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COHOMOLOGIES OF COLOUR LEIBNIZ ALGEBRAS: PRE-SIMPLICIAL APPROACH

ASKAR DZHUMADIL'DAEV

ABSTRACT. Cohomologies for colour Leibniz algebras are defined. Pre-simplicial structure on the cochain complex of colour Leibniz algebras is constructed. In particular, pre-simplicial structures for cohomologies of Leibniz algebra, Lie and Lie super algebras are found.

1. INTRODUCTION

A cochain complex $C^* = \oplus_k C^k$ has a pre-simplicial structure if it has a sequence of endomorphisms $d_i : C^* \to C^{*+1}$, $i \in \mathbf{Z}$, $i \geq 0$ such that $d_j d_i = d_i d_{j-1}$ for any $i < j$. Pre-simplicial endomorphisms are usually used in the construction of a coboundary operator, i.e. in the construction of an endomorphism $d : C^* \to C^{*+1}$ such that $d^2 = 0$. It can be done as an alternating sum $d = \sum_i (-1)^i d_i$. The existence of pre-simplicial structure for cohomologies of associative algebras is well known. We construct pre-simplicial structure for colour Leibniz algebras. In particular, we construct pre-simplicial structure for cohomologies of Leibniz algebras, Lie algebras and Lie super algebras. Corresponding cohomology groups coincide with known cohomology groups for Lie algebras [1], Liebniz algebras [2], [3], Lie super algebras and colour algebras [4], [5].

Actually we construct on the cochain complex $T^*(L, M)$ of colour Leibniz algebras two kinds of operations

$$d : C^k(L, K) \to C^{k+1}(L, M),$$

coboundary operator, and

$$\theta : C^k(L, K) \to C^{k+1}(L, M),$$

such that

$$d\theta^k = \theta^k d,$$
$$d\theta^{2k-1} + \theta^{2k-1} d = \theta^{2k},$$

for any k.

2. Colour Leibniz algebras

Let P be a field and G be a commutative group with bicharacter

$$\chi : G\times G\to P: \quad \begin{array}{l}\chi(f,gh)=\chi(f,g)\chi(f,h),\\ \chi(fg,h)=\chi(f,h)\chi(g,h),\\ \chi(f,g)\chi(g,f)=1.\end{array}$$

Let L be a G-graded algebra with multiplication $[\ ,\ ]$:

$$L=\oplus_{g\in G}L_g,\quad L_gL_h\subseteq L_{gh}.$$

We will write, $|x|=f$, if $x\in L$ is homogeneous, and $x\in L_f$. The G-graded algebra L is colour commutative, if

$$[x,y]=\chi(|x|,|y|)[y,x]$$

and colour skew-commutative, if

$$[x,y]=-\chi(|x|,|y|)[y,x]$$

for any homogeneous $x,y\in L$. A colour skew-commutative algebra L is called colour Lie algebra, if the following identity (Jacobi identity) holds

$$[[x,y],z]=[x,[y,z]]+\chi(|y|,|z|)[[x,z],y].$$

Following J.L. Loday we say that L is colour (left) Leibniz algebra if the following identity (Leibniz identity) is true:

$$[[x,y],z]=[x,[y,z]]-\chi(|x|,|y|)[y,[x,z]].$$

In particular, any colour Lie algebra is a colour Leibniz algebra. If G is additive group $Z/2Z$ and $\chi(f,g)=(-1)^{fg}$, then colour Lie algebras are known as Lie super algebras. We call a Leibniz colour algebra for $G=Z/2Z$ a Leibniz super algebra. When G is a trivial group a colour Leibniz algebra is a Leibniz algebra and a colour Lie algebra is a usual Lie algebra.

