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Algèbres de Lie/*Lie Algebras*

Symmetric (co)homologies of Lie algebras

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Abstract. Cohomologies of Lie algebras are usually calculated using the Chevalley-Eilenberg cochain complex of skew-symmetric forms. We consider two cochain complexes consisting of forms with some symmetric properties. First, cochains $\bar{C}^*(L)$ are symmetric in the last 2 arguments, skew-symmetric in the others and satify moreover some kind of Jacobi condition in the last 3 arguments. In characteristic 0, its cohomologies are isomorphic to the cohomologies of the factor-complex $C^*(L,L')/C^{*+1}(L,K)$. Second, a symmetric version $\bar{C}^*_\lambda(A)$ is defined for an associative algebra A. It is a subcomplex of the cyclic cochain complex. These symmetric cochain complexes are used for the calculation of 3-cohomologies of Cartan Type Lie algebras with trivial coefficients.

(Co)homologies symétriques des algèbres de Lie

Résumé. *Les cohomologies usuelles des algèbres de Lie sont définies par des formes multilinéaires anti-symétriques. Nous considérons deux complexes de cochaînes qui se composent de formes avec certaines propriétés de symétrie. La cohomologie symétrique $\bar{H}^*(L)$ d'une algèbre de Lie L, les cohomologies de Chevalley-Eilenberg du L-module coadjoint et du L-module trivial définissent une suite exacte de cohomologie :*

$$\cdots \to H^{k+2}(L,K) \to H^{k+1}(L,L') \to \bar{H}^k(L) \xrightarrow{\delta} H^{k+3}(L,K) \to \cdots$$

La cohomologie symétrique cyclique $\bar{H}^*_\lambda(A)$ d'une algèbre associative A, la cohomologie cyclique $H^*_\lambda(A)$ et la cohomologie de l'algèbre de Lie $\mathcal{A}$ définissent une suite exacte de cohomologie :

$$\cdots \to \bar{H}^k_\lambda(A) \to H^k_\lambda(A) \to H^{k+1}(\mathcal{A},K) \xrightarrow{\delta} \delta \to \bar{H}^{k+1}_\lambda(A) \to \cdots.$$

Nous utilisons les cohomologies symétriques pour calculer la 3-cohomologie des algèbres de Lie du type de Cartan.

Note présentée par Alain CONNES.

Version française abrégée

On suppose que K est un corps de caractéristique 0, L une K-algèbre de Lie. On désigne par $\bar{C}^k(L)$ l'espace des applications multilinéaires de $L^{\otimes k+2}$ dans K, qui vérifient les identités suivantes :

$$\psi(x_1,\ldots,x_k,x,y) = \operatorname{sgn}\sigma\,\psi(x_{\sigma(1)},\ldots,x_{\sigma(k)},x,y), \quad \forall\sigma\in \mathrm{Sym}_k,$$
$$\psi(x_1,\ldots,x_k,x,y) = \psi(x_1,\ldots,x_k,y,x),$$
$$\psi(x_1,\ldots,x_{k-1},x,y,z) + \psi(x_1,\ldots,x_{k-1},y,z,x) + \psi(x_1,\ldots,x_{k-1},z,x,y) = 0.$$

Soit $\bar{C}^*(L)$ la somme directe des groupes $\bar{C}^k(L)$. L'opérateur $s : \bar{C}^k(L) \to \bar{C}^{k+1}(L)$:

$$s\psi(x_1,\ldots,x_{k+3}) = \sum_{1\le i<j\le k+3, i\neq k+2} (-1)^i \psi(x_1,\ldots,\hat{x}_i,\ldots,[x_i,x_j],x_{j+1},\ldots,x_{k+3})$$

est un opérateur de cobord : $s^2 = 0$. L'espace de cohomologie $\bar{H}^*(L)$ correspondant s'appelle la *cohomologie symétrique de L.*

Soient A une K-algèbre associative, $\mathcal{A}$ l'algèbre de Lie correspondante, $C^*_\lambda(A)$ le complexe de cochaînes cyclique. On désigne par $\bar{C}^*_\lambda(A)$ le sous-complexe de $C^*_\lambda(A)$, tel que $\bar{C}^k_\lambda(A) = \{\psi \in C^k_\lambda(A) : \sum_{\sigma\in \mathrm{Sym}_k}(\operatorname{sgn}\sigma)\sigma\psi = 0\}$.

