Modular Lie algebras: new trends

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Abstract. The article deals with next three problems. 1. What are ways for the origin of parametres in the structure tensor of finite-dimensional simple modular Lie algebra? A both "naive" approach, and a standard way through cohomology are discussed in this connection. 2. A determination of general deformations of modular Lie algebras suggest quite different problems and specific ways for their decision. 3. Presentation of maximal nilpotent subalgebras in Lie algebras of the Cartan type is compared with J.-P.Serre presentation in the classical case. A special attention is turned into Lie algebras over fields of small characteristic. Four tables summerise a lot of calculations.

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1. Introduction. In our talk we shall discuss three topics: tensor products of Lie structures, deformations and defining relations. Common source of these topics is cohomology.

Our results are illustrated by the following observation. All cocycles (cycles) can be of the following three kind.

- Very large characteristic cocycles, i.e., cocycles that will remain cocycles for $p \to \infty$; they can be considered as cocycles of characteristic 0; examples of these kind of cocycles are in the section 3.
- Large charactersictic cocycles, i.e. cocycles that are defined for characterictic p beginning with some N, say $p > 11$, but they iwill not be cocycles for $p \to \infty$; examples of these kind are derivations of form D^p (1-cocycles), and SqD (2-cocycles).
- Small characterisitic cocycles; they have no analogy for large characteristics; local deformation cocycle of nonrestricted Hamiltonian algebra in characteristic $p = 5$ gives us the example of small characteristic cocycles (the corresponding deformation is isomorphic to Melikyan algebra); another example can be get from the list of defining relations additional to Serre relations (Table 1) and from the list of 2-cocycles for Zassenhaus algebra, $p = 2$ (Table 2).

It is impossible to give exact border between characteristics. It depends from the problem under consideration. It depends also from the concrete situation. Moreover, very large characteristic case can be included into the case of large characteristic. Very large characteristic case can be understood as a case of characteristic 0.

Example 1. Serre defining relations for nilpotent subalgebra $\mathcal{L}_+$ of classical Lie algebra are generated by positive root vectors. The border between large and cmall characteristics is 5. There is no difference between large and very large characteristics respectively for the problem on defining relations of $\mathcal{L}_+$. Exact meaning of these statements is the following one: if $p > 3$ then defining relations of $\mathcal{L}_+$ are just the same as in characteristic 0, i.e. $\mathcal{L}_+$ is generated by Serre relations $(ad\, e_i)^{-1-a_{j,i}} e_j = 0$; additional relations are appear in the cases of small characteristics $p = 2, 3$. The list of additional relations is given in the Table 1.

Example 2. Classification problem of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic p: according Kostrikin-Shafarevich conjecture the border between small and large characteristics pass through 7; according Kac conjecture this border is 5. The modification of Kostrikin-Shafarevich conjecture says that there is no border between small characteristics and large characteristics cases. In right understanding of the notion of deformation almost all new simple algebras appeared at the last time in small charactersitics will be deformations of Cartan Type Lie algebras. Note that dimensions of simple Lie algebras in small characteristics, except Brown algebra, are not new. We think that this fact is not occasional. We suggest that deformations should be considered in the sense of Gerstenhaber. More general approach, outlined in [17, 23], especially for Lie algebras over fields of small characteristic, will be out of discussion.

At first, we should like to explain, on an intuitive level, the origin of parametric structure tensor. Parametric families of simple Lie algebras over algebraically closed field are due to characteristic $p > 0$. Nevertheless, as we shall see soon, it is not so easy to construct a family by direct approach.

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2. Tensor product of Lie structures. Basic definitions. Let A, B be two commutative associative algebras over an algebraically closed field k with the common unit 1. Let $[*,*]_A, [*,*]_B$ be Lie algebra structures on A, B (by the Poisson bracket or in an other way). Let us define Lie algebra structure $(A \otimes B,\ [*,*])$ on their tensor product in the following way:

$$[a \otimes b, a' \otimes b'] = \lambda a a' \otimes [b, b']_B + \mu [a, a']_A \otimes b b'. \qquad (1)$$

Here is $\lambda, \mu \in k,\ \lambda\mu \neq 0$; λ and μ do not depend on elements of algebras. Is it possible? When it is the case we shall speak on *the tensor product of Lie structures*:

$$\mathcal{L} = \mathcal{L}(\lambda, \mu)$$

Our aim is to give some useful examples both of positive and negative nature.

