2. Coadjoint representations. Let $\Omega^* = \bigoplus_k \Omega^k$ be the space of exterior differential forms with coefficients in U and with the exterior differential operator δ. Let us recall that

$$\Omega^0 = U\,, \qquad \Omega^k = \langle u_{i_1,\dots,i_k}\delta x_{i_1} \wedge \dots \wedge \delta x_{i_k} \mid i_1 < \dots < i_k\,,\ i_1\,, \dots\,, i_k \in I\rangle\,, \qquad k > 0\,,$$

and the operator $\delta\colon \Omega^k \to \Omega^{k+1}$ is given by

$$\delta u = \sum_i \partial_i(u)\delta x_i\,, \quad u \in \Omega^0\,, \qquad \delta(\omega \wedge \omega') = \delta\omega \wedge \omega' + (-1)^k \omega \wedge \delta\omega'\,, \quad \omega \in \Omega^k.$$

Recall also that the natural action of W_n on U extends to an action of W_n in the space Ω^* by the rules

$$D(\delta u) = \delta(Du)\,, \qquad D(\omega \wedge \omega') = D\omega \wedge \omega' + \omega \wedge D\omega'.$$

Let

$$Z^k(\Omega) = \langle \omega \in \Omega^k \mid \delta\omega = 0\rangle$$

be the space of closed k-forms, let

$$B^k(\Omega) = \langle \delta\eta \mid \eta \in \Omega^{k-1}\rangle$$

be the space of exact k-forms, and let

$$H^*(\Omega) = \bigoplus_k H^k(\Omega)\,, \qquad H^k(\Omega) = Z^k(\Omega)/B^k(\Omega)\,,$$

be the space of de Rham cohomology.

Equip Ω^k with another W_n-module structure:

$$(D)_t\omega = D\omega + t(\operatorname{Div} D)\omega\,, \qquad t \in \mathbb{C}.$$

Denote the module thus obtained by $(\Omega^k)_t$. Also, let U_t be the twisted K_{n+1}-module defined on $U = \mathbb{C}[[x_i^{\pm 1} \mid i = 0\,, \pm 1\,, \dots\,, \pm n]]$ by the rule

$$(u)_t v = [u\,, v] + (t-2)\partial_0(u)v.$$

Proposition 1. *Coadjoint L-modules L' and corresponding pairings $(L, L') \to \mathbb{C}$ are given by the formulas*

$$L = W_n, \qquad L' = (\Omega_n^1)_1, \qquad \Big(\sum_i u_i\partial_i, \sum_j v_j\delta x_j\Big) = \pi\Big(\sum_i u_i v_i\Big),$$

$$L = S_n, \qquad L' = B^2(\Omega), \qquad \Big(D_{i,j}(u), \delta\Big(\sum_s v_s\delta x_s\Big)\Big) = \pi(u(\partial_i(v_j) - \partial_j(v_i))),$$

$$L = H_n, \qquad L' = H_n, \qquad (u, v) = \pi(uv),$$

$$L = K_{n+1}, \qquad L' = U_{-2(n+2)}, \qquad (u, v) = \pi(uv).$$

The proof is similar to the case $p > 0$ [3].

Corollary. *The Lie algebras S_2 and H_n possess nondegenerate symmetric invariant forms.*

In the sequel the following analog of the Poincaré lemma is useful.

Proposition 2. *The space $H^k(\Omega)$ of de Rham k-cohomology is generated by the classes of differential forms $x_{i_1}^{-1}\cdots x_{i_k}^{-1}\delta x_{i_1}\wedge\cdots\wedge\delta x_{i_k}$, $i_1 < \cdots < i_k$, these forms constitute a basis, and there is an isomorphism*

$$H^k(\Omega) \cong \Lambda^k\mathbb{C}^{|I|},$$

where $|I|$ is the number of elements of the set I.