La suite de complexes

$$0 \to C^{*+2}(L,K) \to C^{*+1}(L,L') \to \bar{C}^*(L) \to 0$$

est exacte et définit une suite exacte de cohomologie :

$$\cdots \to H^{k+2}(L,K) \to H^{k+1}(L,L') \to \bar{H}^k(L) \to H^{k+3}(L,K) \to \cdots$$

La suite de complexes

$$0 \to \bar{C}^*_\lambda(A) \to C^*_\lambda(A) \to C^{*+1}(\mathcal{A},K) \to 0$$

est exacte et définit une suite exacte de cohomologie :

$$\cdots \to \bar{H}^k_\lambda(A) \to H^k_\lambda(A) \to H^{k+1}(\mathcal{A},K) \to \bar{H}^{k+1}_\lambda(A) \to \cdots$$

Nous utilisons les complexes de cochaînes symétriques pour calculer les 3-cohomologies des algèbres de Lie du type de Cartan $L = W_n, S_n (n > 2)$, H_n, K_{n+1}; nous montrons que, si $L = W_1$ (resp. W_2, S_n et H_n), alors $\dim H^3(L,\mathbf{C}) = 1$ (resp. 2, $(n^3-n)/6 + 1$ et $2n^2 + n + 1$). Dans les autres cas, $H^3(L,\mathbf{C}) = 0$.

We are interested in the cohomologies of two-sided Cartan Type Lie algebras over a field of characteristic 0. In [1], the 2-cohomology groups of such algebras with coefficients in the trivial module is computed. It was proved that, except for the Virasoro algebra, there are two new series of Lie algebras of vector fields that have nontrivial central extensions. We compute here the 3-cohomology groups for two-sided Lie algebras of formal vector fields.

Let $\tilde{L}$ be a Lie algebra of one of the following series:

$$\begin{aligned}
&W_n = \mathrm{Der}\,\mathbf{C}[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]],\\
&\tilde{S}_n = \{D \in W_{n+1} : D(\omega_S) = 0\}, \quad \omega_S = dx_0 \wedge \ldots \wedge dx_n,\\
&\tilde{H}_n = \{D \in W_{2n} : D(\omega_H) = 0\}, \quad \omega_H = dx_{-1} \wedge dx_1 + \ldots + dx_{-n} \wedge dx_n,\\
&K_{n+1} = \{D \in W_{2n+1} : D\omega_K = \omega_K\}, \quad \omega_K = dx_0 + dx_{-1} \wedge dx_1 + \ldots + dx_{-n} \wedge dx_n.
\end{aligned}$$

By a *Cartan type Lie algebra*, we understand a simple Lie algebra of these series. Namely, W_n and K_{n+1} are simple, but $S_n = [\tilde{S}_n, \tilde{S}_n], H_n = [\tilde{H}_n, \tilde{H}_n]/\mathbf{C}$. Lie algebra of special type S_n are generated by derivations without divergence. Let $\partial_i = d/dx_i, \theta = (1, \ldots, 1), 0 = (0, \ldots, 0)$. For $u \in U = \mathbf{C}[[x_i^{\pm 1}, i \in I]]$, we denote by $\pi(u)$ its coefficient at $x^{-\theta}$. The Hamiltonian algebra $\tilde{H}_n$ can be defined on the space $U = \mathbf{C}[[x_{\pm 1}^{\pm 1}, \ldots, x_{\pm n}^{\pm 1}]]$ by the Poisson bracket $[u, v] = \sum_{i=1}^{n} \partial_{-i}(u)\partial_i(v) - \partial_{-i}(v)\partial_i(u)$. Then $H_n = \{x^\alpha \in U : \alpha \neq 0, -\theta\}$. Let $\Delta = 2 - \sum_{i=\pm 1}^{\pm n} x_i\partial_i$.

THEOREM 1. – *Let L be a simple Lie algebra of Cartan series W_n, S_n $(n > 2)$, H_n, K_{n+1}. Then $H^3(L, \mathbf{C}) = 0$, except in the following cases:*

$$\begin{aligned}
L = W_1 \quad & H^3(L, \mathbf{C}) \cong H^2(L, L') = \{d \ln x \wedge \partial^3\} \cong \mathbf{C};\\
L = W_2 \quad & H^3(L, \mathbf{C}) \cong H^2(L, L') = \{\psi_{11}^W : (u\partial_i, v\partial_j) \mapsto d[\partial_i(u), \partial_j(v)],\\
& \psi_{12}^W : (u\partial_i, v\partial_j) \mapsto d\partial_j(u) \wedge d\partial_i(v)\} \cong \mathbf{C}^2;\\
L = S_n \quad & H^3(L, \mathbf{C}) \cong H^2(L, L') = \{\psi_1^S : (u\partial_i, v\partial_j) \mapsto d\partial_j(u) \wedge d\partial_i(v)\}\\
& + \{ad \ln x_i \wedge ad \ln x_j \wedge ad \ln x_s, i < j < s\} \cong \mathbf{C}^{(n^2-1)n/6+1};\\
L = H_n \quad & H^3(L, \mathbf{C}) \cong \bar{H}^2(L, L) = \{ad \ln x_i \wedge ad \ln x_j, i < j, ad\, x^{-\theta} \wedge ad \ln x_i,\\
& \Delta \wedge ad\, x^{-\theta}\} \cong \mathbf{C}^{2n^2+n+1}.
\end{aligned}$$