Anticommutativity of the product (1), linearly extendable on the all elements of $A \otimes B$, is evident. The Jacoby identity may be the obstacle to the extendability of Lie structures on their tensor product. As $\lambda\mu \neq 0$, we have

$$[[f \otimes u, g \otimes v], h \otimes w] + [[h \otimes w, f \otimes u], g \otimes v] + [[g \otimes v, h \otimes w], f \otimes u] = 0 \quad (2)$$

$(f, g, h \in A,\ u, v, w \in B)$ precisely when

$$\begin{aligned} &[fg, h] \otimes [u, v]w + [f, g]h \otimes [uv, w] \\ &+[hf, g] \otimes [w, u]v + [h, f]g \otimes [wu, v] \\ &+[gh, f] \otimes [v, w]u + [g, h]f \otimes [vw, u] = 0 \end{aligned} \quad (3)$$

(the left part of the expression (3) enteres with coefficient $\lambda\mu$ in the left part of (2); coefficients λ^2, μ^2 will be under null members, as a consequence of the Jacoby identity in algebras A, B).

The natural question arises: when the identity (3) is true and what properties of input structures are inherited? If we have no restrictions on A, B, the problem will be too general. In our situation the case $\operatorname{char} k > 0,\ \dim_k A < \infty,\ \dim_k B < \infty$ is the most suitable one. More precisely, we pay attention to Lie algebras of Cartan type, using notation and the main definitions from the survey paper [16]. So, A, B (in associative sense) will be algebras of divided powers.

3. Tensor product of Lie structures. Some calculations and results.

Example 3. $W_1(m) = \langle x^{(i)} \,|\, m \rangle_k$. Divided powers $x^{(i)},\ 0 \leq i \leq p^m - 1$, $p = \operatorname{char} k$, are multiplied as usual: $x^{(i)}x^{(j)} = \binom{i+j}{j} x^{(i+j)}$. A derivative f' of the element $f \in W_1(m)$ is by definition $(x^{(i)})' = x^{(i-1)}$. A commutation $[f, g] = fg' - f'g$ set on $W_1(m)$ the structure of simple Lie algebra, which is known as the Zassenhaus algebra (if $p = 2$, the ideal $\langle x^{(i)} \,|\, i \leq 2^m - 2 \rangle$ is simple).

A direct verification shows that $W_1(m) \otimes W_1(n)$ is a Lie algebra; the calculations connected with (3) are lengthy but straightforward. It is rather interesting to look at the structure of it. For the simplicity we restrict ourself to the case of the Witt algebra $W_1 = W_1(1)$, when we can take

$$A := W_1 = \langle 1, x, \ldots, x^{p-1} \,|\, x^p = 0 \rangle,\ (x^k)' = kx^{k-1}.$$

Let $B := \langle y^i \,|\, 0 \leq i \leq p - 1 \rangle$ be another copy of W_1. Also, we put $\mu = 1$, so that

$$L := W_1 \otimes W_1 = \langle x^i \otimes y^k \,|\, 0 \leq i, k \leq p - 1 \rangle,$$

$$[x^i \otimes y^k, x^j \otimes y^k] = (j - i)x^{i+j-1} \otimes y^{k+l} + \lambda(l - k)x^{i+j} \otimes y^{k+l-1}. \quad (4)$$

Having in mind (4), we consider elements

$$T_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda^k x^k \otimes y^{n-k}.$$

Specifically,

$$T_0 = 1 \otimes 1,\ T_1 = 1 \otimes y - \lambda x \otimes 1.$$

It is not so hard to see that

$$L = \bigotimes_{n=0}^{p-1} L_{-n},\ L_{-n} = \langle T_n, xT_n, \ldots, x^{p-1}T_n \rangle,$$

What is more important, for any $f, g \in A$ we have:

$$[fT_m, gT_n] = \sum_{k,l} (-1)^{k+l} \binom{m}{k} \binom{n}{l} \lambda^{k+l} [fx^k \otimes y^{m-k}, gx^l \otimes y^{n-l}]$$

$$= \cdots \text{(very tedious calculations)} = [f, g] \cdot T_{m+n},$$

so that the preset parameter $\lambda \in k$ disappeared. In fact,

$$L_0 = W_1,\ [L_0, L_n] = L_n,\ [L_m, L_n] = L_{m+n},$$

and L is a direct sum of adjoint W_1-modules. Furthermore, $\oplus_{n\geq 1} L_n$ is a nilpotent radical of L.

