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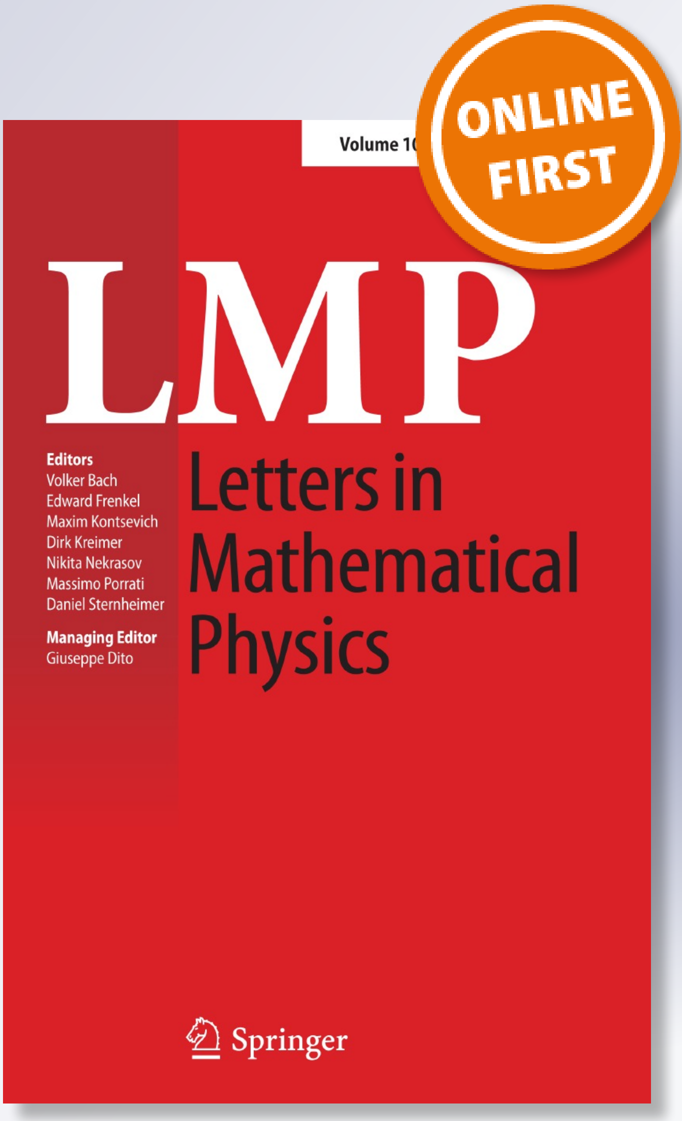

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$2p$-Commutator on Differential Operators of Order p

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Abstract. We show that a space of one variable differential operators of order p admits non-trivial $2p$-commutator and the number $2p$ here can not be improved.

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Keywords. Weyl algebra, differential operator, N-commutator, polynomial identity, Amitzur–Levitzki identity.

Let A be an associative algebra over a field K of characteristic 0. Let $f = f(t_1,\dots,t_n)$ be some non-commutative associative polynomial. Say that $f=0$ is *identity* on A if $f(a_1,\dots,a_n)=0$ for any substitutions $t_i := a_i \in A$. Let s_n be a skew-symmetric associative non-commutative polynomial

$$s_n(t_1,\dots,t_n)=\sum_{\sigma\in\mathrm{Sym}_n} \mathrm{sign}\,\sigma\; t_{\sigma(1)}\cdots t_{\sigma(n)}.$$

For example,

$$s_2(t_1,t_2)=t_1t_2-t_2t_1=[t_1,t_2]$$

is a Lie commutator.

Suppose that an associative commutative algebra U has k commuting derivations $\partial_1,\dots\partial_k$. A linear span of linear operators of a form $u\partial_{i_1}\dots\partial_{i_p}$, where $1\le i_1,\dots,i_p\le k$, is denoted $D_k^{(p)}(U)$. Let $D_k(U)=\cup_{p\ge 0}D_k^{(p)}(U)$ be space of differential operators on U generated by derivations $\partial_1,\dots,\partial_k$. In case of $k=1$ we reduce notation ∂_1 to ∂.

It is known that $D_k(U)$ can be endowed by a structure of associative algebra. A multiplication of the algebra $D_k(U)$ is given as a composition of differential operators. For example, if $k=1$, then

$$u\partial^p\cdot v\partial^l=\sum_{s=0}^{p}\binom{p}{s}u\partial^s(v)\partial^{p+l-s}.$$

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Certainly this construction can be easily generalized for algebras with several derivations.

We can consider $D_k^{(p)}(U)$ as a space of differential operators of order p. Well known, that any differential operator of first order is a derivation and a space of derivations $\mathrm{Der}(U)=D_k^{(1)}(U)$ forms Lie algebra under commutator,

$$\begin{aligned}u\partial_i, v\partial_j \in D_k^{(1)}(U) &\Rightarrow s_2(u\partial_i, v\partial_j)=u\partial_i \cdot v\partial_j - v\partial_j \cdot u\partial_i\\ &\Rightarrow s_2(u\partial_i, v\partial_j)=u\partial_i(v)\partial_j - v\partial_j(u)\partial_i \in D_k^{(1)}(U).\end{aligned}$$

Main example of the algebra of differential operators appears in the case $U=K[x_1,\ldots,x_k]$ and $\partial_i=\partial/\partial x_i$, $i=1,\ldots,k$, are partial differential operators. Recall that action of ∂_i on a monom $x^\alpha=x_1^{\alpha_1}\cdots x_k^{\alpha_k}$, where $\alpha=(\alpha_1,\ldots,\alpha_k)\in \mathbf{Z}_0^k$, is defined by

$$\partial_i x^\alpha = \alpha_i x^{\alpha-\epsilon_i}.$$

Here $\mathbf{Z}_0$ is a set of non-negative integers and $\epsilon_i=(0,\ldots,0,1,0,\ldots,0)\in \mathbf{Z}_0^k$ (all components of ϵ except i-th are 0).

Denote by A_k an algebra of differential operators on polynomials algebra $K[x_1,\ldots,x_k]$ generated by k commuting derivations $\partial_1,\ldots,\partial_k$. The algebra A_k is called k-th *Weyl* algebra. Let $A_k^{(p)}=\langle u\partial^\alpha \,||\alpha|=p\rangle$ be subspace of A_k consisting differential operators of p-th order.

Let us consider $A_k^{(p)}$ as N-ary algebra under N-ary multiplication s_N,

$$s_N(X_1,\ldots,X_N)=\sum_{\sigma\in \mathrm{Sym}_N} \mathrm{sign}\,\sigma\, X_{\sigma(1)}\cdots X_{\sigma(N)}.$$

In general this notion is not correct. Might happen that s_N is not well-defined on $A_k^{(p)}$,

$$s_N(X_1,\ldots,X_n)\notin A_k^{(p)}$$

for some $X_1,\ldots,X_N\in A_k^{(p)}$. We say that $A_k^{(p)}$ *admits N-commutator s_N*, if

$$s_N(X_1,\ldots,X_N)\in A_k^{(p)}$$

for any $X_1,\ldots,X_N\in A_k^{(p)}$.

In [3] it was proved that the space of differential operators of first order $A_n^{(1)}$ in addition to Lie commutator s_2 admits (n^2+2n-2)-commutator and that $s_N=0$ is identity if $N\geq n^2+2n$. Let Mat_n be an algebra of $n\times n$ matrices. Amitzur–Levitzky theorem states that Mat_n satisfies the identity $s_{2n}=0$ and it is a minimal identity [1]. Note that Weyl algebra has no polynomial identity except associativity. So, to construct non-trivial identities we have to consider smaller subspaces of Weyl algebra.