Here we use the notation of [5]. Our calculations of cohomology groups of trivial module are based on the connection between cohomologies of trivial module and coadjoint module. Namely, in [4], the homomorphism of cochain complexes $C^{*+1}(L, K) \to C^*(L, L')$ was constructed. Denote the corresponding factor-complex by $\tilde{C}^*(L)$. Any short exact sequence of cochain complexes has long cohomological sequence. In our case this means that the following long sequence α of cohomology groups is exact:

$$0 \to H^2(L, K) \to H^1(L, L') \to \tilde{H}^1(L) \to H^3(L, K) \to H^2(L, K) \to \tilde{H}^2(L) \to \cdots$$

In particular, there exists a monomorphism $H^2(L, K) \to H^1(L, L')$. The descriptions of central extensions in The existence of the long homological sequence α was also observed in [10] and [14].

Here we observe that the factor-complex $\tilde{C}^*(L)$ is equivalent to the cochain complex $\bar{C}^*(L)$ (if the characteristic of the field is 0) with coboundary operator s that we construct below.

Let L be a Lie algebra over a field K and M a L-module; let $T^k(L, M) = \mathrm{Hom}(L^{\otimes k}, M)$ and $C^k(L, M) = \mathrm{Hom}(\bigwedge^k L, M)$, and let $C^*(L, M) = \oplus C^k(L, M)$ be the Chevalley-Eilenberg cochain complex with coboundary operator d. Define an action of the symmetric group Sym_k on $T^k(L, M)$ by the rule $\sigma\psi(x_1, \ldots, x_k) = \psi(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. Let σ_i be the transposition $(i, i+1)$, and $\nu(i, k)$ be cyclic permutation $(i, i+1, \ldots, k)$. Let $J : T^{k+2}(L, K) \to T^{k+2}(L, K)$, be such that

$$J\psi = (\nu(k, k+2) + \nu(k, k+2)^2 + \nu(k, k+2)^3)\psi.$$

DEFINITION. – We define $\bar{C}^k(L)$ as

$$\left\{\psi \in T^{k+2}(L,K) : \begin{cases} \sigma_i\psi = -\psi & \text{if } 1 \le i < k, \\ \sigma_i\psi = \psi & \text{if } i = k+1 \end{cases} \quad \text{and} \quad J\psi = 0\right\}$$

For an associative algebra A, we denote by $\mathcal{A}$ its Lie algebra with commutator $[x,y] = xy - yx$, by $C^*_\lambda(A)$ and $H^*_\lambda(A)$ its cyclic cochain complex and cohomologies under cyclic coboundary operator b respectively, (*see* [12]). Recall that $C^k_\lambda(A) = \{\psi \in T^{k+1}(A,K) : \nu(1,k+1)\psi = (-1)^k\psi\}$.

DEFINITION. – $\bar{C}^k_\lambda(A) = \{\psi \in C^k_\lambda(A) : \sum_{\sigma \in \mathrm{Sym}_{k+1}, \sigma(k+1)=k+1} \mathrm{sgn}\,\sigma\,\sigma\psi = 0\}$.

Define an operator $s : \bar{C}^k(L) \to \bar{C}^{k+1}(L)$ by the rule

$$s\psi(x_1,\ldots,x_{k+3}) = \sum_{1\le i<j\le k+3, i\ne k+2} (-1)^i\psi(x_1,\ldots,\hat{x}_i,\ldots,x_{j-1},[x_i,x_j],\ldots,x_{k+3}).$$

PROPOSITION 1. – *The operator s is well defined and $s^2 = 0$. If $p = \mathrm{char}K$ is 0, then for any $k > 0$ there is an isomorphism of cohomology groups $\tilde{H}^k(L) \cong \bar{H}^{k-1}(L)$. If $p \ne 2,3$, then the same is true for $k = 1,2$.*

Notice that s is similar to the Loday coboundary operator, (*see* [12] and [13]). Consider L as a Leibniz algebra and M as a symmetric module over L. It was observed in [12] that $H^2(L,M)$ is a subspace of Loday cohomology group $HL^2(L,M)$. Thus we may consider

$$\mathcal{H}^2(L,M) := HL^2(L,M)/H^2(L,M)$$

as a space of pure Leibniz extensions of L. Non-commutative extensions of Cartan Type Lie algebras are described in [6].