Example 4. If f and g are functions of $2s$ variables $x_1, x_{s+1}, \ldots, x_s, x_{2s}$, then the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{s} \left(\partial_i f \partial_{s+i} g - \partial_{s+i} f \partial_i g \right),$$

introduced at first in analytical mechanics, equip the fuction space with the Hamiltonian Lie algebra structure. It is connected with the nondegenerate differential form

$$\omega = \sum_{i=1}^{s} dx_i \wedge dx_{s+1}.$$

The geometric quantization on a corresponding symplectic manifold leads to a formal deformation of the Poisson bracket. After all this circumstance finds a reflection in the modular case.

Let us consider the simplest case of two variables:

$$A = H_1(m, n; \omega) = \langle x^{(i)} y^{(j)} \mid 0 \leq i < p^m, 0 \leq j < p^n \rangle,$$

$$B = H_1(q, r; \omega) = \langle z^{(k)} t^{(l)} \mid 0 \leq k < p^q, 0 \leq l < p^r \rangle.$$

Here $\omega = dx \wedge dy$ is for A, and $\omega = dz \wedge dt$ is for B, so that

$$[a, a']_A = \partial_x a \partial_y a' - \partial_y a \partial_x a', \quad [b, b']_B = \partial_z b \partial_t b' - \partial_t b \partial_z b' \tag{5}$$

(generally speaking, we should take in (5) opposite signs, but it is not so essential). Thus, we have:

$$[x^{(i)}y^{(j)} \otimes z^{(k)}t^{(l)}, x^{(i')}y^{(j')} \otimes z^{(k')}t^{(l')}] \tag{6}$$

$$= \lambda\binom{i+i'}{i}\binom{j+j'}{j}\left\{\binom{k+k'-1}{k'}\binom{l+l'-1}{l} - \binom{k+k'-1}{k}\binom{l+l'-1}{l'}\right\}\cdot$$

$$\cdot x^{(i+i')}y^{(j+j')} \otimes z^{(k+k'-1)}t^{(l+l'-1)}$$

$$+\mu\binom{k+k'}{k}l+l'l\left\{\binom{i+i'-1}{i'}\binom{j+j'-1}{j} - \binom{i+i'-1}{i}\binom{j+j'-1}{j'}\right\}\cdot$$

$$\cdot x^{(i+i'-1)}y^{(j+j'-1)} \otimes z^{(k+k')}t^{(l+l')}.$$

Put $\bar{x} = x^{(p^m-1)}, \bar{y} = y^{(p^n-1)}, \ldots$. It is not so hard to see that $\bar{x}\bar{y} \otimes \bar{z}\bar{t} \notin [\mathcal{L}(\lambda, \mu), \mathcal{L}(\lambda, \mu)]$. Indeed, if we take, for example $p > 2,\ i + i' = p^m - 1, j + j' = p^n, k + k' = p^q, l + l' = p^r$, then $k + k' + l + l' \equiv 0 \pmod 2$, and (6) results in $[x^{(i)}y^{(j)} \otimes z^{(k)}t^{(l)}, x^{(i')}y^{(j')} \otimes z^{(k')}t^{(l')}] = \lambda(-1)^{i+j}\left\{(-1)^{k'+l} - (-1)^{k+l'}\right\}\bar{x}\bar{y} \otimes \bar{z}\bar{t} = 0$.

By putting

$$e(i, j, k, l) := \frac{1}{\sqrt{\mu^{i+j}\lambda^{k+l}}}x^{(i)}y^{(j)} \otimes z^{(k)}t^{(l)}$$

we get basis with multiplication, free of any parametres. If we shall throw out $\bar{x}\bar{y} \otimes \bar{z}\bar{t}$ and factorize upon the ideal, generated by $1 \otimes 1$, then we came to the Hamiltonian simple Lie algebra

$$\mathcal{L}(\lambda, \mu) \cong \mathcal{L}(1, 1) \cong H_2,$$

corresponding to the differential form $\omega(x, y, z, t) = dx \wedge dy + dz \wedge dt$.