3. Modules of colour Leibniz algebras

Let L be (left) Leibniz algebra. We call a G-graded space $M=\oplus_{f\in G}M_j$ a module over L if there are given bilinear maps $L\times M\to M$, $(x,m)\mapsto [x,m]$ and $M\times L\to M$, $(m,x)\mapsto [m,x]$ such that

$$L_fM_g\subseteq M_{fg},\quad M_gL_f\subseteq M_{gf},$$

$$[[x,y],m]=[x,[y,m]]-\chi(|x|,|y|)[y,[x,m]],\quad (LLM)$$

$$[[x,m],y]=[x,[m,y]]-\chi(|x|,|m|)[m,[x,y]],\quad (LML)$$

$$[[m,x],y]=[m,[x,y]]-\chi(|m|,|x|)[x,[m,y]],\quad (MLL)$$

for any $x, y \in L$, $m \in M$, $g \in G$. Here the notation $|m| = f$ means that $m \in M$ is homogeneous, and $m \in M_j$. Notice that

$$|[x, y]| = |x||y|,\ |[x, m]| = |x||m|,\ |[m, x]| = |m||x|.$$

For a L-module M

$$M^{ann} := \{a(x, m) := [x, m] + \chi(|x|, |m|)[m, x] | x \in L, m \in M\}$$

has a module structure over L:

$$[x, a(y, m)] = a([x, y], m) + \chi(x, y)a(y, [xm]),$$

$$[a(y, m), x] = 0,$$

for any $x, y \in L$, $m \in M$. In particular, the natural adjoint on L endows it the structure of a module and

$$L^{ann} = \{a(x, y) = [x, y] + \chi(x, y)[y, x] | x, y \in L\}$$

lies in a right center of L.

Let us have a L-module M. Denote by $T^k(L, M)$, $k > 0$, a space of polylinear maps $\psi : L \times \ldots \times L \to M$ with k-arguments, $T^0(L, M) = M$, $T^k(L, M) = 0$, $k < 0$ and $T^*(L, M) = \oplus_k T^k(L, M)$. For $\psi \in T^k(L, M)$ set $|\psi| = f$, if

$$\psi(L_{f_1}, \ldots, L_{f_k}) \subseteq M_{f f_1 \ldots f_k}.$$

Define a G-gradiation on $T^*(L, M)$ by the rule

$$T^*(L, M) = \oplus_{f \in G} T^*(L, M)_f,$$

$$T^*(L, M)_f = \{\psi \in T^*(L, M) | |\psi| = f\}.$$

4. Pre-simplicial structure for colour Leibniz cohomology

Let L be a colour Leibniz algebra and M be a L-module. For a given $\psi \in T^k(L, M)$, $x_1, \ldots, x_{k+2} \in L$ set

$$\chi_i = \chi(|\psi||x_1| \ldots |x_{i-1}|, |x_i|),$$

$$\chi_j(\hat{i}) = \chi(|\psi||x_1| \ldots |\hat{x_i}| \ldots |x_{j-1}|, |x_j|), i < j,$$

$$\chi_{i,j} = \chi(|x_i|, |x_{i+1}| \ldots |x_{j-1}|), i < j,$$

$$\chi_{i,s}(\hat{j}) = \chi(|x_i|, |x_{i+1}| \ldots |\hat{x_j}| \ldots |x_s|), i < j < s.$$

Consider the operators

$$\theta_i, \eta_i : T^*(L,M) \to T^{*+1}(L,M)$$
$$\theta_{j,i} : T^*(L,M) \to T^{*+2}(L,M), i < j,$$

defined on $\psi \in T^k(L,M)$ by the following formulas

$$(\theta_i\psi)(x_1,\dots,x_{k+1}) = \chi_i[x_i, \psi(x_1,\dots,\hat{x_i},\dots,x_{k+1})], i < k+1,$$
$$(\theta_{k+1}\psi)(x_1,\dots,x_{k+1}) = -[\psi(x_1,\dots,x_k), x_{k+1}],$$
$$(\theta_i\psi) = 0, i > k+1,$$

$$\eta_i\psi(x_1,\dots,x_{k+1}) = \sum_{i<j} \chi_{i,j}\psi(x_1,\dots,\hat{x_i},\dots,[x_i,x_j],\dots,x_{k+1}),$$
$$\eta_i\psi = 0, i > k,$$

$$\theta_{j,i}\psi(x_1,\dots,x_{k+2}) = \chi_i\chi_j[[x_j,x_i], \psi(x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{k+2})],$$
$$i < j < k+2,$$

$$\theta_{k+2,i}\psi(x_1,\dots,x_{k+2}) = -\chi(|x_i|,|x_{i+1}|\dots|x_{k+2}|)[\psi(x_1,\dots,\hat{x_i},\dots,x_{k+1}),[x_{k+2},x_i]].$$