3. First cohomology groups.

Theorem 2. *For a Lie algebra L of Cartan type, the 1-cohomology space $H^1(L, L')$ of the coadjoint representation is trivial, except in the following cases:*

L	basic cocycles	$\dim H^1(L, L')$
W_1	$u\partial \mapsto \partial^3(u)\delta x$	1
S_n	$D \mapsto D(x_i^{-1}x_j^{-1}\delta x_i\wedge\delta x_j)$, $D_{i,j}(u) \mapsto (-1)^s\delta(\Delta(u)\delta x_s)$, $\quad (i-j)(i-s)(j-s)\neq 0$, $n=2$	$(n+1)n/2 + \delta_{n,2}$
H_n	$u \mapsto x_i^{-1}\partial_{-i}(u)$, $\quad i = \pm 1, \dots, \pm n$, $u \mapsto \{x^\theta, u\}$, $\Delta\colon x^\alpha \mapsto \left(2 - \sum_i \alpha_i\right)x^\alpha$	$2(n+1)$

Corollary. *For $L = S_2$, the space of outer derivations $\operatorname{Out} L$ is four-dimensional, and the classes of derivations $\operatorname{ad} x_i^{-1}x_j^{-1}\partial_s$, $(i-j)(i-s)(j-s)\neq 0$, and $\operatorname{ad}\sum_i x_i\partial_i$ constitute a basis; for $L = H_n$, this space is $2(n+1)$-dimensional, and the classes of derivations $x_i^{-1}\partial_{-i}$, $\operatorname{ad} x^\theta$, and Δ constitute a basis.*

It can be proved that other simple Lie algebras of Cartan type, except S_n, possess no outer derivations.

Before passing to the proof of Theorem 2, we recall some facts on cohomologies and on Lie algebras of Cartan type. Let M be an L-module, let H be a Cartan subalgebra in L, and let $C^*(L, M) = \bigoplus_k C^k(L, M)$ be the standard cochain complex of the Lie algebra L with coefficients in M and with coboundary operator d; let $Z^k(L, M) = \langle\psi \in C^k(L, M) \mid d\psi = 0\rangle$ be the space of k-cocycles, let $B^k(L, M) = \langle d\varphi \mid \varphi \in C^{k-1}(L, M)\rangle$ be the space of k-coboundaries, and let $H^*(L, M) = \bigoplus_k H^k(L, M)$, $H^k(L, M) = Z^k(L, M)/B^k(L, M)$, be the cohomology space of the L-module M. For example, the space $Z^2(L, \mathbb{C})$ of 2-cocycles of the trivial module ($Da = 0$ for all $D \in L$, $a \in \mathbb{C}$) is generated by bilinear skew-symmetric mappings $\psi\colon L\times L \to \mathbb{C}$ such that

$$\psi(D_1, [D_2, D_3]) + \psi(D_2, [D_3, D_1]) + \psi(D_3, [D_1, D_2]) = 0,$$

and the space $B^2(L, \mathbb{C})$ of 2-coboundaries consists of skew-symmetric bilinear mappings $df\colon L\times L \to \mathbb{C}$ constructed from the linear functionals $f\colon L \to \mathbb{C}$ by the rule

$$df([D_1, D_2]) = -f([D_1, D_2]).$$

If L and M are semisimple H-modules, then the H-module $C^*(L, M) \cong \Lambda^*(L') \otimes M$ is also semisimple. Therefore $C^*(L, M)$ can be decomposed into the direct sum of H-submodules $C^*_\lambda(L, M)$ formed by H-*proper* cochains whose proper values (weights) λ are linear functions on H. Moreover, $C^*_\lambda(L, M) = \bigoplus_k C^k_\lambda(L, M)$ forms a cochain subcomplex with zero cohomology for $\lambda \neq 0$. Thus we have

$$H^*(L, M) \cong H^*_0(L, M), \qquad H^k_0(L, M) = Z^k_0(L, M)/B^k_0(L, M).$$

In the sequel, replacing $C^*(L, L')$ by $C^*_0(L, L')$, we shall assume that all cochains $f \in C^1(L, L')$ under consideration satisfy the additional condition