The aim of our paper is to establish that the space of one variable differential operators of order p admits $2p$-commutator. The number $2p$ here can not be improved: if $N>2p$, then $s_N=0$ is identity on $A_1^{(p)}$; if $N<2p$, then s_N is not well-defined on $A_1^{(p)}$; if $N=2p$, then s_N is well-defined on $A_1^{(p)}$ and non-trivial. Obtained $2p$-ary algebra $A_1^{(p)}$ under multiplication s_{2p} is simple and left-commutative. In particular, the $2p$-algebra $(A_1^{(p)}, s_{2p})$ is homotopical $2p$-Lie. To formulate exact result we have to introduce some definitions.

Let us given an n-ary algebra (A,ψ) with n-ary skew-symmetric multiplication $\psi:\wedge^n A\to A$. Say that A has $(2n-2,1)$-*type identity* (in [4] it is called $(n-1)$-left commutative) if it satisfies the identity

$$\sum_{\sigma\in S^{(2n-2,1)}} \operatorname{sign}\sigma\,\psi(a_{\sigma(1)},\dots,a_{\sigma(n-1)},\psi(a_{\sigma(n)},\dots,a_{\sigma(2n-2)},a_{2n-1}))=0$$

Say that (A,ω) satisfies $(1,2n-2)$-*type identity*, if

$$\sum_{\sigma\in S^{(1,2n-2)}} \operatorname{sign}\sigma\,\psi(a_1,a_{\sigma(2)},\dots,a_{\sigma(n-1)},\psi(a_{\sigma(n)},\dots,a_{\sigma(2n-1)}))=0,$$

for any $a_1,\dots,a_{2n-1}\in A$. Here

$$S^{(2n-1,1)}=\{\sigma\in S_{n-1,n}|\sigma(2n-1)=2n-1\},$$
$$S^{(1,2n-1)}=\sigma\in S_{n-1,n}|\sigma(1)=1\},$$

where

$$S_{n-1,n}=\{\sigma\in S_{2n-1}|\sigma(1)<\cdots\sigma(n-1),\ \sigma(n)<\cdots<\sigma(2n-1)\}$$

is a set of shuffle $(n-1,n)$-permutations on the set $\{1,2,\dots,2n-1\}$. Call n-algebra (A,ψ) *left-commutative* if it satisfies the $(2n-2,1)$-type identity. Similarly, it is called *right-commutative* if it has the $(1,2n-2)$-type identity. In fact, these two notions are equivalent (Lemma 22).

Say that (A,ψ) is *homotopical n-Lie* [5] if it satisfies the following identity

$$\sum_{\sigma\in S_{n-1,n}} \operatorname{sign}\sigma\,\psi(a_{\sigma(1)},\dots,a_{\sigma(n-1)},\psi(a_{\sigma(n)},\dots,a_{\sigma(2n-1)}))=0.$$

For k-ary algebra (A,ψ) with k-multiplication $\psi:\wedge^k A\to A$ and for a subspace $I\subseteq A$ say that I is *ideal* of A, if $\psi(a_1,\dots,a_{k-1},b)\in I$, for any $a_1,\dots,a_{k-1}\in A, b\in I$. Say that A is *simple,* if it has no ideal except 0 and A.

In our paper, we prove the following result.

THEOREM 1. *Let $A_1=D(K[x])$ be one variable Weyl algebra over a field K of characteristic 0. Then*

- *$s_{2p+1}=0$ is a polynomial identity on $A_1^{(p)}$.*
- *Any polynomial identity of degree no more than $2p$ follows from the associativity one*
- *s_N is not well-defined on $A_1^{(p)}$ if $N<2p$*

- *s_{2p} is well-defined and non-trivial operation on $A_1^{(p)}$*
- *For any $u_1,\dots,u_{2p}\in K[x]$, the following formula holds*

$$s_{2p}(u_1\partial^p,\cdots,u_{2p}\partial^p)=\lambda_p\begin{vmatrix} u_1 & u_2 & \cdots & u_{2p}\\ \partial(u_1) & \partial(u_2) & \cdots & \partial(u_{2p})\\ \vdots & \vdots & \cdots & \vdots\\ \partial^{2p-1}(u_1) & \partial^{2p-1}(u_2) & \cdots & \partial^{2p-1}(u_{2p})\end{vmatrix}\partial^p,$$

  where λ_p is a positive integer
- *the $2p$-algebra $(A_1^{(p)},s_{2p})$ is simple and left-commutative.*

COROLLARY 2. *If $k>2p$, then $s_k=0$ is a polynomial identity on $A_1^{(p)}$.*

COROLLARY 3. *The $2p$-algebra $(A_1^{(p)},s_{2p})$ is right-commutative*

Proof. It follows from Lemma 22.

COROLLARY 4. *The $2p$-algebra $(A_1^{(p)},s_{2p})$ is homotopical $2p$-Lie.*

Proof. By Corollary 2.2 of [4] the algebra $(A_1^{(p)},s_{2p})$ is homotopical n-Lie.

COROLLARY 5. *Any polynomial identity of Weyl algebra A_n follows from the associativity identity.*

This result follows also from results of [7].

Proof. Suppose that A_n has some polynomial identity $g=0$ that does not follow from associativity identity. We can assume that g is multi-linear. Suppose that it has degree $\deg g=d$. Then $g=0$ induces a polynomial identity for any subspace of A_n. In particular, $g=0$ is identity on $A_1^{(p)}$. Take p such that $2p>d$. We obtain contradiction with the minimality of identity $s_{2p}=0$ for $A_1^{(p)}$.

COROLLARY 6. *Let U be an associative commutative algebra with a derivation ∂. Then s_{2p} is a $2p$-commutator of $D^{(p)}(U)$ and $s_N=0$ is identity on $D^{(p)}(U)$ for any $N>2p$.*

Remark 1. In [9], identities of Lie algebras of vector fields are considered. In [6,8], growth of Lie algebras of vector fields on the line generated by two vector fields is studied. In other words, they consider Lie algebras of differential operators of order 1. In our paper we consider one variable differential operators of order $p>1$. They are not close under composition. They are not close under Lie commutator. They do not form Lie algebra and they are not associative algebras. We study not only identities, but also their pre-identities. We show that pre-identities generate non-trivial N-commutators.

Remark 2. One can ask about N-commutators for the case $n > 1$. It seems that situation in general case is more complicated. Amitsur–Levitzki theorem states that $s_{2n}=0$ is an identity for the matrix algebra Mat_n. Let us formulate this theorem in terms of differential operators.

We consider two kinds of multiplications on differential operators. Let $W_n = D_n^{(1)}(U)$ be a space of differential operators of first order. The first kind multiplication is defined on W_n by the rule

$$a\partial_i \circ b\partial_j = b\partial_j(a)\partial_i.$$

The algebra $(W_n, \circ)$ is right-symmetric,

$$X\circ(Y\circ Z)-(X\circ Y)\circ Z = X\circ(Z\circ Y)-(X\circ Z)\circ Y, \qquad \forall X, Y, Z \in W_n.$$

It has subalgebra generated by differential operators of a form $x_i\partial_j$, where $i, j = 1, \ldots, n$. As a subalgebra of right-symmetric algebra it is certainly right-symmetric, but in this case the right-symmetric identity is not minimal. The algebra $W_{n,0} = \langle x_i\partial_j | 1 \le i, j \le n\rangle$ is not only right-symmetric, but also associative,

$$X\circ(Y\circ Z)-(X\circ Y)\circ Z = 0, \qquad \forall X, Y, Z \in L_0.$$

The algebra $W_{n,0}$ is isomorphic to the associative matrix algebra Mat_n. Amitsur–Levitzki found identity for the subalgebra $W_{n,0}$ of right-symmetric algebra W_n. So, we see that Amitsur–Levitzki theorem is in fact a result about identities of right-symmetric algebras. Generalization of identities of $W_{n,0}$ for whole right-symmetric algebra W_n was studied in [2].