Let

$$d_i = \theta_i - \eta_i,\ i = 0,1,2,\dots$$

Theorem 4.1. *For any $i < j$ the following relations are true:*

$$d_jd_i = d_id_{j-1}.$$

Corollary 4.2. *The operator $d : T^*(L,M) \to T^{*+1}(L,M)$ given by the rule*

$$d = \sum_i (-1)^i d_i$$

is a coboundary operator: $d^2 = 0$.

Proof of Theorem 4.1:

$$d_jd_i = (\theta_j-\eta_j)(\theta_i-\eta_i) = \theta_j\theta_i-\eta_j\theta_i-\theta_j\eta_i+\eta_j\eta_i = \text{(lemma 4.3, see below)}$$
$$= \theta_i\theta_{j-1}+\theta_{j,i}-\theta_i\eta_{j-1}-\eta_i\theta_{j-1}-\theta_{j,i}+\eta_i\eta_{j-1} = (\theta_i-\eta_i)(\theta_{j-1}-\eta_{j-1}) = d_id_{j-1}. \quad \square$$

Lemma 4.3. *For $i < j$ we have the following relations:*

(i) $\theta_j\theta_i = \theta_i\theta_{j-1} + \theta_{j,i}$,
(ii) $\theta_j\eta_i = \eta_i\theta_{j-1} + \theta_{j,i}$,
(iii) $\eta_j\eta_i = \eta_i\eta_{j-1}$,
(iv) $\eta_j\theta_i = \theta_i\eta_{j-1}$.

Proof: Notice that χ_i depends on ψ, but

$$|\theta_i\psi| = |\theta_{j-1}\psi| = |\psi|,$$

that is why

$$\chi_i = \chi_i(\psi) = \chi_i(\theta_s\psi),$$

for any $s = 1, \dots, k+2$.

(i) Let $\psi \in T^k(L, M)$. If $j < k+2$, then

$$(\theta_j\theta_i\psi)(x_1, \dots, x_{k+2}) = \chi_j[x_j, (\theta_i\psi)(x_1, \dots, \hat{x_j}, \dots, x_{k+2})] =$$
$$\chi_i\chi_j[x_j, [x_i, \psi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}', \dots, x_{k+2})]],$$

and

$$(\theta_i\theta_{j-1}\psi)(x_1, \dots, x_{k+2}) = \chi_i[x_i, (\theta_{j-1}\psi)(x_1, \dots, \hat{x_i}, \dots, x_{k+2})] =$$
$$\chi_i\chi_j\chi(|x_j|, |x_i|)[x_i, [x_j, \psi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{k+2})]]$$

Thus

$$(\theta_j\theta_i\psi - \theta_i\theta_{j-1}\psi)(x_1, \dots, x_{k+2}) = \chi_j\chi_i[[x_j, x_i], \psi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{k+2})] =$$
$$(\theta_{j,i}\psi)(x_1, \dots, x_{k+2}).$$

If $j = k+2$, $i < k+1$, then

$$(\theta_j\theta_i\psi)(x_1, \dots, x_{k+2}) = -[\theta_i\psi(x_1, \dots, x_{k+1}), x_{k+2}] =$$
$$-\chi_i[[x_i, \psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1})], x_{k+2}] =$$

(due to (LML))

$$-\chi_i[x_i, [\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1}), x_{k+2}]] +$$
$$\chi(|x_i|, |\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1})|)\chi_i[\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1}), [x_i, x_{k+2}]] =$$
$$-\chi_i[x_i, [\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1}), x_{k+2}]] +$$
$$\chi(|x_i|, |x_{i+1}| \dots |x_{k+1}|)[\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1}), [x_i, x_{k+2}]].$$