$$hf(D) = f([h, D]), \qquad \forall h \in H, \ \forall D \in L. \tag{1}$$

It is easy to see that the actions of standard Cartan subalgebras H in L are semisimple in L and L' and that the standard basis vectors are proper with respect to these actions. Let $L = \bigoplus_\alpha L_\alpha$, $L' = \bigoplus_\alpha L'_\alpha$ be the corresponding decompositions. To be more precise, the root system forms an infinite abelian group, and we identify it with $\mathbb{Z}^{\dim H} = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. Denote by ε'_i the linear functionals $H \to \mathbb{C}$ such that $\varepsilon'_i(h_j) = \delta_{i,j}$ (the Kronecker symbol). Then the root subspaces L_α and the mappings $\mathcal{T}\colon L \to H'$, $\mathcal{T}'\colon L' \to H'$ that map the basis vectors from L and L' into their roots, are given as follows.

$$L = W_n, \qquad H = \langle h_i = x_i\partial_i \mid i \in I = \{1, \dots, n\}\rangle, \qquad L_\alpha = \langle x^\beta \partial_j \mid \beta_i - \delta_{i,j} = \alpha_i, \ i, j \in I\rangle,$$
$$\mathcal{T}(x^\beta\partial_j) = \sum_s (\beta_s - \delta_{s,j})\varepsilon'_s, \qquad \mathcal{T}'(x^\beta \delta x_j) = \sum_s (\beta_s + \delta_{j,s} + 1)\varepsilon'_s;$$
$$L = S_n, \qquad H = \langle h_i = x_i\partial_i - x_0\partial_0 \mid i = 1, \dots, n\rangle,$$
$$L_\alpha = \langle D_{i,j}(x^\beta) \stackrel{\text{def}}{=} \partial_i(x^\beta)\partial_j - \partial_j(x^\beta)\partial_i \mid \beta - \varepsilon_i - \varepsilon_j \equiv \alpha \,(\operatorname{mod} \theta), \ i, j \in I\rangle,$$
$$\mathcal{T}(D_{i,j}(x^\beta)) = \sum_{s=0}^{n} (\beta_s - \delta_{i,s} - \delta_{j,s})\varepsilon'_s \,(\operatorname{mod} \theta'), \qquad \mathcal{T}'(\delta(x^\beta \delta x_j)) = \sum_{s=0}^{n} (\beta_s + \delta_{j,s})\varepsilon'_s \,(\operatorname{mod} \theta');$$
$$L = H_n, \qquad H = \langle h_i = x_{-i}x_i \mid i = 1, \dots, n\rangle, \qquad L_\alpha = \langle x^\beta \mid \beta_s - \beta_{-s} = \alpha_s, \ s = 1, \dots, n\rangle,$$
$$\mathcal{T}(x^\beta) = \sum_{s=1}^{n} (\beta_s - \beta_{-s})\varepsilon'_s, \qquad \mathcal{T}'(x^\beta) = \mathcal{T}(x^\beta);$$
$$L = K_{n+1}, \qquad H = \langle h_0 = x_0, \ h_i = x_{-i}x_i \mid i = 1, \dots, n\rangle,$$
$$L_\alpha = \left\langle x^\beta \,\middle|\, 2 - 2\beta_0 - \sum_{s\neq 0} \beta_s = \alpha_0, \ \beta_s - \beta_{-s} = \alpha_s, \ s = 1, \dots, n \right\rangle,$$
$$\mathcal{T}(x^\beta) = \Big(2 - 2\beta_0 - \sum_{s\neq 0}\beta_s\Big)\varepsilon'_0 + \sum_{s=1}^{n} (\beta_s - \beta_{-s})\varepsilon'_s, \qquad \mathcal{T}'(x^\beta) = \mathcal{T}(x^\beta) - 2(n+3)\varepsilon'_0.$$

Here in the case $L = S_n$, we mean that L (in particular, H) is embedded in W_{n+1} and the root α is taken as the corresponding root of W_{n+1} modulo $\theta' = -\sum_{s=0}^{n} \varepsilon'_s$. Thus, L is a multigraded Lie algebra and L' is a multigraded L-module:

