8. Classification of symplectic vector bundles N over $\mathbf{C}P^1$, their Lagrangian subspaces, and linear SG-processes. By symplectic vector bundles of weight k (over $\mathbb{C}P^1$) we will mean the normal bundles of sections in relative symplectic spaces of the corresponding weight $k = 0, \pm 1, \pm 2, \dots$ (over the base $\mathbb{C}P^1$) modulo the natural equivalence relation. These bundles have the following properties. **I.** *Each symplectic bundle N can be decomposed into the skew-orthogonal direct sum of symplectic 2-bundles of weight k, $N = \sum_{i=1}^{r} \mathcal{O}(a_i) + \mathcal{O}(k - a_i)$.* Denote by $n(i)$ the numbers $\min\{a_j, k - a_j\}$ $(1 \le i, j \le r)$ arranged in increasing order. **II.** *Any Lagrangian linear subspace $L \subset N$ is equivalent to a fiber of one of the 2^r subbundles $N^\varepsilon = \sum_{i=1}^{r} \varepsilon_i \mathcal{O}(a_i) + (1 - \varepsilon_i)\mathcal{O}(k - a_i)$, where $\varepsilon_i = 0, 1$.* **III.** *An elementary linear SG-process $\widehat{N} \to N$ along L reduces k by 1 (Lemma 1) and transforms the numbers a_i according to the rule $\hat{a}_i = a_i - (1 - \varepsilon_i)$, $1 \le i \le r$.* This implies **IV.** *A linear SG-process $\widehat{N} \to N$ of height h (see §5) between two symplectic 2r-bundles in item I exists if and only if $\hat{k} = k - h$ and $n(i) - h \le \hat{n}(i) \le n(i, h) := \min\{n(i), [k/2 - h/2]\}$ for all i, $0 < i \le r$.*

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Odd Central Extensions of Lie Superalgebras

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The central extensions of Lie superalgebras calculated in [1, 7–10] are even. We present a characterization of odd central extensions of Lie superalgebras in terms of cohomology of Lie algebras. The calculation of the second cohomology group (more precisely, space) of a Lie superalgebra with coefficients in an odd module is reduced to calculating the first cohomology group of the even part of this superalgebra with coefficients in the dual module of the odd part. This simple fact permits us to interpret many of the known

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central extensions of Lie algebras as odd supercentral extensions of the corresponding Lie superalgebras. For example, the central extensions of Lie algebras that are Lie algebras of Hamiltonian vector fields, Lie algebras of differential and pseudodifferential operators, Lie algebras of generalized Jacobi matrices, and the Zassenhaus modular algebra have superextensions only in the odd case.

Proposition. *Let L be a Lie superalgebra with even part L_0 and odd part L_1, let P be the ground field of characteristic $p \geq 0$, and let L_1' be the dual L_0-module of the L_0-module L_1. The space of odd central extensions $H_1^2(L, P)$ is isomorphic to the subspace of $H^1(L_0, L_1')$ generated by the classes of cocycles $f \in Z^1(L_0, L_1')$ such that*

$$f([u,v])w + f([v,w])u + f([w,u])v = 0 \qquad \forall\, u, v, w \in L_1.$$

By analogy with Lie algebra assigned to an associative algebra, we can assign to each associative algebra A a Lie superalgebra $L(A)$: its even and odd parts regarded as vector spaces coincide with A, the multiplication in the even part is introduced by the ordinary commutator, the multiplication in the odd part is defined as the product $[u, v] = (uv + vu)/2$, and the action of the even part on the odd one coincides with the adjoint action. With each 2-cochain ψ of the Lie algebra A we can associate an odd 2-cochain $l(\psi)$ of the Lie superalgebra $L(A)$ by setting $l(\psi)(a, b) = 0$ if the elements $a, b \in A$ as elements of the Lie superalgebra $L(A)$ have the same parity and $l(\psi)(a, b) = \psi(a, b)$ if the parities are distinct.

Let I be the set of indices $\pm 1, \ldots, \pm n$ and let

$$U = \mathbb{C}[[t_i^{\pm 1} \mid i \in I]] = \left\{ \sum_\alpha \lambda_\alpha t^\alpha \;\middle|\; \lambda_\alpha \in \mathbb{C},\ \alpha \in \Gamma,\ t^\alpha = \prod_i (t_i)^{\alpha_i} \right\}$$

be the algebra of formal Laurent series with finite positive part (the number of nonzero λ_α, $\alpha \in \Gamma^+$, is finite), where

$$\Gamma = \{\alpha = (\alpha_{-n}, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{Z},\ i \in I\}, \qquad \Gamma^+ = \{\alpha \in \Gamma \mid \alpha_i \geq 0,\ i \in I\}.$$

Let $\partial_i \colon u \mapsto du/dt_i$ be partial derivation with respect to the ith variable. We endow the associative commutative algebra U with the Poisson bracket $\sum_i \partial_{-i} \wedge \partial_i$. The resulting Lie algebra (we denote it by H_n) is called the Poisson algebra. Note that the Lie superalgebra $L(H_n)$ is well defined if we adopt the identification $(L(H_n))_1 = H_n$.