On the other hand, for $j = k+2$, $i < k+1$,

$$(\theta_i\theta_{j-1}\psi)(x_1, \dots, x_{k+2}) = \chi_i[x_i, (\theta_{k+1}\psi)(x_1, \dots, \hat{x_i}, \dots, x_{k+2})] =$$
$$-\chi_i[x_i, [\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1}), x_{k+2}]],$$

thus

$$(\theta_j\theta_i\psi)(x_1, \dots, x_{k+2}) = (\theta_i\theta_{j-1}\psi)(x_1, \dots, x_{k+2}) +$$
$$\chi_{i,k+1}[\psi(x_1, \dots, \hat{x_i}, \dots, x_{k+1}), [x_i, x_{k+2}]]$$

If $i = k+1, j = k+2$, then

$$(\theta_j\theta_i\psi)(x_1, \dots, x_{k+2}) = [[\psi(x_1, \dots, x_k), x_{k+1}], x_{k+2}] =$$

(due to (MLL))

$$[\psi(x_1, \dots, x_k), [x_{k+1}, x_{k+2}]] - \chi_{k+1}[x_{k+1}, [\psi(x_1, \dots, x_k), x_{k+2}]],$$

and

$$(\theta_i\theta_{j-1}\psi)(x_1,\dots,x_{k+2}) = \chi_i[x_i,(\theta_{j-1}\psi)(x_1,\dots,\hat{x_{k+1}},x_{k+2})] =$$
$$-\chi_{k+1}[x_{k+1},[\psi(x_1,\dots,x_k),x_{k+2}]].$$

Thus, (i) is proved completely.

(ii) If $j<k+2$, then

$$(\theta_j\eta_i\psi)(x_1,\dots,x_{k+2}) = \chi_j[x_j,(\eta_i\psi)(x_1,\dots,\hat{x_j},\dots,x_{k+2})] =$$
$$\chi_j\sum_{i<i'<j}\chi_{i,i'}[x_j,\psi(\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots,\hat{x_j},\dots)]+$$
$$\chi_j\sum_{i<j<i'}\chi_{i,i'}(\hat{j})[x_j,\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{i'-1},[x_i,x_{i'}],\dots)],$$

and

$$(\eta_i\theta_{j-1}\psi)(x_1,\dots,x_{k+2}) = \sum_{i<i'}\chi_{i,i'}(\theta_{j-1}\psi)(\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots)] =$$
$$\sum_{i<i'<j}\chi_{i,i'}\chi_j[x_j,\psi(\dots,\hat{x_i},\dots,[x_i,x_{i'}],\dots,\hat{x_j},\dots)]+$$
$$\sum_{i<j<i'}\chi_{i,i'}\chi_j(\hat{i})[x_j,\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{i'-1},[x_i,x_{i'}],\dots)]+$$
$$a_{i,j}[[x_i,x_j],\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots)],$$

where

$$a_{i,j} = \chi_{i,j}\chi(|\psi||x_1|\dots||\hat{x_i}|\dots x_{j-1},|[x_i,x_j]|).$$

Since

$$\chi_j\chi_{i,i'}(\hat{j}) = \chi_{i,i'}\chi_j(\hat{i}),$$
$$a_{i,j}[x_i,x_j] = -\chi_i\chi_j[x_j,x_i],$$

we have

$$((\theta_j\eta_i-\eta_i\theta_{j-1}-\theta_{j,i})\psi)(x_1,\dots,x_{k+2}) = 0,\ j<k+2.$$