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}, \qquad L_\alpha \circ L'_\beta \subseteq L'_{\alpha+\beta}.$$

Let $U^+ = \mathbb{C}[[x_i \mid i \in I]]$ be the subalgebra in U consisting of formal power series, and let L^+ be the corresponding subalgebra (a Lie algebra of Cartan type of the first kind) in L with the standard grading

$L^+ = \bigoplus_{t\geq -2} L_t^+$ in L^+:

$$L^+ = W_n^+ = \langle x^\beta \partial_j \mid \beta_i \geq 0,\ i \in I\rangle, \qquad L_t^+ = \left\langle x^\beta \partial_j \,\middle|\, \sum_i \beta_i = t+1 \right\rangle,$$

$$L^+ = S_n^+ = S_n \cap W_{n+1}^+, \qquad L_t^+ = S_n \cap (W_{n+1}^+)_t,$$

$$L^+ = H_n^+ = \langle x^\beta \,\big|\, \beta \neq (0,\dots,0),\ \beta_i \geq 0,\ i \in I\rangle, \qquad L_t^+ = \left\langle x^\beta \,\middle|\, \sum_i \beta_i = t+2 \right\rangle,$$

$$L^+ = K_{n+1}^+ = \langle x^\beta \mid \beta_i \geq 0,\ i \in I\rangle, \qquad L_t^+ = \left\langle x^\beta \mid \sum_{i\neq 0} \beta_i + 2\beta_0 = t+2 \right\rangle.$$

Introduce the subalgebra $J(L) = \bigoplus_{t<0} L_t^+$. Note that $J(L) = L_{-1}^+$ except for the case $L = K_{n+1}$, when we have $J(L) = L_{-1}^+ + L_{-2}^+$.

Recall that for a subalgebra Q and an L-module M, the space of relative cochain complex $C^*(L, Q, M)$ consists of cochains $\psi \in C^*(L, M)$ such that $\psi(D, \dots) = 0$, provided at least one of the arguments D belongs to Q. Recall also that the 0-cohomology space $H^0(Q, M)$ coincides with the space of invariants $M^Q = \langle m \in M \mid qm = 0,\ \forall q \in Q\rangle$.

Lemma 1. *The following relations are valid:*

$$H^0(J(W_n), W_n') = (\Lambda^1)_1 = \langle \delta x_i \mid i \in I\rangle, \qquad \mathcal{T}'(\delta x_i) = \varepsilon_i' + \theta',$$

$$H^0(J(S_n), S_n') = \Lambda^2 = \langle \delta x_i \wedge \delta x_j \mid i, j \in I\rangle, \qquad \mathcal{T}'(\delta x_i \wedge \delta x_j) = \varepsilon_i' + \varepsilon_j' \pmod{\theta'},$$

$$H^0(J(H_n), H_n') = \langle x_i \mid i \in I\rangle, \qquad \mathcal{T}'(x_i) = \operatorname{sgn} i\, \varepsilon_i',$$

$$H^0(J(K_{n+1}), K_{n+1}') = \langle 1\rangle, \qquad \mathcal{T}'(1) = -2(n+2)\varepsilon_0'.$$

Lemma 2. *For a simple Lie algebra L of Cartan type (of the second kind) and its subalgebra L^+ we have*

$$Z^1(L, L^+, L') = 0.$$

Proof. Let $\psi \in Z^1(L, L^+, L')$. Then $\operatorname{Ker}\psi = \langle D \in L \mid \psi(D) = 0\rangle \supseteq L^+$. Below we shall construct the elements $A_i \in \operatorname{Ker}\psi$, $i \in I$, that, together with L^+, generate L. The relation $\psi(A_i) = 0$ will follow from the cocycle condition of the form $d\psi(A_i, A_i^+) = 0$ for elements $A_i^+ \in L^+$ such that $[A_i, A_i^+] \in L^+$.