It is well known that for $s = 1$ along with $\omega_0 := \omega = dx \wedge dy$ there exist more forms:

$$\omega_1 = (\exp\, y) \cdot dx \wedge dy;\ \omega_2 = (1 - \bar{x}\bar{y}) \cdot dx \wedge dy,$$

which are result in filtered (nongraded!) deformations $(H_1, \omega_1),\ (H_1, \omega_2)$ of (H_1, ω_2). By extending our direct checking we arrive to the conclusion that the tensor products

$$(H_1, \omega_0) \otimes (H_1, \omega_0),\ (H_1, \omega_0) \otimes (H_1, \omega_2),\ (H_1, \omega_2) \otimes (H_1, \omega_2)$$

give simple Lie algebras, while

$$(H_1, \omega_0) \otimes (H_1, \omega_1),\ (H_1, \omega_1) \otimes (H_1, \omega_2),\ (H_1, \omega_1) \otimes (H_1, \omega_1)$$

have not Lie algebra structure. Alike, tensor products

$$H_1 \otimes W_1,\ H_1 \otimes K_1,\ K_1 \otimes K_1$$

(where K_1 is the contact Lie algebra, corresponding to the form $\omega = dz + ydx - xdy$) do not lead to a Lie structure.

As a whole, the construction under consideration gives rather interesting examples of Lie algebras. In principle, a tensor product $H_s \otimes H_t$ with $s > 1$ would give a parametric family.

4. Deformations. The following conjecture was proved by R.Block, H.Strade, R.L. Wilson [1, 24, 25].

Conjecture 1. *Any finite-dimensional Lie algebra of simple Lie algebra over an algebraically closed field of characteristic $p \geq 7$ is isomorphic to either*

- *classical Lie algebra;*
- *Lie algebra of Cartan Type;*
- *deformations;*

Here we would like to be more precise what deformations we mean. There are at least three kinds of deformations applicable respectively to simple Lie algebras. It is well known that Cartan type Lie algebras are defined by differential forms. By deformations one can understand deformation of corresponding differential forms. Deformations of differential forms are described in [15, 26, 21, 28]. We know that all Cartan Type Lie algebras have long filtration. One can understand deformations in the sence of filtered deformations. In [8] such deformations are called $\{L_i\}$-deformations. Problem in this case means that for given graded algebra with homogeneous subspaces L_i one should find filtered algebra

$$\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_r$$

such that $\mathrm{gr}\mathcal{L} \cong L$. Filtered deformations are described in [15]. The last kind of deformations — Gerstenhaber deformations [13] — are applicable to any algebra A. Special properties of algebras like existence of filtration or differential forms are not necessary. Gerstenhaber deformations are ruled by two cohomology groups, $H^2(A, A)$ and $H^3(A, A)$. More exactly, if A is a Lie algebra with vector space V over a field F and commutator $V \times V \to V, \quad (a, b) \mapsto [a, b]$ then Gerstenhaber deformation of A is defined on the vector space $V_t := V \otimes F((t))$ by multiplication

$$[a, b]_t = [a, b] + tf_1(a, b) + t^2 f_2(a, b) + \cdots ,$$

where

$$f_1, f_2, \ldots \in C^2(V, V).$$

Jacobi identities on the new commutator $[\ ,\ ]_t$ are reduced to a serie of conditions

$$\sum_{i=1}^{k} f_i \star f_{k-i} = -d\, f_k, \qquad k = 1, 2, \ldots,$$

where

$$f \star g(a,b,c) = f(g(a,b),c) + f(g(b,c),a) + f(g(c,a),b),$$

$$d\, h(a,b,c) = h([a,b],c) + h([b,c],a) + h([c,a],b)$$

$$+[h(a,b),c] + [h(b,c),a] + [h(c,a),b],$$

for $f, g, h \in C^2(V,V)$. The first deformation term f_1 is called *local deformation.* From deformation equations it follows that any local deformation should be 2-cocycle of the adjoint module:

$$f_1 \in Z^2(A,A).$$

If we get any cocycle ψ instead of f_1 one can ask about constructing of prolongations $f_2, f_3, \ldots,$ that satisfy deformation equations. In other words in each step k one should check whether the following 3-cocycle

$$\sum_{i=1}^{k} f_i \star f_{k-i} \in Z^3(A,A)$$

will be a coboundary

$$\sum_{i=1}^{k} f_i \star f_{k-i} = -d\, f_k \in B^3(A,A).$$

If it is the case for any k then we say that the local deformation $f_1 = \psi$ has prolongation to global deformation. In this sense, studying of Gerstenhaber deformations is reduced to the question on calculating second cohomology group $H^2(A,A)$ and third cohomology group $H^3(A,A)$. Experience shows that all filtered deformations and deformations of differential forms of Cartan Type Lie algebras can be realized as a Gerstenhaber defomations. We are not sure whether in general all Gerstenhaber deforations should be filtered deformations for some filtration.