The second kind multiplication is a composition of differential operators. It is an associative multiplication. But under composition W_n and $W_{n,0}$ are not close. For example, composition of operators $x_1\partial_1 \cdot x_1\partial_1 = x_1^2\partial_1^2 + x_1\partial_1$ is not an operator of first order. Appears natural problem about identities of the space $L_{n,p}$ generated by operators of a form $x^\alpha\partial^\beta$, where $\alpha, \beta \in \mathbf{Z}_0^n$, and $|\alpha| = |\beta| = p$. As a vector space $L_{n,1} = W_{n,0}$. One checks that $s_{2n}=0$ is identity for the subspace $L_{n,1}$, if $n=1,2,3$. For $n=4$ this statement is wrong. One checks that $s_9=0$ is not identity and that $s_{10}=0$ is identity for $L_{4,1}$. Moreover, s_{2n-1}, as $(2n-1)$-ary operation on differential operators are not well-defined operations for $L_{n,1}$, if $n \le 4$. It seems that it is a common situation: if $n>1$, the space $L_{n,p}$ has non-trivial N-commutator if and only if $N=2$ and $p=1$.

Proof of Theorem 1 is based on super-Lagrangians calculus. We do it in next section. The key observation here is the following fact: If X is a base element of super-Lagrangians algebra, then $\partial(X)$ is a linear combination of base elements with *non-negative* integer coefficients. This result allows us to construct a non-trivial part of $s_{2p-1}(X_1, \ldots, X_{2p-1})$ of order more than p. This result allows also to prove that $s_{2p}(X_1, \ldots, X_{2p})$ is a non-trivial differential operator of order p for some differential operators $X_1, \ldots, X_{2p}$ of order p.

1. Super-Lagrangians Algebra

Let $\mathbf{Z}_0$ be set of non-negative integers, E set of sequences with non-negative integer components, and

$$E_k = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) | 0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k,\ \alpha_i \in \mathbf{Z}_0\},$$
$$E_{k,0} = \{\alpha \in E_k | \alpha_1 = 0\},$$
$$E_k(l) = \{\alpha \in E_k || \alpha| = \sum_{i=1}^{k} \alpha_k = l\},$$
$$E_{k,0}(l) = \left\{\alpha \in E_{k,0} || \alpha| = \sum_{i=1}^{k} \alpha_k = l\right\}.$$

We endow E_k by lexicographic order, $\alpha \le \beta$ if $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}$, but $\alpha_i < \beta_i$. This order is prolonged to order on E by $\alpha < \beta$ if $\alpha \in E_k, \beta \in E_l, k < l$.

Let us consider Grassman algebra $\mathcal{U}$ generated by formal symbols $\partial^i(a)$, where $i \in \mathbf{Z}_0$. We suppose that the generator a is odd and the derivation ∂ is even. So, elements $\partial^i(a)$ are odd for any $i \in \mathbf{Z}_0$.

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in E_k$ set

$$a^\alpha = \partial^{\alpha_1}(a_1) \cdots \partial^{\alpha_k}(a_k).$$

The algebra $\mathcal{U}$ is super-commutative and associative,

$$a^\alpha a^\beta = (-1)^{kl} a^\beta a^\alpha.$$
$$a^\alpha (a^\beta a^\gamma) = (a^\alpha a^\beta) a^\gamma,$$

for any $\alpha \in E_k, \beta \in E_l, \gamma \in E_s$. In particular, $a^\alpha a^\beta = 0$, if α and β have common components. For example,

$$a^{(2,3,5)} a^{(1,3)} = 0, \quad a^{(1,2,3,5)} a^{(0,4)} = -a^{(0,1,2,3,4,5)}.$$

Let $\mathcal{L}$ be an algebra of super-differential operators on $\mathcal{U}$ under composition. Then operators of a form $a^\alpha \partial^i$, where $\alpha \in E, i \in \mathbf{Z}_0$, collect a base of $\mathcal{L}$. Composition of operators is defined as usual

$$u\partial^k \cdot v\partial^l = \sum_{i=0}^{k} \binom{k}{i} u\partial^i(v) \partial^{k+l-i},$$

where elements $u\partial^i(v) \in \mathcal{U}$ are calculated in terms of super-multiplication in super-algebra $\mathcal{U}$. For example, if $X = a^{(2,4,5)}\partial^2$ and $Y = a^{(0,1,3)}\partial^3$, then

$$\begin{aligned}
\partial(a^{(0,1,3)}) &= a^{(0,2,3)} + a^{(0,1,4)},\\
\partial^2(a^{(0,1,3)}) &= \partial(\partial(a^{(0,1,3)})) = \partial(a^{(0,2,3)} + a^{(0,1,4)})\\
&= a^{(1,2,3)} + a^{(0,2,4)} + a^{(0,2,4)} + a^{(0,1,5)} = a^{(1,2,3)} + 2a^{(0,2,4)} + a^{(0,1,5)},
\end{aligned}$$

and

$$\begin{aligned}
X\cdot Y &= a^{(2,4,5)}a^{(0,1,3)}\partial^5 + 2a^{(2,4,5)}\partial(a^{(0,1,3)})\partial^4 + a^{(2,4,5)}\partial^2(a^{(0,1,3)})\partial^3\\
&= a^{(0,1,2,3,4,5)}\partial^5 + 2a^{(2,4,5)}(a^{(0,2,3} + a^{(0,1,4)})\partial^4\\
&\quad + a^{(2,4,5)}(a^{(1,2,3)} + 2a^{(0,2,4)} + a^{(0,1,5)})\partial^3\\
&= a^{(0,1,2,3,4,5)}\partial^5,
\end{aligned}$$

since

$$a^{(2,4,5)}a^{(0,2,3)} = a^{(2,4,5)}a^{(0,1,4)} = a^{(2,4,5)}a^{(1,2,3)} = a^{(2,4,5)}a^{(0,2,4)} = a^{(2,4,5)}a^{(0,1,5)} = 0.$$

Let $X = \sum_{i=k}^{l} X_i \in \mathcal{L}$, where $X_i = (\sum_{\alpha\in E}\lambda_{\alpha,i}a^{\alpha})\partial^i$, $k \le i \le l$ and $X_k \neq 0$. Take $\beta \in E$ such that $\lambda_{\beta,k} \neq 0$ and $\lambda_{\alpha,k} = 0$ if $\alpha > \beta$. So, X has highest term $\lambda_{\beta,k}x^{\beta}\partial^k$. Call it leader of X and denote leader(X). For example,

$$X = 2a^{(0,1,5)}\partial^2 + 5a^{(0,2,3)}\partial^3 - 3a^{(0,2,4)}\partial^2 \Rightarrow \text{leader}(X) = -3a^{(0,2,4)}\partial^2.$$

Denote by U_k a linear span of base elements a^{α}, where $\alpha \in E_k$. Similarly define linear spaces $U_{k,0}$ $U_k(n)$ and $U_{k,0}(n)$ as linear span of base elements a^{α}, where correspondingly $\alpha \in E_{k,0}$, $\alpha \in E_k(n)$, and $\alpha \in E_{k,0}(n)$

Let $U_k^+ \subset U_k$ and $U_k^+(n) \subset U_k(n)$ are subsets generated by linear combinations of e^{α} with non-negative integer coefficients,

$$\begin{aligned}
U_k^+ &= \left\{\sum_{\alpha\in E_k}\lambda_{\alpha}a^{\alpha} \,|\, \lambda_{\alpha}\in\mathbf{Z}_0\right\},\\
U_k^+(n) &= \left\{\sum_{\alpha\in E_k(n)}\lambda_{\alpha}a^{\alpha} \,|\, \lambda_{\alpha}\in\mathbf{Z}_0\right\}.
\end{aligned}$$

Note that $U_k^+, U_k^+(n)$ are semigroups under addition,

$$0 \in U_k^+, \quad 0 \in U_k^+(n),$$

and

$$\begin{aligned}
u, v \in U_k^+ &\Rightarrow u + v \in U_k^+,\\
u, v \in U_k^+(n) &\Rightarrow u + v \in U_k^+(n).
\end{aligned}$$

Let

$$\begin{aligned}
L_k &= \langle a^{\alpha}\partial^i \,|\, \alpha \in E_k, i \in \mathbf{Z}_0\rangle,\\
L_k(n) &= \langle a^{\alpha}\partial^i \,|\, i + |\alpha| = n, \alpha \in E_k, i \in \mathbf{Z}_0\rangle.
\end{aligned}$$

Denote by $\mathcal{L}^{(\geq p)}$ a space of differential operators of order no less than p.