Theorem 1. *Let $P = \mathbb{C}$. Every even central extension of the Lie superalgebra $L(H_n)$ is splittable. The space of odd central extensions $H_1^2(L(H_n), \mathbb{C})$ is $2n$-dimensional, and the classes of cocycles $l(\phi_i)$, where ϕ_i are basis cocycles of the space $H^2(H_n, \mathbb{C})$ [3], form its basis.*

The space U can be endowed with a different structure of associative algebra, which is not commutative:

$$u \circ v = \sum_{\alpha \in \Gamma_n^+} \frac{\partial_-^\alpha(u)\,\partial_+^\alpha(v)}{\alpha!},$$

where

$$\Gamma_n = \{\alpha = (\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{Z},\ i = 1, \ldots, n\}, \qquad \Gamma_n^+ = \{\alpha \in \Gamma_n \mid \alpha_i \geq 0\},$$

$$\partial_-^\alpha = \prod_{i=1}^n \partial_{-i}^{\alpha_i}, \quad \partial_+^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^n (\alpha_i)!$$

The new algebra (denote it by $Pdop_n$) is isomorphic to the algebra of pseudodifferential operators in n variables. Let Dop_n be the subalgebra of $Pdop_n$ generated by the elements of the form $t_-^\alpha t_+^\beta$, $\alpha \in \Gamma_n$, $\beta \in \Gamma_n^+$. It is isomorphic to the algebra of differential operators in n variables $t_{-1}, \ldots, t_{-n}$.

Theorem 2. *The Lie superalgebras $L(Dop_n)$ and $L(Pdop_n)$ have no nontrivial even central extensions. The classes of cocycles $l(\psi_i)$, $i = 1, \ldots, n$ (for the definition of ψ_i see [4]), form a basis of the space $H_1^2(L(Dop_n), \mathbb{C})$. In particular, the space of odd central extensions of the Lie superalgebra $L(Dop_n)$ is*

n-dimensional. The space of odd central extensions of the Lie superalgebra $L(Pdop_n)$ is $2n$-dimensional, and the classes of cocycles $l(\psi_i)$, $i = \pm 1, \dots, \pm n$, form its basis.

The algebra of generalized Jacobi matrices gl_J consists of matrices $X = (x_{i,j})$ with a finite number of nonzero diagonals: $x_{i,j} = 0$ whenever $|i - j| \gg 0$. It is proved in [2, 11] that the space $H^2(gl_J, \mathbb{C})$ is one-dimensional and the class of the cocycle ρ defined by the rule $\rho(X, Y) = \mathrm{tr}[R, X]Y$, where $R = (r_{i,j})$ is a matrix such that $r_{i,i} = 1$ for $i \geq 0$ and $r_{i,j} = 0$ otherwise, determines its basis.

Theorem 3. *The space of odd central extensions of the Lie superalgebra $L(gl_J)$ is one-dimensional, and the class of the cocycle $l(\rho)$ of the Lie superalgebra $L(gl_J)$ forms its basis. The cocycle ρ of the Lie algebra gl_J has no even superextension.*

Let U be an associative commutative algebra with derivation ∂. Let us endow its algebra of derivations with structure of a Lie algebra by setting $[u\partial, v\partial] = u\partial(v)\partial - v\partial(u)\partial$. Denote the resulting Lie algebra by W. We introduce a W-module U_q in the space U by the rule $(u\partial)_q v = u\partial(v) + q\partial(u)v$, $q \in P$. Note that the adjoint W-module is isomorphic to U_{-1}. It can readily be shown that the graded space $L = W + U_{-1/2}$ becomes a Lie superalgebra if we define the symmetric mapping $U_{-1/2} \times U_{-1/2} \to W$ by means of the commutative multiplication in U. For $U = \mathbb{C}[t^{\pm 1}]$ we obtain the Ramon superalgebra without center. The Virasoro–Gelfand–Fuks cocycle can be extended to the even cocycle

$$\phi(u\partial, v\partial) = \pi(\partial^3(u)v), \qquad \phi(u, v) = -2\pi(\partial^2(u)v).$$

Here $\pi(u)$ is the coefficient in x^{-1} in the element u for the case $p = 0$ and in $x^{p^m - 1}$ for the case $p > 0$ and $U = O_1(m)$ is the divided fraction algebra. For more detailed presentation of modular Lie algebras and their cohomology see [5, 6].

Theorem 4. *Assume that $p \geq 3$. Then the space of even extensions of the Lie superalgebra $L(m)$ is one-dimensional and is generated by the class of the Ramon–Neveux–Schwarz cocycle ϕ for $p \geq 5$; in the case $p = 3$ it is $(m-1)$-dimensional and is generated by the classes of cocycles α_k, $0 < k < m$, defined as*

$$\alpha_k(u\partial, v\partial) = \pi(\partial^{p^k}(u)v), \qquad \alpha_k(u, v) = \pi(\partial^{p^k - 1}(u)v);$$

for $p > 5$ the space of odd extensions of the Lie superalgebra $L(m)$ is trivial; for $p = 5$ it is $(m-1)$-dimensional and is generated by the classes of cocycles β_k, $0 < k \leq m$, given by the formulas

$$\beta_k(u\partial, v) = \pi(\partial^{p^k}(u)v);$$

for $p = 3$ it is one-dimensional and is generated by the class of a cocycle γ such that

$$\gamma(u\partial, v) = \pi(\partial(u)v).$$

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