Let L' be the coadjoint L-module. Consider the folowing operators:

$$a_0^+ : C^{k+2}(L,K) \to C^{k+1}(L,L'), \quad \text{with } a_0^+\alpha(x_1,\ldots,x_{k+1})(x_{k+2}) = \alpha(x_1,\ldots,x_{k+2}),$$

$$a_1^+ : C^{k+1}(L,L') \to \bar{C}^k(L), \quad \text{with } a_1^+\beta = \beta + \sigma_{k+1}\beta,$$

$$a_0^- : \bar{C}^{k+1}(L) := \ker a_1^+ \to C^{k+2}(L,K), \quad \text{with } (a_0^-\beta)(x_1,\ldots,x_{k+1},x) = \beta(x_1,\ldots,x_{k+1})(x),$$

$$a_1^- : \bar{C}^{k-1}(L) \to C^k(L,L'), \quad \text{with } (a_1^-\gamma)(x_1,\ldots,x_k)(x) = \sum_{1\le i\le k} (-1)^i\gamma(\ldots,\hat{x}_i,\ldots,x_k)(x_i,x).$$

The sequence of cochain complexes

$$0 \to (C^{*+2}(L,K),d) \xrightarrow{a_0^+} (C^{*+1}(L,L'),d) \xrightarrow{a_1^+} (\bar{C}^*(L),s) \to 0$$

is exact and has the following exact cohomological sequence

$$\cdots \to H^{k+2}(L,K) \xrightarrow{a_0^+} H^{k+1}(L,L') \xrightarrow{a_1^+} \bar{H}^k(L) \xrightarrow{\delta = a_0^- d a_1^-} H^{k+3}(L,K) \to \cdots.$$

The space $\bar{C}^0(L)$ coincides with the space of symmetric bilinear forms $S^2(L,K)$, and $\bar{H}^0(L)$ coincides with the space of symmetric invariant bilinear forms.

For an associative algebra A, we definet wo operators

$$q^+ : C^k_\lambda(A) \to C^k(\mathcal{A},\mathcal{A}'),\ q^+\psi(x_1,\ldots,x_k)(x) = \sum_{\sigma\in \mathrm{Sym}_k} \mathrm{sgn}\,\sigma\,\psi(x_{\sigma(1)},\ldots,x_{\sigma(k)},x),$$

$$q^- : \bar{C}^k(\mathcal{A},\mathcal{A}') \to C^k_\lambda(A),\quad q^-\psi(x_1,\ldots,x_{k+1}) = \psi(x_1,\ldots,x_k)(x_{k+1}).$$

As an easy consequence of results in [12] we obtain that the sequence of cochain complexes

$$0 \to (\bar{C}^*_\lambda(A),b) \to (C^*_\lambda(A),b) \overset{a_0^- q^+}{\to} (C^{*+1}(\mathcal{A},K),-d) \to 0$$

is exact and has the following exact cohomological sequence

$$\cdots \to \bar{H}^k_\lambda(A) \to H^k_\lambda(A) \to H^{k+1}(\mathcal{A},K) \xrightarrow{\delta=q^- s a^+} \bar{H}^{k+1}_\lambda(A) \to \cdots.$$

PROPOSITION 2. – *If $p \neq 2$ and $p \neq 3$, then*
1. $H^1(L,K) \cong H^0(L,L')$.
2. $H^2(L,K) \cong \bar{H}^1(L) = \{[\psi] : \psi \in Z^1(L,L'), \psi(x)(y) + \psi(y)(x) = 0\}$ *is a direct summand of* $H^1(L,L')$. *If all symmetric invariant forms on L are trivial, then* $H^2(L,K) \cong H^1(L,L')$.
3. *Let* $\bar{H}^2(L) = \{[\psi] : \psi \in Z^2(L,L'), \psi(x,y)(z) + \psi(x,z)(y) = \tau([x,y],z) + \tau(y,[x,z])$, *for some* $\tau \in S^2(L,K)\}$. *In* $\bar{H}^2(L)$ *one can find a basis consisting of classes of cocycles from* $\bar{Z}^2(L)$. *If all symmetric invariant forms on L are trivial, then* $H^3(L,K) \cong \bar{H}^2(L)$. *If L is simple and has nondegenerate symmetric invariant form* $(\ ,\ )$, *then* $H^3(L,K)$ *has direct summand isomorphic to* $\bar{H}^2(L)$ *and the additional summand is generated by the cohomological class of the cocycle* $\mathcal{D}(\ ,\ )$, *where* $[\mathcal{D}] : S^2(L,K)^L \to H^3(L,K)$ *is a Koszul homomorphism:* $\mathcal{D}(\ ,\ )(x,y,z) = (x,[y,z])$, (see [11]). *There is an isomorphism* $\mathcal{H}^2(L,K) \cong \ker[\mathcal{D}]$.
4. *If A is an associative algebra, then* $\bar{H}^k_\lambda(A) = 0, k = 0,1$, *and* $\bar{H}^2_\lambda(A) = \{\psi \in S^3(A,K) : \psi(x,y,zt) + \psi(xy,z,t) = \psi(t,x,yz) + \psi(tx,y,z)\}$.