PROPOSITION 7. *For any* $p \geq 0$ *the subspace* $\mathcal{L}^{(\geq p)}$ *generates left-ideal on the algebra* $\mathcal{L}$,

$$\mathcal{L}\mathcal{L}^{(p)} \subseteq \mathcal{L}^{(p)}.$$

Algebras $\mathcal{U}$ *and* $\mathcal{L}$ *are graded,*

$$U_k(n)U_l(m) \subseteq U_{k+l}(n+m),$$
$$L_k(n)L_l(m) \subseteq L_{k+l}(n+m),$$
$$U_k(n)L_l(m) \subseteq L_{k+l}(n+m),$$
$$L_k(n)U_l(m) \subseteq L_{k+l}(n+m),$$

for any $k, l, n, m \in \mathbf{Z}_0$.

Proof. Evident.

LEMMA 8. *Let* $p \geq 0$. *If* $u \in U_k(n)$, *then* $a\partial^p(u) \in U_{k+1,0}(n+p)$. *Moreover, if* $u \in U_k^+(n)$, *then* $a\partial^p(u) \in U_{k+1,0}^+(n+p)$.

Proof. Our Lemma is an easy consequence of the following statements:

$$u \in U_k(n) \Rightarrow \partial(u) \in U_k(n+1),$$
$$u \in U_k^+(n) \Rightarrow \partial(u) \in U_k^+(n+1).$$

To prove these statements we use induction on p.

For $p=0$ our statement is trivial. Let $p=1$. If $u = a^\alpha = \partial^{\alpha_1}(a)\cdots\partial^{\alpha_k}(a)$, then by Leibniz rule $\partial(u)$ is a sum of monoms of a form

$$u_i = \partial^{\alpha_1}(a)\cdots\partial^{\alpha_{i-1}}(a)\partial^{\alpha_i+1}(a)\partial^{\alpha_{i+1}}(a)\cdots\partial^{\alpha_k}(a), \qquad 1 \leq i \leq k.$$

If $\alpha_{i+1} = \alpha_i + 1$, then by super-commutativity condition $u_i = 0$. If $\alpha_{i+1} > \alpha_i + 1$, then u_i is a base monom. Therefore, if $\alpha \in E_k(n)$, then $\partial(a^\alpha)$ is a linear combination of base monoms a^β, where $\beta \in E_k(n+1)$ with coefficients that are equal to 0 or 1. Hence

$$u \in U_k(n) \Rightarrow \partial(u) \in U_k(n+1),$$
$$u \in U_k^+(n) \Rightarrow \partial(u) \in U_k^+(n+1).$$

So, base of induction is valid.

Suppose that

$$u \in U_k(n) \Rightarrow \partial^{p-1}(u) \in U_k(n+p-1).$$

Then as we established above

$$\partial^p(u) = \partial(\partial^{p-1}(u)) \in U_k(n+p)$$

By similar reasons

$$u \in U_k^+(n) \Rightarrow \partial^{p-1}(u) \in U_k^+(n+p-1) \Rightarrow \partial^p(u) = \partial(\partial^{p-1}(u)) \in U_k^+(n+p).$$

LEMMA 9. *For any $k \in \mathbf{Z}_0$ the k-th power $(a\partial^p)^k \in \mathcal{L}$ is a linear combination with non-negative integer coefficients of operators of a form $a^\alpha \partial^i$, where $\alpha \in E_k$, $|\alpha| + i = pk$ and $i \geq p$.*

Proof. By grading property of $\mathcal{U}$ and $\mathcal{L}$ (Proposition 7) it is clear that $(a\partial^p)^k$ is a linear combination of super-differential operators of a form $a^\alpha \partial^i$, where $\alpha \in E_k(pk - i)$ and $i \geq p$. By Lemma 8 coefficients are non-negative integers.

LEMMA 10. *If $N > 2p$, then $(a\partial^p)^N = 0$ and*

$$(a\partial^p)^{2p} = \lambda_p a^{(0,1,2\ldots,2p-1)} \partial^p,$$

for some non-negative integer λ_p.

Proof. If $\alpha \in E_N$, and $N = 2p+1$, then

$$|\alpha| \geq \sum_{i=0}^{N-1} i = N(N-1)/2 = (2p+1)p = pN.$$

Therefore, by Lemma 9 $(a\partial)^{2p+1} = 0$. So, $(a\partial^p)^N = 0$, if $N > 2p$.

If $N = 2p$ and $\alpha \in E_N$ then by the same reasons,

$$|\alpha| \geq p(2p-1),$$

and

$$(a\partial^p)^N = \mathrm{leader}((a\partial^p)^N) = \lambda_p a^{(0,1,\ldots,2p-1)} \partial^p,$$

for some $\lambda_p \in \mathbf{Z}_0$.

To prove Theorem 1 we have to establish that $\lambda_p > 0$. It will be done in next section.

2. Positivity of λ_p

LEMMA 11. *Let $\delta(k)$ be maximal element in $E_{k+1,0}(pk)$. Then*

$$\delta(k) = \begin{cases} (0, p-l, p-l+1, \ldots, p-1, \ p+1, \ldots, p+l-1, p+l), & \text{if } k = 2l, \\ (0, p-l, p-l+1, \ldots, p-1, p, p+1, \ldots, p+l-1, p+l), & \text{if } k = 2l+1. \end{cases}$$

Proof. Note first of all that $\delta(k)\in E_{k+1,0}(pk)$. Indeed,

$$|\delta(k)|=\begin{cases} 2pl, & \text{if } k=2l \text{ is even,}\\ p(2l+1), & \text{if } k=2l+1 \text{ is odd}\end{cases} \Rightarrow |\delta(k)|=pk.$$

Suppose that $\beta\geq\delta(k)$ for some $\beta=(\beta_1,\ldots,\beta_{k+1})\in E_{k+1,0}(pk)$. Then

$$\beta_2\geq p-l,$$

where $l=\lfloor n/2\rfloor$.

For $1<i\leq k$ let us call $\beta_{i+1}-\beta_i$ as i-rise of β and denote $r_i(\beta)$. If $r_i(\beta)\geq 3$ for some $1<i\leq k$, then we can find $\gamma\in E_{k+1,0}(pk)$ such that $\beta<\gamma$. Take for example, $\gamma_j=\beta_j$, if $j\neq i,i+1$ and $\gamma_i=\beta_i+1, \gamma_{i+1}=\beta_{i+1}-1$. Therefore,

$$r_i(\beta)\leq 2, \qquad 1<i\leq k.$$

If $r_i(\beta)=2$ for some i then $r_j(\beta)=1$ for any $j\neq i$, $1<j\leq k$. Let us prove it by contradiction. Suppose that $r_i(\beta)=2$ and $r_j(\beta)=2$ for $i\neq j, 1<i,j\leq k$. Then there exists $\mu\in E_{k+1,0}(pk)$, such that $\beta<\mu$. Take for example, $\mu_s=\beta_s$, if $s\neq i,j$, and $\mu_i=\beta_i+1$, $\mu_{j+1}=\beta_{j+1}-1$.

Let $k=2l+1$. If $r_{s+1}(\beta)>1$, for some $0\leq s\leq k-1$. then $\beta_{2+s}>p-l+s$. Therefore, $|\beta|>\sum_{i=p-l}^{p+l} i>pk$. Hence, $r_i(\beta)=1$ for any $1<i\leq$, and, $\beta=\delta(k)$.