If $j=k+2$, then

$$(\theta_j\eta_i\psi)(x_1,\dots,x_{k+2}) = [(\eta_i\psi)(x_1,\dots,x_{k+1}),x_{k+2}]$$

and

$$(\eta_i\theta_{j-1}\psi)(x_1,\dots,x_{k+2}) =$$
$$\sum_{i<i'}\chi_{i,i'}(\theta_{j-1}\psi)([x_1,\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots,x_{k+2}) =$$
$$-\sum_{i<i'<k+2}\chi_{i,i'}[\psi([x_1,\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots,x_{k+1}),x_{k+2}]-$$
$$\chi_{i,k+2}[\psi(x_1,\dots,\hat{x_i},\dots,x_{k+1}),[x_i,x_{k+2}]] =$$
$$-[(\eta_i\psi)(x_1,\dots,x_{k+1}),x_{k+2}]+$$
$$\chi(|x_i|,|x_{i+1}|\dots|x_{k+2}|)[\psi(x_1,\dots,\hat{x_i},\dots,x_{k+1}),[x_{k+2},x_i]] =$$

$$(\theta_j\eta_i\psi - \theta_{j,i}\psi)(x_1,\dots,x_{k+2}).$$

Thus, relation (ii) is proved.

(iii) We have

$$(\eta_j\eta_i\psi)(x_1,\dots,x_{k+2}) = \sum_{j<j'}\chi_{j,j'}(\eta_i\psi)(\dots,\hat{x_j},\dots,x_{j'-1},[x_j,x_{j'}],\dots) =$$

$$A_1+A_2+A_3+A_4,$$

where

$$A_1 = \sum_{i<i',j<j'}\sum_{i'<j}\chi_{j,j'}\chi_{i,i'}\psi(\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots,\hat{x_j},\dots,x_{j'-1},[x_j,x_{j'}],\dots),$$

$$A_2 = \sum_{i<i',j<j'}\sum_{j<i'<j'}\chi_{j,j'}\chi_{i,i'}(\hat{j})\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{i'-1},[x_i,x_{i'}],\dots,x_{j'-1},[x_j,x_{j'}],\dots),$$

$$A_3 = \sum_{i<i',j<j'=i'}\chi_{j,j'}\chi_{i,i'}(\hat{j})\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{j'-1},[x_i,[x_j,x_{j'}]],\dots),$$

$$A_4 = \sum_{i<i',j<j'}\sum_{j'<i'}\chi_{j,j'}\chi_{i,i'}\psi(\dots,\hat{x_i},\dots,\hat{x_{j'}},\dots,x_{j'-1},[x_j,x_{j'}],\dots,x_{i'-1},[x_i,x_{i'}],\dots)$$

and

$$(\eta_i\eta_{j-1}\psi)(x_1,\dots,x_{k+2}) = \sum_{i<i'}\chi_{i,i'}(\eta_{j-1}\psi)(\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots) =$$

$$B_1+B_2+B_3+B_4+B_5,$$

where

$$B_1 = \sum_{i<i',j<j'}\sum_{i'<j}\chi_{i,i'}\chi_{j,j'}\psi(\dots,\hat{x_i},\dots,x_{i'-1},[x_i,x_{i'}],\dots,\hat{x_j},\dots,x_{j'-1},[x_j,x_{j'}],\dots),$$

$$B_2 = \sum_{i<i',j<j'}\sum_{j<i'<j'}\chi_{i,i'}\chi_{j,j'}\chi(|x_j|,|x_i|)\times$$

$$\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{i'-1},[x_i,x_{i'}],\dots,x_{j'-1},[x_j,x_{j'}],\dots),$$

$$B_3 = \sum_{i<i',j<j',i'=j'}\chi_{i,j'}\chi_{j,j'}\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{i'-1},[x_j,[x_i,x_{i'}]],\dots),$$

$$B_4 = \sum_{i<i'=j<j'}\chi_{i,j}\chi_{j,j'}\chi(|x_i|,|x_{j+1}|\dots|x_{j'-1}|)\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{j'-1},[[x_i,x_j],x_{j'}],\dots)$$

$$B_5 = \sum_{i<i',j<j'}\sum_{j'<i'}\chi_{i,i'}\chi_{j,j'}\psi(\dots,\hat{x_i},\dots,\hat{x_{j'}},\dots,x_{j'-1},[x_j,x_{j'}],\dots,x_{i'-1},[x_i,x_{i'}],\dots).$$