For $L = W_n$ set $A_i = x_i^{-1}\partial_i$, $A_i^+ = x_i^3\partial_i$. Since

$$\mathcal{T}(x_i^{-1}\partial_i) = -2\varepsilon_i', \qquad x_i^3\partial_i(L'_{-2\varepsilon_i'}) \subseteq \langle (-3 + 2\delta_{s,i}) x_s^{-1} x^\theta \delta x_s \mid s \in I\rangle,$$

we see that the kernel of the mapping $x_i^3\partial_i : L'_{-2\varepsilon_i'} \to L_0'$ is zero. Hence, the condition $\operatorname{Ker}\psi \supset L^+$ and (1) imply the relation $0 = d\psi(A_i, A_i^+) = A_i^+\psi(A_i) \implies \psi(A_i) = 0$. Similarly, in other cases it is sufficient to set

$$L = S_n, \quad A_i = x_i^{-1}\partial_i + x_i^{-2}x_j\partial_j, \quad A_i^+ = x_i^3\partial_j,\ x_i^4\partial_j, \quad i \neq j,\ i, j \in I,$$

$$L = H_n, \quad A_i = x_i^{-1}, \quad A_i^+ = x_{-i}x_i^3,$$

$$L = K_{n+1}, \quad A_i = x_i^{-1}, \quad A_i^+ = x_{-i}x_0,\ x_{-i}x_i^3.$$

Lemma 3. *For a simple Lie algebra L of Cartan type (of the second kind) we have $H^1(L, J(L), L') = 0$, except for the case $L = W_1$, where $H^1(W_1, J(W_1), W_1') = \langle \partial^3 : u\partial \mapsto \partial^3(u)\delta x \mid u \in U\rangle$ is one-dimensional.*

Proof. Let $\psi \in Z^1(L, J(L), L')$. By Lemma 2, we need consider only the case $\psi \notin Z^1(L, L^+, L')$. Then there exists a $t \geq 0$ such that $\psi(D) \neq 0$ for a certain basis vector $D \in L_t$, but $\psi(\overline{D}) = 0$ for all $\overline{D} \in L_{\overline{t}}$, $0 \leq \overline{t} < t$. Hence the cocycle condition $d\psi(l, D) = 0$, $l \in J(L)$, implies $l\psi(D) = -\psi([l, D]) + D\psi(l) = 0$. Thus, from (1) and Lemma 1 it follows that $\psi(D) \in (L')^{J(L)}$.

On the other hand, for the nilpotent subalgebra $\mathcal{L}_1 = \bigoplus_{t\geq 1} L_t^+$ we have

$$H^1(\mathcal{L}_1, \mathbb{C}) = \mathcal{L}_1/[\mathcal{L}_1, \mathcal{L}_1] = \begin{cases} L_1 + L_2 & \text{for } L = W_1, \\ L_1 & \text{in other cases}, \end{cases}$$

and

$$d\psi(\overline{D}, \widetilde{D}) = 0, \quad \overline{D} \in L_{\bar{t}}, \quad \widetilde{D} \in L_{\tilde{t}}, \quad 0 \leq \bar{t}, \tilde{t} < t \quad \Longrightarrow \quad \psi([\overline{D}, \widetilde{D}]) = 0.$$

Thus, from (1) and Lemma 1 it follows that only the following cases are possible: $L = W_1$, $D = x^3\partial$ and $L = H_n$, $D = x_{-s}x_s x_i$ for $s \neq -i$. Indeed, in the first case there arises a nontrivial cocycle ψ such that $\psi(u\partial) = \partial^3(u)\delta x$. The second case is impossible: the condition $d\psi(x_{-s}^2, x_s^2 x_i) = 0$ implies $\psi(x_{-s}x_s x_i) = [x_{-s}^2, \psi(x_s^2 x_i)] \in \langle x_i \rangle$, i.e., $\psi(D) = 0$.