Let $k=2l$. If $r_{s+1}(\beta)>1$, for some $0\leq s\leq l-1$, then $\beta_{2+s}>p-l+s$. Therefore,

$$\sum_{i=1}^{s+1}\beta_i\geq\sum_{i=1}^{s+1}\delta(k)_i,$$
$$\sum_{j=s+2}^{l+1}\beta_s>\sum_{j=s+2}^{l+1}\delta(k)_j,$$
$$\sum_{t=l+2}^{2l+1}\beta_t\geq\sum_{t=l+2}^{2l+1}\delta(k)_t.$$

Hence,

$$|\beta|=\sum_{i=1}^{s+1}\beta_i+\sum_{j=s+2}^{l+1}\beta_j+\sum_{t=l+2}^{2l+1}\beta_t>\sum_{i=1}^{s+1}\delta(k)_i+\sum_{j=s+2}^{l+1}\delta(k)_j+\sum_{t=l+2}^{2l+1}\delta(k)_t$$
$$=|\delta(k)|=pk.$$

If $r_{s+1}(\beta)>1$, for some $l<s\leq k+1$, then $\beta_{2+s}>p-l+s$, and,

$$\sum_{i=1}^{s+1}\beta_i\geq\sum_{i=1}^{s+1}\delta(k)_i,$$
$$\sum_{j=s+2}^{2l+1}\beta_j>\sum_{j=s+2}^{2l+1}\delta(k)_i.$$

Therefore,

$$|\beta| = \sum_{i=1}^{s+1} \beta_i + \sum_{j=s+2}^{2l+1} \beta_j > \sum_{i=1}^{s+1} \delta(k)_i + \sum_{j=s+2}^{2l+1} \delta(k)_j = |\delta(k)| = pk.$$

Hence

$$r_{s+1}(\beta) > 1 \Rightarrow s = l,$$

and $\beta = \delta(k)$.

Recall that $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbf{Z}_0^k$ is called *composition* of n with length k if $\sum_{i=1}^k \alpha_i = n$. Denote by $C_k(n)$ set of compositions of n of length k. For $\alpha \in C_k(n)$ denote by $\mathrm{sort}(\alpha)$ the composition α written in non-decreasing order. Note that $\mathrm{sort}(\alpha)$ gives us a partition of n. For example, $\mathrm{sort}((2, 0, 2, 3, 1)) = (0, 1, 2, 2, 3)$. For $\sigma = (0, \sigma_2, \dots, \sigma_{k+1}) \in E_{k+1,0}(n)$ set $\bar{\sigma} = (\sigma_2, \dots, \sigma_k) \in E_k(n)$.

For $\alpha \in E_k, \beta \in E_{k+1}$ set

$$M(\alpha,\ \beta) = \{\gamma \in E_k | \mathrm{sort}(\alpha + \gamma) = \bar{\beta}\}.$$

For $\alpha \in \mathbf{Z}_0^k, \beta \in \mathbf{Z}_0^l$ define $\alpha \smile \beta \in \mathbf{Z}_0^{k+l}$ as a prepend α to β

$$\alpha \smile \beta = (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l).$$

Let

$$\mathbf{0}_0 = (),$$
$$\mathbf{0}_i = \underbrace{(0, 0, \dots, 0)}_{i \text{ times}}, \qquad i > 0.$$

For $\alpha \in \mathbf{Z}_0^k$ set

$$\binom{|\alpha|}{\alpha} = \prod_{i=1}^k \binom{\alpha_1 + \cdots + \alpha_k}{\alpha_1, \dots, \alpha_k} = \frac{(\alpha_1 + \cdots + \alpha_k)!}{\alpha_1! \cdots \alpha_k!}.$$

Let

$$G_0 = \{()\},$$
$$G_k = \{(i) \smile \mathbf{0}_{i-1} \smile \alpha | \alpha \in G_{k-i}, \quad i = 1, 2, \dots, k\}, \qquad k > 0.$$

EXAMPLE.

$$G_1 = \{(1)\}, \quad G_2 = \{(2, 0), (1, 1)\}, \quad G_3 = \{(3, 0, 0), (2, 0, 1), (1, 2, 0), (1, 1, 1)\}.$$

LEMMA 12. *If* $k = 2l - 1$ *is odd,*

$$M(\delta(k-1), \delta(k)) = \{(p - l + i) \smile \mathbf{0}_{i-1} \smile \alpha \smile \mathbf{0}_{l-1} | \alpha \in G_{l-i}, i = 1, 2, \dots, l\}.$$

If $k = 2l$ *is even,*

$$M(\delta(k-1), \delta(k)) = \{(p - l) \smile \mathbf{0}_{l-1} \smile \alpha | \alpha \in G_l\}.$$

Proof. Evident.

EXAMPLE. If $p=5$, then

$$M(\delta(2),\delta(3))=M((0,4,6),(0,4,5,6))=\{(4,1,0),(5,0,0)\},$$
$$M(\delta(3),\delta(4))=M((0,4,5,6),(0,3,4,6,7))=\{(3,0,1,1),(3,0,2,0)\}.$$

LEMMA 13.

$$\sum_{\alpha\in G_k}\mathrm{sign}\,(\alpha+(0,1,\dots,k-1))\binom{k}{\alpha}=1$$

Proof. Induction on k. For $k=1$ our statement is evident. Suppose that it is true for $k-1$. Note that

$$G_k=\cup_{i=1}^{k}\{(i)\smile\mathbf{0}_{i-1}\smile G_{k-i}\}.$$

For $\alpha\in G_{k-i}$,

$$(i)\smile 0_{i-1}\smile\alpha+(0,1,\dots,k-1)=(i,1,2,\dots,i-1,\alpha_1+i,\dots,\alpha_{k-i}+k-1),$$

and,

$$\mathrm{sign}\,((i)\smile 0_{i-1}\smile\alpha+(0,1,\dots,k-1))=(-1)^{i-1}\mathrm{sign}\,(\alpha+(0,1,\dots,k-i-1)).$$

Further, for $\alpha\in G_{k-i}$,

$$\binom{k}{(i)\smile 0_{i-1}\smile\alpha}=\binom{k}{(i)\smile\alpha}=\binom{k}{i}\binom{k-i}{\alpha}.$$

Therefore,

$$\sum_{\alpha\in G_k}\mathrm{sign}\,(\alpha+(0,1,\dots,k-1))\binom{k}{\alpha}=$$
$$\sum_{i=1}^{k}\sum_{\alpha\in G_{k-i}}(-1)^{i-1}\mathrm{sign}\,(\alpha+(0,1,\dots,k-i-1))\binom{k}{I}\binom{k-i}{\alpha}=$$
$$\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\sum_{\alpha\in G_{k-i}}\mathrm{sign}\,(\alpha+(0,1,\dots,k-i-1))\binom{k-i}{\alpha}=$$

(by inductive suggestion)

$$\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}=1.$$

LEMMA 14.

$$\sum_{i=0}^{l-1}(-1)^i\binom{p}{i}=(-1)^{l-1}\binom{p-1}{l-1}.$$

Proof. Induction on l. If $l=1$, then our statement is evident. Suppose that it is true for $l-1\geq 1$. Then

$$\begin{aligned}\sum_{i=0}^{l-1}(-1)^i\binom{p}{i}&=\sum_{i=0}^{l-2}(-1)^i\binom{p}{i}+(-1)^{l-1}\binom{p}{l-1}\\&=(-1)^{l-1}\binom{p-1}{l-2}+(-1)^{l-1}\binom{p}{l-1}\\&=(-1)^{l-1}\left(\binom{p}{l-1}-\binom{p-1}{l-2}\right)=(-1)^{l-1}\binom{p-1}{l-1}.\end{aligned}$$

LEMMA 15. *If* $k=2l-1$, *then*

$$\sum_{i=1}^{l}\sum_{\alpha\in G_{l-i}}\operatorname{sign}((p-l+i)\smile\mathbf{0}_{i-1}\smile\alpha\smile\mathbf{0}_{l-1}$$
$$+(0,1,\dots,l-1,l,\dots,2l-2))\binom{p}{(p-l+i)\smile\alpha}=\binom{p-1}{l-1}.$$

If $k=2l$, *then*

$$\sum_{\alpha\in G_l}\operatorname{sign}(\alpha+(0,1,\dots,l-1))\binom{p}{(p-l)\smile\alpha}=\binom{p}{l}.$$