It is easy to see that

$$A_1 = B_1,\ A_4 = B_5.$$

Since

$$\chi_{j,j'}\chi_{i,i'}(\hat{j}) = \chi_{i,i'}\chi(|x_j|,|x_i|)\chi_{j,j'},$$

we have the following equality

$$A_2 = B_2.$$

Since

$$\chi_{i,j}\chi_{j,j'}\chi(|x_i|,|x_{j+1}|\dots|x_{j'-1}|)=\chi_{j,j'}\chi_{i,j'}(\hat{j}),$$
$$\chi_{i,j}\chi_{j,j'}\chi(|x_i|,|x_{j+1}|\dots|x_{j'-1}|)\chi(|x_i|,|x_j|)=\chi_{i,j'}\chi_{j,j'},$$

due to be Leibniz identity

$$A_3=B_3+B_4.$$

Thus, relation (iii) is proved.

(iv) If $j\geq k+2$, then (iv) is trivial:

$$\eta_j\theta_i\psi=0=\theta_i\eta_{j-1}\psi.$$

Assume that $j<k+2$. Then

$$(\eta_j\theta_i\psi)(x_1,\dots,x_{k+2})=\sum_{j<j'}\chi_{j,j'}(\theta_i\psi)(x_1,\dots,\hat{x_j},\dots,x_{j'-1},[x_j,x_{j'}],\dots,x_{k+2})=$$
$$\sum_{j<j'}\chi_{j,j'}\chi_i[x_i,\psi(x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{j'-1},[x_j,x_{j'}],\dots)=$$
$$\chi_i[x_i,\sum_{j<j'}\chi_{j,j'}\psi(\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_{j'-1},[x_j,x_{j'}],\dots)=$$
$$\chi_i[x_i(\eta_{j-1}\psi)(\dots,\hat{x_i}),\dots]=(\theta_i\eta_{j-1}\psi)(x_1,\dots,x_{k+2}).$$

Thus, relation (iv) is proved. □

Corollary 4.4. *Let*

$$\theta=\sum_i(-1)^i\theta_i,\ \eta=\sum_i(-1)^i\eta_i,\ d=\sum_i(-1)^i d_i.$$

Then

$$d^2=0, \tag{1}$$

$$\eta^2=0, \tag{2}$$

and for any $k=1,2,\dots$ *we have the following formulas*

$$d\theta^{2k}=\theta^{2k}d, \tag{3}$$

$$d\theta^{2k-1}+\theta^{2k-1}d=\theta^{2k}. \tag{4}$$

Proof: Since $\eta_i,\ i=0,1,2,\dots$ and $d_i,\ i=0,1,2,\dots$ are precosimplicial systems, $\eta^2=0$ and $d^2=0$. We have

$$d=\theta-\eta,$$

thus

$$0=d^2=\theta^2-\theta\eta-\eta\theta+\eta^2,$$
$$\theta\eta+\eta\theta=\theta^2,$$

and

$$d\theta+\theta d=(\theta-\eta)\theta+\theta(\theta-\eta)=\theta^2.$$

Thus, relation (4) for $k = 1$ is proved. From this follows that

$$\theta^2 d = (\theta d + d\theta)d = d(\theta d) = d(\theta d + d\theta) = d\theta^2.$$

Thus, (3) for $k = 1$ is also proved. From the inductive assumption for $k-1$ follows that

$$d\theta^{2k} = d\theta^2\theta^{2(k-1)} = \theta^2 d\theta^{2(k-1)} = \theta^{2k}d.$$

$$d\theta^{2k-1} + \theta^{2k-1}d = \theta^2(d\theta^{2k-3} + \theta^{2k-3}d) = \theta^{2k}.$$

5. Pre-simplicial structure for Leibniz and Lie cohomologies

The notation of cohomology of a Lie algebra was defined by Chevalley-Eilenberg [1]. Let L be a Lie algebra and M is L-module. A cochain complex $C^*(L,M) = \oplus_k C^k(L,M)$ consists of polylinear skew-symmetric maps $\psi \in C^k(L,M)$, $\psi : L \times \cdots \times L \to M$, if $k > 0$, and $C^0(L,M) = M$, $C^k(L,M) = 0$, $k < 0$. The Chevalley-Eilenberg coboundary operator

$$C^*(L,M) \to C^{*+1}(L,M)$$

is defined on $\psi \in C^k(L,M)$ by the formula

$$d\psi(x_1,\dots,x_{k+1}) = \sum_{i<j} \psi([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{k+1}) +$$
$$\sum_i (-1)^{i+1}[x_i,\psi(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]$$

(the notation $\hat{x}_i$ means that x_i is omitted).