Proof of Theorem 2. For $L = W_n$ by (1) we have:

$$\psi \in Z_0^1(L, L') \implies \psi(\partial_i) = \sum_j \lambda_{i,j} x_i^{-1} x_j^{-1} x^\theta \delta x_j = \partial_i\Big(\sum_j -\lambda_{i,j}(1+\delta_{i,j})^{-1} x_j^{-1} x^\theta \delta x_j\Big),$$

because $\mathcal{T}(\partial_i) = -\varepsilon_i'$, $L'_{-\varepsilon_i'} = \langle x_i^{-1} x_j^{-1} x^\theta \delta x_j \mid j \in I\rangle$. Moreover, $0 = d\psi(\partial_i, \partial_j) = \partial_i\psi(\partial_j) - \partial_j\psi(\partial_i)$, $i \neq j \implies \lambda_{i,j} = \lambda_{j,j}/2$, $i \neq j$. Finally, for $L = W_n$ we have

$$\psi(\partial_i) = -\partial_i\Big(\sum_j (\lambda_{j,j}/2) x_j^{-1} x^\theta \delta x_j\Big).$$

In other words, if we set $f = \sum_j (\lambda_{j,j}/2) x_j^{-1} x^\theta \delta x_j$, then the cocycle $\varphi = \psi + df$ is contained in $Z^1(L, L_{-1}, L')$. It remains to apply Lemma 3 to achieve the proof in the case $L = W_n$.

Similarly, making use of condition (1), the cocycle conditions $d\psi(\bar{l}, \tilde{l}) = 0$, $\bar{l}, \tilde{l} \in J(L)$, and the following facts:

$$L = S_n, \quad \mathcal{T}(\partial_i) = -\varepsilon_i' (\operatorname{mod} \theta'),$$

$$d(x_i^{-1}x_j^{-1}\delta x_i \wedge \delta x_j)\partial_i = \delta(x_i^{-1}x_j^{-1}\delta x_j), \quad \operatorname{ad} h(\partial_i) = -\partial_i \quad \Big(h = \sum_i x_i\partial_i\Big),$$

$$L = H_n, \quad \mathcal{T}(x_i) = \operatorname{sgn} i\,\varepsilon_i', \quad x_{-i}^{-1}\partial_i(x_i) = x_{-i}^{-1}, \quad \operatorname{ad} x^\theta(x_i) = \pm x^{\theta-\varepsilon_{-i}}, \quad \Delta(x_i) = x_i, \qquad i \in I,$$

$$L = K_{n+1}, \quad \mathcal{T}(1) = 2\varepsilon_0',$$

we find that for some $f \in C^1(L, L')$ the cocycle $\varphi = \psi + df$ satisfies the following conditions:

$$\begin{aligned} &L = S_n, && \varphi(\partial_i) \in \langle \delta_{n,2}\partial_i, x_i^{-2}x_j^{-1}\delta x_i \wedge \delta x_j \mid j \in I\rangle, \ i \in I, \\ &L = H_n, && \varphi(x_i) \in \langle x_{-i}^{-1}, x^{\theta-\varepsilon_{-i}}, x_i\rangle, \ i \in I, \\ &L = K_{n+1}, && \varphi(1) = 0. \end{aligned}$$

In other words, adding to φ a certain linear combination of cocycles described in Theorem 2, we obtain a cocycle belonging to the space $Z^1(L, J(L), L')$. Hence Lemma 3 implies that the cocycle classes listed in the statement of Theorem 2 generate $H^1(L, L')$. It remains to show that they really form a basis.