Proof. Let $k=2l-1$. For $\alpha\in G_{l-i}$ let $\Gamma(\alpha)\in\mathbf{Z}_0^{2l-1}$ be defined as

$$\Gamma(\alpha)=(p-l+i)\smile\mathbf{0}_{i-1}\smile\alpha\smile\mathbf{0}_{l-1}+(0,p-l+1,\dots,p-1,p+1,\dots,p+l-1).$$

Note that

$$\begin{aligned}\Gamma(\alpha)=(&p-l+i,p-l+1,\dots,p-l+i-1,\alpha_1+p-l+i,\dots,\alpha_{l-i}\\&+p-1,p+1,\dots,p+l-1).\end{aligned}\tag{1}$$

By (1)

$$\begin{aligned}\operatorname{sort}(\Gamma(\alpha))=(&p-l+1,\dots,p-l+i-1,p-l+i)\\&\smile\operatorname{sort}(\alpha_1+p-l+i,\dots,\alpha_{l-i}+p-1,p+1,\dots,p+l-1).\end{aligned}$$

Hence,

$$\operatorname{sort}(\Gamma(\alpha))=\overline{\delta(k)},\qquad \alpha\in G_{l-i}$$
$$\Updownarrow$$
$$\begin{aligned}&\operatorname{sort}(\alpha_1+p-l+i,\dots,\alpha_{l-i}+p-1,p+1,\dots,p+l-1)\\&\quad=(p-l+i+1,\dots,p-1,p,p+1,\dots,p+l-1).\end{aligned}$$

Therefore, the condition $\mathrm{sort}(\Gamma(\alpha)) = \overline{\delta(k)}$ is equivalent to the condition

$$\mathrm{sort}(\alpha_1 + p - l + i, \dots, \alpha_{l-i} + p - 1) = (p - l + i + 1, \dots, p - 1, p). \tag{2}$$

By (1)

$$\begin{aligned}\mathrm{sign}\,\Gamma(\alpha) = (-1)^{i-1}\mathrm{sign}\,(&p - l + 1, \dots, p - l + i, \alpha_1 + p - l + i, \dots, \alpha_{l-i}\\ &+ p - 1, p + 1, \dots, p + l - 1).\end{aligned}$$

Therefore, by (2)

$$\begin{aligned}\mathrm{sign}\,\Gamma(\alpha) &= (-1)^{i-1}\mathrm{sign}\,(\alpha_1 + p - l + i, \dots, \alpha_{l-i} + p - 1)\\ &= (-1)^{i-1}\mathrm{sign}\,(\alpha_1, \alpha_2 + 1, \dots, \alpha_{l-i} + l - i - 1).\end{aligned} \tag{3}$$

Hence,

$$\sum_{i=1}^{l} \sum_{\alpha \in G_{l-i}} \mathrm{sign}\,\Gamma(\alpha) \binom{p}{(p - l + i) \smile \alpha} =$$

[by (3)]

$$\sum_{i=1}^{l} \sum_{\alpha \in G_{l-i}} (-1)^{i-1}\mathrm{sign}\,(\alpha + (0, 1, \dots, l - i - 1)) \binom{p}{l-i} \binom{l-i}{\alpha} =$$
$$\sum_{i=1}^{l} (-1)^{i-1} \binom{p}{l-i} \sum_{\alpha \in G_{l-i}} \mathrm{sign}\,(\alpha + (0, 1, \dots, l - i - 1)) \binom{l-i}{\alpha} =$$

(by Lemma 13)

$$\sum_{i=1}^{l} (-1)^{i-1} \binom{p}{l-i} =$$
$$\sum_{j=0}^{l-1} (-1)^{l-j-1} \binom{p}{j} =$$

(by Lemma 14)

$$\binom{p-1}{l-1}.$$

So, our Lemma in case of odd k is proved.

Let $k=2l$. Then

$$\sum_{\alpha\in G_l} \operatorname{sign}(\alpha+(0,1,\ldots,l-1))\binom{p}{(p-l)\smile\alpha}=$$
$$\sum_{\alpha\in G_l} \operatorname{sign}((\alpha_1,\alpha_1+1,\ldots,\alpha_l+l-1)\binom{p}{l}\binom{l}{\alpha}=$$
$$\binom{p}{l}\sum_{\alpha\in G_l} \operatorname{sign}((\alpha_1,\alpha_1+1,\ldots,\alpha_l+l-1)\binom{l}{\alpha}=$$

(by Lemma 12)

$$\binom{p}{l}.$$

Our Lemma is proved completely.

LEMMA 16. *Let μ_k be coefficient at $a^{\delta(k-1)}$ of the element $a\partial^p(a^{\delta(k-2)})$, if $k>1$, and $\mu_1=1$. If $1\le k\le 2p$, then*

$$\mu_k=\begin{cases}\binom{p}{l}, & \text{if } \ k=2l+1 \text{ is odd,}\\[2ex] \binom{p-1}{l-1}, & \text{if } \ k=2l \text{ is even.}\end{cases}$$

Proof. Follows from Lemmas 12 and 15.

EXAMPLE. If $p=5$, then

k	$\delta(k-1)$	μ_k
1	(0)	1
2	$(0,5)$	1
3	$(0,4,6)$	5
4	$(0,4,5,6)$	4
5	$(0,3,4,6,7)$	10
6	$(0,3,4,5,6,7)$	6
7	$(0,2,3,4,6,7,8)$	10
8	$(0,2,3,4,5,6,7,8)$	4
9	$(0,1,2,3,4,6,7,8,9)$	5
10	$(0,1,2,3,4,5,6,7,8,9)$	1

The following two lemmas can be proved in a similar way as Lemmas 12 and 15.

LEMMA 17. *Let $\delta_1(k)$ be maximal element in $E_{k+1,0}(pk-1)$. Then*

$$\delta_1(k) = \begin{cases} (0,\, p-l,\, p-l+2, \ldots, p+l-1), & \textit{if } k=2l, \\ (0,\, p-l,\, p-l+1, \ldots, p,\ p+2, \ldots, p+l), & \textit{if } k=2l+1. \end{cases}$$

LEMMA 18. *Let* γ_k *be coefficient at* $a^{\delta_1(k-1)}$ *of* $a\partial^{p-1}(a^{\delta(k-2)})$. *Then*

$$\gamma_k = p\binom{p-1}{\lfloor (k-2)/2 \rfloor}$$

if $2 \le k \le 2p-1$.

LEMMA 19. *Let* ν_k *be coefficient at* $a^{\delta(k-1)}$ *of the element* $(a\partial^p)^{k-1}(a)$. *Then*

$$\text{leader}((a\partial^p)^k) = \nu_k a^{\delta(k-1)} \partial^p.$$

Proof. Follows from Lemma 11.

LEMMA 20. *For any* $0 \le k \le 2p$,

$$\nu_k \ge \mu_k \nu_{k-1}.$$

(Definition of μ_k *see Lemma 16, and definition of* ν_k *see Lemma 19).*

Proof. By Lemmas 8 coefficient at $a^{\delta(k-1)}$ of the element $(a\partial^p)^{k-1}(a)$ is a non-negative integer that is no less than another non-negative integer $(a\partial^p)^{k-1}(\nu_{k-1}a^{\delta(k-2)})$. By Lemma 16 the last number is equal to $\nu_{k-1}\mu_k$.