Remark. – The origin of our constructions is the Weyl Algebra $\wedge^*(L) \otimes S^*(L)$. Set $C^{k,r}(L) = C^{k-r}(L,S^r(L))$, $C^{*,r}(L) = \oplus_k C^{k,r}(L)$. The following locally finite sequence of cochain complexes

$$0 \to C^{*,0}(L) \overset{a_0^+}{\to} C^{*,1}(L) \overset{a_1^+}{\to} C^{*,2}(L) \overset{a_2^+}{\to} \cdots$$

is exact. In particular, the cochain subcomplexes $\bar{C}^{*,r} = \ker a_r^+$ form an exact sequence of cochain complexes:

$$0 \to \bar{C}^{*,r}(L) \to C^{*,r}(L) \to \bar{C}^{*,r+1}(L) \to 0.$$

Therefore the following exact cohomological sequence holds

$$0 \to \bar{H}^{k,r}(L) \to H^{k,r}(L) \to \bar{H}^{k,r+1}(L) \to 0$$

Our sequences arise from these sequences for $r = 0$ and $r = 1$.

The proof of Theorem 1 follows from the description of the second cohomology group of Cartan type Lie algebras (with coefficients in the irreducible tensor modules (*see* [5]), in particular in coadjoint modules) and from Proposition 2. One can obtain exact formulas for cocycles from $\bar{Z}^2(L)$, whose cohomological classes give the basis of $\bar{H}^2(L)$. Consider, for example, the cases $L = W_n, n = 1,2$. For $L = W_1$, the basic 3-cocycle in $Z^3(W_1,K)$ corresponding to $d\ln x \wedge \partial^3$ is the Gelfand-Fuchs cocycle (*see* [9]): the value of this cocycle on $(u\partial, v\partial, w\partial)$ is the determinant

$$\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$$

at $x = 0$ (here $u = u(x), v = v(x), w = w(x), u' = \partial u/\partial x$).

For $L = W_2$, we have:

$$a_0^- \psi_{11}^W(X,Y,Z) = \pi([\mathrm{Div}(X), \mathrm{Div}(Y)]\,\mathrm{Div}(Z)),$$

(where $[\cdot,\cdot]$ is Poisson bracket). But $a_1^+\psi_{12}^W \neq 0$, and the construction of

$$\psi_2 \in \bar{Z}^2(L), \qquad \psi_2 \in [\psi_{12}^W]$$

is slightly more complicated:

$$\psi_2(u\partial_1, v\partial_1) = \left(\frac{2}{3}\partial_2[\partial_1 u, \partial_1 v] - \partial_1([\partial_1 u, \partial_2 v] + [\partial_2 u, \partial_1 v])\right)dx_2 - \frac{4}{3}\partial_1[\partial_1 u, \partial_1 v]dx_1,$$

$$\psi_2(u\partial_1, v\partial_2) = \left(-\partial_1[\partial_2 u, \partial_1 v] + \frac{2}{3}\partial_1[\partial_1 u, \partial_2 v] - \partial_2[\partial_1 u, \partial_1 v]\right)dx_1 \cdots$$

$$\cdots + \left(-\partial_2[\partial_2 u, \partial_1 v] + \frac{2}{3}\partial_2[\partial_1 u, \partial_2 v] - \partial_1[\partial_2 u, \partial_2 v]\right)dx_2,$$

$$\psi_2(u\partial_2, v\partial_2) = \left(\frac{2}{3}\partial_1[\partial_2 u, \partial_2 v] - \partial_2([\partial_2 u, \partial_1 v] + [\partial_1 u, \partial_2 v])\right)dx_1 - \frac{4}{3}\partial_2[\partial_2 u, \partial_2 v]dx_2.$$

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