J.-L. Loday [2] noticed that the main property of the coboundary operator $d^2 = 0$ follows from the Leibniz identity (skew-symmetry condition is not necessary) if the formula for d is rewritten in the following way

$$d\psi(x_1,\dots,x_{k+1}) = \sum_{i<j} \psi(x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,[x_i,x_j],x_{j+1},\dots,x_{k+1}) +$$

$$\sum_{i=1}^{k} (-1)^{i+1}[x_i,\psi(x_1,\dots,\hat{x}_i,\dots,x_{k+1})] + (-1)^{k+1}[\psi(x_1,\dots,x_k),x_{k+1}].$$

A vector space M is called a module over a Leibniz algebra L if there are defined left action $M \times L \to M$, $(m,x) \mapsto [m,x]$ and right action $L \times M \to M$, $(x,m) \mapsto [x,m]$, such that

$$[[m,x],y] = [m,[x,y]] - [x,[m,y]], \quad (MLL)$$

$$[[x,m],y] = [x,[m,y]] - [m,[x,y]], \quad (LML)$$

$$[[x,y],m] = [x,[y,m]] - [y,[x,m]], \quad (LLM)$$

The L-module M is called symmetric if

$$[x,m] + [m,x] = 0, \, \forall x \in L, \, \forall m \in M.$$

In particular, if L is a Lie algebra, any L-module will be symmetric module in the category of Leibniz algebras. Let L be a Leibniz algebra, M be a L-module and $T^*(L,M) = \oplus_k T^k(L,M)$ be cochain complex with Loday coboundary operator d, where $T^k(L,M) = Hom(L^{\otimes k}, M)$, $k > 0$, and $T^0(L,M) = M$, $T^k(L,M) = 0$, $k < 0$. Denote by $HL^*(L,M) = \oplus_k HL^k(L,M)$ its cohomology groups, where $HL^k(L,M) = ZL^k(L,M)/BL^k(L,M)$ (k-cohomology groups, more exactly,spaces), $ZL^k(L,M) = \{\psi \in T^k(L,M) : d\psi = 0\}$ (k-space of cycles) and $BL^k(L,M) = \{d\eta : \eta \in T^{k-1}(L,M)\}$ (k-space of coboundaries).

We notice that a Loday cochain complex $(T^*(L,M), d)$ has a structure of pre-simplicial complex. In our knowledge this observation is new even for Lie algebras. For $i = 1, 2, \ldots$ we define operators

$$d_i : T^*(L,M) \to T^{*+1}(L,M)$$

on $\psi \in T^k(L,M)$ by the rules

$$d_i\psi(x_1, \ldots, x_{k+1}) = [x_i, \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1})] -$$
$$\sum_{i<j} \psi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, [x_i, x_j], x_{j+1}, \ldots, x_{k+1}), \, 1 \le i \le k,$$
$$d_{k+1}\psi(x_1, \ldots, x_{k+1}) = [\psi(x_1, \ldots, x_k), x_{k+1}],$$
$$d_i\psi(x_1, \ldots, x_{k+1}) = 0, \, k+1 < i.$$

Theorem 5.1. *If $i < j$, then $d_j d_i = d_i d_{j-1}$. In particular, $d := \sum_i (-1)^i d_i$ is a coboundary operator of $T^*(L,M) : d^2 = 0$.*

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Institute of Mathematics, Academy of Sciences of Kazakhstan, Pushkin street 125, Almaty, Kazakhstan