EXAMPLE. Let $p=3$. Then

$$\mu_1 = 1,\ \mu_2 = 1,\ \mu_3 = 3,\ \mu_4 = 2,\ \mu_5 = 3,\ \mu_6 = 1$$

and

$$\begin{aligned}
&(a\partial^3)^2 = 3a^{(0,1)}\partial^5 + 3a^{(0,2)}\partial^4 + a^{(0,3)}\partial^3,\\
&\text{leader}((a\partial^3)^2) = a^{(0,3)}\partial^3, \quad \nu_2 = 1,\\
&(a\partial^3)^3 = 18a^{(0,1,2)}\partial^6 + 27a^{(0,1,3)}\partial^5 + 15a^{(0,1,4)}\partial^4 + 3a^{(0,1,5)}\partial^3\\
&\qquad\qquad +9a^{(0,2,3)}\partial^4 + 3a^{(0,2,4)}\partial^3,\\
&\text{leader}((a\partial^3)^3) = 3a^{(0,2,4)}\partial^3, \quad \nu_3 = 3,\\
&(a\partial^3)^4 = 126a^{(0,1,2,3)}\partial^6 + 189a^{(0,1,2,4)}\partial^5 + 99a^{(0,1,2,5)}\partial^4 + 18a^{(0,1,2,6)}\partial^3\\
&\qquad\qquad +75a^{(0,1,3,4)}\partial^4 + 24a^{(0,1,3,5)}\partial^3 + 6a^{(0,2,3,4)}\partial^3,\\
&\text{leader}((a\partial^3)^4) = 6a^{(0,2,3,4)}\partial^3, \quad \nu_4 = 6,\\
&(a\partial^3)^5 = 432a^{(0,1,2,3,4)}\partial^5 + 432a^{(0,1,2,3,5)}\partial^4 + 108a^{(0,1,2,3,6)}\partial^3 + 90a^{(0,1,2,4,5)}\partial^3,\\
&\text{leader}((a\partial^3)^5) = 90a^{(0,1,2,4,5)}\partial^3, \quad \nu_5 = 90,\\
&(a\partial^3)^6 = 90a^{(0,1,2,3,4,5)}\partial^3.\\
&\text{leader}((a\partial^3)^6) = (a\partial^3)^6 = a^{(0,3)}\partial^3, \quad \nu_6 = 90.
\end{aligned}$$

LEMMA 21. *For any* $X_1, \ldots, X_N \in A_1^{(p)}$,

$$s_N(X_1, \ldots, X_N) = 0,$$

if $N > 2p$ *and*

$$s_{2p}(\partial^p, x\partial^p, x^2/2\,\partial^p, \ldots, x^{2p-1}/(2p-1)!\,\partial^p) = \lambda_p \partial^p.$$

Proof. Suppose that $X_i = u_i \partial^p$, where $u_i \in K[x]$. Let us make specialization of a in super-algebra $\mathcal{U}$. Take $a = (\sum_{i=1}^N u_i \xi_i)\partial^p$, where ξ_i are odd super-generators. Then

$$(a\partial^p)^N = s_N(u_1\partial^p, \ldots, u_N\partial^p)\xi_1 \cdots \xi_N.$$

By Lemma 10 $(a\partial^p)^N = 0$, if $N > 2p$. Therefore, $s_N = 0$ is identity if $N > 2p$.

Now consider the case $N = 2p$. Set $a = \sum_{i=0}^{2p-1} x^i/i!\xi_{i+1}$ where ξ_i are odd elements and ∂ acts on x^i as usual, $\partial(x^i) = ix^{i-1}$. Then

$$(a\partial^p)^{2p} = s_{2p}(\partial^p, x\partial^p, x^2/2\partial^p, \ldots, x^{2p-1}/(2p-1)!\partial^p)\xi_1\xi_2 \cdots \xi_{2p}$$

Further,

$$\begin{aligned} a^{(0,1,2\ldots,2p-1)} &= \partial^0(a)\partial^1(a) \cdots \partial^{2p-1}(a) \\ &= \left(\sum_{i=0}^{2p-1} x^i/i!\xi_{i+1}\right)\left(\sum_{i=0}^{2p-1} x^{i-1}/(i-1)!\xi_{i+1}\right) \cdots (\xi_{2p-1} + x\xi_{2p})\xi_{2p} \\ &= \xi_1\xi_2 \cdots \xi_{2p}. \end{aligned}$$

Therefore, by Lemma 10

$$\begin{aligned} &s_{2p}(\partial^p, x\partial^p, x^2/2\,\partial^p, \ldots, x^{2p-1}/(2p-1)!\partial^p)\xi_1\xi_2 \cdots \xi_{2p} = (a\partial^p)^{2p} \\ &\quad = \lambda_p \xi_1\xi_2 \cdots \xi_{2p}\partial^p. \end{aligned}$$

Hence

$$s_{2p}(\partial^p, x\partial^p, x^2/2\,\partial^p, \ldots, x^{2p-1}/(2p-1)!\,\partial^p) = \lambda_p \partial^p.$$

3. Equivalence of Left-Commutative and Right-Commutative Identities

LEMMA 22. $(2n-2, 1)$-*type and* $(1, 2n-2)$-*type identities are equivalent.*

Proof. We have to prove that any n-algebra (A, ψ) with $(2n-2, 1)$-type identity

$$lcom = 0,$$

where

$$lcom(t_1,\dots,t_{2n-1}) = \sum_{\sigma\in S^{(2n-2,1)}} \operatorname{sign}\sigma\,\psi(t_{\sigma(1)},\dots,t_{\sigma(n-1)},\psi(t_{\sigma(n)},\dots,t_{\sigma(2n-2)},t_{\sigma(2n-1)})),$$

satisfies the identity

$$rcom=0,$$

where

$$rcom(t_1,\dots,t_{2n-1}) = \sum_{\sigma\in S^{(1,2n-2)}} \operatorname{sign}\sigma\;\psi(t_1,t_{\sigma(2)},\dots,t_{\sigma(n-1)},\psi(t_{\sigma(n)},\dots,t_{\sigma(2n-1)})),$$

and vice versa, any n-ary algebra with identity $rcom=0$ satisfies also the identity $lcom=0$.

Let us prove that

$$n\,rcom(t_1,\dots,t_{2n-1}) = rcom_1(t_1,\dots,t_{2n-1}), \tag{4}$$

$$(n-1)\,lcom(t_1,\dots,t_{2n-1}) = lcom_1(t_1,\dots,t_{2n-1}), \tag{5}$$

where

$$\begin{aligned} rcom_1(t_1,\dots,t_{2n-1}) &= \sum_{i=2}^{2n-1}(-1)^{i+1}\,lcom(t_1,\dots,\hat{t_i},\dots,t_{2n-1},t_i)\\ &\quad -(n-1)\,lcom(t_2,\dots,t_{2n-1},t_1),\\ lcom_1(t_1,\dots,t_{2n-1}) &= \sum_{i=1}^{2n-2}(-1)^{i+1}\,rcom(t_i,t_1,\dots,\hat{t_i},\dots,t_{2n-1})\\ &\quad -(n-2)\,rcom(t_{2n-1},t_1,\dots,t_{2n-2}). \end{aligned}$$

Note that $rcom(t_1,\dots,t_{2n-1})$ and $rcom_1(t_1,\dots,t_{2n-1})$ are skew-symmetric under $2n-2$ variables $t_2,\dots,t_{2n-1}$. Therefore, it is enough to prove that coefficients at $\psi(t_1,\dots,t_{n-1},\psi(t_n,\dots,t_{2n-2},t_{2n-1}))$ and $\psi(t_2,\dots,t_n,\psi(t_1,t_{n+1},\dots,t_{2n-1}))$ of $rcom(t_1,\dots,t_{2n-1})$ and $rcom_1(t_1,\dots,t_{2n-1})$ are equal.

It is easy to see that, if $n\le i\le 2n-1$, then the coefficient at $\psi(t_1,\dots,t_{n-1},\psi(t_n,\dots,t_{2n-1}))$ of

$$(-1)^{i+1}\,lcom(t_1,\dots,\hat{t_i},\dots,t_{2n-1},t_i)$$

is equal to 1. If $1\le i<n$, then this coefficient is 0. Therefore, the coefficient at $\psi(t_1,\dots,t_{n-1},\psi(t_n,\dots,t_{2n-1}))$ of $rcom_1(t_1,\dots,t_{2n-1})$ is equal to n.