Since the case $L = W_1$ is well known, we begin with the case $L = S_n$. Consider the exact cohomological sequence of the corresponding short exact sequence

$$0 \to B^2(\Omega) \to Z^2(\Omega) \to H^2(\Omega) \to 0.$$

We have

$$0 \to H^0(L, B^2(\Omega)) \to H^0(L, Z^2(\Omega)) \to H^0(L, H^2(\Omega)) \to H^1(L, B^2(\Omega)) \to \dots.$$

By Proposition 2, the classes of differential forms $x_i^{-1}x_j^{-1}\delta x_i \wedge \delta x_j$, $i < j$, constitute a basis of the space $H^2(\Omega)$. Moreover, $B^2(\Omega)^L = Z^2(\Omega)^L = 0$ and the action of L in $H^2(\Omega)$ is zero, so $H^0(L, H^2(\Omega)) = H^2(\Omega)$. Thus, the Bokshtein homomorphism

$$d\colon H^0(L, H^2(\Omega)) = H^2(\Omega) = \langle x_i^{-1}x_j^{-1}\delta x_i \wedge \delta x_j \mid i < j\rangle \to H^1(L, B^2(\Omega))$$

is a monomorphism. This implies that the classes of cocycles

$$D \mapsto [D\,,\, x_i^{-1} x_j^{-1} \delta x_i \wedge \delta x_j]\,, \qquad i<j\,,$$

form a basis of $H^1(S_n\,,\, B^2(\Omega))$, $n \geq 3$. In the case $n=2$ we note that the rule $D_{i,j}(u) \mapsto (-1)^s \delta(u\delta x_s)$ defines an isomorphism of modules

$$S_2 \cong B^2(\Omega) \qquad ((i-j)(i-s)(j-s) \neq 0).$$

This isomorphism maps the classes of cocycles mentioned above into the exterior derivations $\operatorname{ad} x_i^{-1} x_j^{-1} \partial_s$ $((i-j)(i-s)(j-s) \neq 0)$. The centralizer of the subalgebra S_2 in W_3 is zero, and this implies that the latter derivations together with the derivation $\operatorname{ad} h$ $(h=\sum_i x_i \partial_i)$ really constitute a basis in $\operatorname{Out} S_2$.

Let $L=H_n$. Suppose that for a certain linear combination $\sigma = \sum_i a_i x_{-i}^{-1} \partial_i + b \operatorname{ad} x^\theta + c\Delta$ we have $\sigma = du$, $u \in L$. Since the projection of the subspace $\partial_{-j}(L)$ onto the line x_{-j}^{-1} is zero and L contains no elements of the form x^θ , it is evident that

$$\sigma(x_j) = a_j x_{-j}^{-1} \pm b x^{\theta - \varepsilon_{-j}} + c x_j = \operatorname{sgn} j\, \partial_{-j}(u) \implies a_j = b = c = 0\,, \qquad j \in I. \quad \square$$

Proof of Theorem 1. From [2, Prop. 1] it follows that there exists an embedding $H^2(L\,,\,\mathbb{C}) \to H^1(L\,,\,L')$. To be more precise, $H^2(L\,,\,\mathbb{C})$ is isomorphic to the subspace $H^1(L)$ generated by the classes of cocycles $\psi \in Z^1(L\,,\,L')$ satisfying the relations $(\psi(l_1)\,,\,l_2) + (\psi(l_2)\,,\,l_1) = 0$. It is easy to see that all cocycles mentioned in the statement of Theorem 2, except $\operatorname{ad} h$ $(L=S_2)$, and Δ $(L=H_n)$, satisfy these relations. Consider the exceptional cases. Evidently,

$$[h\,,\, D_{i,j}(x^\alpha)] = (|\alpha|-2) D_{i,j}(x^\alpha)\,, \qquad |\alpha| = \sum_s \alpha_s.$$

Thus, the expression for $L=S_2$

$$([h\,,\, D_{i,j}(x^\alpha)]\,,\, D_{i,s}(x^\beta)) + (D_{i,j}(x^\alpha)\,,\, [h\,,\, D_{i,s}(x^\beta)]) = \pm 2(|\alpha|+1)\,\delta_{\alpha+\beta,\,\theta+\varepsilon_i}$$

is not identically zero. The same is true for the case $L=H_n$:

$$(\Delta(x^\alpha)\,,\, x^\beta) + (x^\alpha\,,\, \Delta(x^\beta)) = 2(n+2)(-1)^{|\alpha|} \delta_{\alpha+\beta,\,\theta}.$$

The theorem is proved.

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