Further, if $n\le i\le 2n-1$, then the coefficient at $\psi(t_2,\dots,t_n,\psi(t_1,t_{n+1},\dots,t_{2n-1}))$ of

$$(-1)^{i+1}\,lcom(t_1,\dots,\hat{t_i},\dots,t_{2n-1},t_i)$$

is equal to 0. If $1 \le i < n$, then this coefficient is 1. Therefore, the coefficient at $\psi(t_2, \ldots, t_n, \psi(t_1, t_{n+1}, \ldots, t_{2n-1}))$ of $rcom_1(t_1, \ldots, t_{2n-1})$ is equal to 0.

Hence, relation (4) is proved completely.

By similar arguments one establishes (5).

Relations (4) and (5) show that identities $rcom$ and $lcom$ are equivalent.

4. Proof of Theorem 1

By Lemma 21 $s_N = 0$ is identity on $A_1^{(p)}$ if $N > 2p$. By Lemma 20

$$\lambda_p = \nu_{2p} \ge \mu_{2p} \cdots \mu_2 \nu_1 > 0.$$

Therefore, by Lemma 21 $s_{2p} = 0$ is not polynomial identity and s_{2p} induces on $A_1^{(p)}$ a non-trivial $2p$-commutator.

By Lemma 20 for any $1 \le k \le 2p-2$

$$\nu_k \ge \mu_k \cdots \mu_2 \nu_1 > 0.$$

Therefore, by Lemmas 8, 17 and 18 the differential $(p+1)$-th order parts of $(a\partial^p)^k$ are non-zero for any $2 \le k \le 2p-1$. Therefore, s_k is not well-defined on $A_1^{(p)}$.

Suppose that $A_1^{(p)}$ has identity of degree no more than $2p$. Then it has skew-symmetric multi-linear consequence. In particular, it has a skew-symmetric polynomial identity of degree $2p$. But $s_{2p} = 0$, as we mentioned above, is not identity. Contradiction.

Suppose that I is a non-trivial ideal of $A_1^{(p)}$ under $2p$-commutator s_{2p}. Take $0 \ne X = u\partial^p \in I$ with minimal degree $s = \deg u$. Let us prove that $s = 0$ and $X = \eta\partial^p \in I$ for some $0 \ne \eta \in K$. Suppose that it is not true, and $s > 0$. If $s \ge 2p-1$, then by Lemma 21

$$s_{2p}(\partial^p, x\partial^p, \ldots, x^{2p-2}\partial^p, X) = \lambda_p \binom{s}{2p-1} \prod_{i=0}^{2p-1} i!\, x^{s-2p+1}\partial^p \in I,$$

or,

$$x^{s-2p+1}\partial^p \in I.$$

We obtain contradiction with minimality of s. If $0 < s < 2p-1$, then

$$s_{2p}(\partial^p, x\partial^p, \ldots, x^{s-1}\partial^p, X, x^{s+1}\partial^p, \ldots, x^{2p-1}\partial^p) = \lambda_p \prod_{i=0}^{2p-1} i!\, \partial^p \in I,$$

or,

$$\partial^p \in I.$$

Once again we obtain contradiction with minimality of s.

So, we establish that $X=\eta\partial^p\in I$, for some $0\neq\eta\in K$. Then for any $l\geq 0$,

$$s_{2p}(X, x\partial, \ldots, x^{2p-2}\partial^p, x^{l+2p-1}\partial^p)=\eta\lambda_p\binom{l+2p-1}{2p-1}\prod_{i=0}^{2p-1} i!\,x^l\partial^p\in I.$$

In other words, $x^l\partial^p\in I$ for any $l\geq 0$. This means that $I=A_1^{(p)}$. So, $(A_1^{(p)}, s_{2p})$ is simple $2p$-algebra.

By Theorem 1.1 (ii) of [4] the algebra $(A_n(p), s_{2p})$ is left-commutative. Presentation of $2p$-commutator as a Vronskian up to scalar λ_p follows from Lemma 21.

5. Expressions for λ_p

In this section, we give some formulas for λ_p. For $s>0$ let us define a polynomial

$$f_s(x_1,\ldots,x_{2p-1}) = \frac{\sum_{\sigma\in Sym_{2p}} \operatorname{sign}\sigma\,\left(x_{\sigma(1)}(x_{\sigma(1)}+x_{\sigma(2)})\cdots(x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(2p-1)})\right)^s}{\prod_{1\leq i<j\leq 2p}(x_i-x_j).}$$

Then $f_s(x_1,\ldots,x_{2p-1})$ is a symmetric polynomial of degree $(2p-1)(s-p)$. In particular, $f_p(x_1,\ldots,x_{2p-1})=\lambda_p$ is constant. The number λ_p appears also in calculating $2p$-commutator,

$$s_{2p}(u_1\partial^p,\cdots,u_{2p}\partial^p)=\lambda_p\begin{vmatrix} u_1 & u_2 & \cdots & u_{2p}\\ \partial(u_1) & \partial(u_2) & \cdots & \partial(u_{2p})\\ \vdots & \vdots & \cdots & \vdots\\ \partial^{2p-1}(u_1) & \partial^{2p-1}(u_2) & \cdots & \partial^{2p-1}(u_{2p})\end{vmatrix}\partial^p.$$

Then

$$\lambda_p=\frac{\sum_{\sigma\in Sym_{2p}} \operatorname{sign}\sigma\,(\sigma(1)(\sigma(1)+\sigma(2))\cdots(\sigma(1)+\sigma(2)+\cdots+\sigma(2p-1)))^p}{\prod_{1\leq i<j\leq 2p}(i-j).}$$

For example,

$$\lambda_1=1, \lambda_2=2, \lambda_3=90, \lambda_4=586, 656, \lambda_5=1, 915, 103, 977, 500.$$
$$\lambda_6=7, 886, 133, 184, 567, 796, 056, 800.$$

Another way to calculate λ_p. Let $\mathcal{M}_p$ be a set of matrices $M=(m_{i,j})$ of order $(2p-1)\times(2p-1)$ such that

- $m_{i,j}\in\mathbf{Z}_0$
- $m_{i,j}=0$ if $i>j$
- Sums by rows are constant, $\sum_{j=1}^{2p-1} m_{i,j}=p$ for any i
- Sums by columns $r_j=\sum_{i=1}^{2p-1} m_{i,j}$, are positive and different for all $j=1,2,\ldots,2p-1$.

In particular,

$$M=(m_{i,j})\in\mathcal{M}_p\Rightarrow m_{1,1}=r_1>0 \quad \text{and} \quad m_{2p-1,2p-1}=p.$$

For $M\in\mathcal{M}_p$ denote by $r(M)$ the permutation $r_1\ldots r_{2p-1}$ constructed by column sums.

EXAMPLE. $p=2$. Then

$$\mathcal{M}_2=\left\{A=\begin{pmatrix}1&1&0\\0&1&1\\0&0&2\end{pmatrix},B=\begin{pmatrix}1&0&1\\0&2&0\\0&0&2\end{pmatrix},C=\begin{pmatrix}1&1&0\\0&2&0\\0&0&2\end{pmatrix},D=\begin{pmatrix}2&0&0\\0&1&1\\0&0&2\end{pmatrix}\right\}.$$
$$r(A)=123,\ r(B)=123,\ r(C)=132,\ r(D)=213.$$

If $M\in\mathcal{M}_p$, then a sequence $r_1\ldots r_{2p-1}$ induces a permutation, where $r_i=\sum_j m_{i,j}$ are sums by columns. In particular, $1\le r_i\le 2p-1$ for any $1\le i\le 2p-1$. Then

$$\lambda_p=\sum_{M\in\mathcal{M}_p}\operatorname{sign} r(M)\prod_{i=1}^{2p-1}\binom{p}{m_{i,1},\ldots,m_{i,2p-1}},$$
$$\lambda_p=\frac{p!^{2p-1}}{\prod_{j=1}^{2p-1} j!}\sum_{M\in\mathcal{M}_p}\operatorname{sign} r(M)\prod_j\binom{r_j}{m_{1,j},\ldots,m_{j,j}}.$$

Here

$$\binom{n}{n_1,\ldots n_k}=\frac{n!}{n_1!\cdots n_k!}$$

is a multinomial coefficient.

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