Our modification of Kostrikin-Shafarevich conjecture is the following one. *Deformations should be understand as a Gerstenhaber deformations.* If this is the case, the number of new simple Lie algebras that appear in small characteristics will be restricted essentially. This modified conjecture is supported not only by the observation that dimensions of new simple Lie algebras in $p = 2, 3$ on the whole are known ones, i.e. dimensions of four series of Cartan Type Lie algebras. In some cases one can construct exact realisation of some new simple Lie algebras as a Gerstenhaber deformations. Let us give realisation of Melikyan algebra (recall that all simple Lie algebras in characteristic 5 except this algebra satisfy

Kostrikin-Shafarevich conjecture) as a deformation of Hamiltonian algebra. Another interpretation of Melikyan algebra is in the paper [18].

5. Deformations. Melikyan algebras as a deformation of Hamiltonian algebras. Let $L = H_2(n_x, n_y)$ be the Hamiltonian algebra over a field F of characteristic $p = 5$ of dimension $p^{n_x+n_y}$, i.e. as a vector space L coinsides with a space of divided power algebra $O_2(n_x, n_y)$ and multiplication is given by

$$[u, v] = \partial_x(u)\partial_y(v) - \partial_y(u)\partial_x(v).$$

(we changed here a sign on the opposite one). Consider cochains $\psi_1, \psi_2 \in Z^2(L, L)$, such that

$$\psi_1 = \partial_x^2 \wedge \partial_x^3, \qquad \psi_2 = \Delta_1 \wedge \partial_x^5,$$

where

$$\Delta_1(u) = u - x\partial_x(u).$$

Then for

$$[\ ,\ ]_{\varepsilon_1,\varepsilon_2} = [\ ,\ ] + \varepsilon_1\psi_1 + \varepsilon_2\psi_2,$$

we have

$$\mathrm{Jac}([\ ,\ ]_{\varepsilon_1,\varepsilon_2}) = (3\varepsilon_1^2 + \varepsilon_1\varepsilon_2)\partial_x^5 \wedge \partial_x^3 \wedge \partial_x^2.$$

In particular, $\psi_1, \psi_2 \in Z^2(L, L)$. The 3-cocycle $\partial_x^5 \wedge \partial_x^3 \wedge \partial_x^2 \in C^3(L, L)$ is nontrivial: if

$$\partial_x^5 \wedge \partial_x^3 \wedge \partial_x^2 = d\omega$$

for some $\omega \in C^2(L, L)$ then we will have

$$1 = \partial_x^5 \wedge \partial_x^3 \wedge \partial_x^2(x^5, x^3, x^2) = [\omega(x^5, x^3), x^2] + [\omega(x^3, x^2), x^5] + [\omega(x^2, x^5), x^3],$$

that is impossibile because of $1 \notin [L, L]$. So, the local deformation $\varepsilon_1\psi_1 + \varepsilon_2\psi_2$ can be prolongated if and only if

$$\varepsilon_1(3\varepsilon_1 + \varepsilon_2) = 0.$$

We obtain Lie algebra defined on the space $O_2(n_x, n_y), p = 5$, by multiplication

$$[\ ,\ ]_\varepsilon = \partial_x \wedge \partial_y + \varepsilon(\partial_x^2 \wedge \partial_x^3 + 2\partial_x^0 \wedge \partial_x^5 - 2x\partial_x \wedge \partial_x^5).$$

Denote it by $L(\varepsilon)$.

Notice that the cocycle $\psi = \psi_1 + 2\psi_2$ is nontrivial, if $n_x > 1$, and trivial, if $n_x = 1$. If $n_x = 1$, then

$$\psi = d\omega,$$

where

$$\omega(uy^k) = \partial^4(u)y^{k+1}, \;\; u = u(x).$$

If $n_x > 1$ then the condition $\psi = d\omega_1$ gives us that

$$2 = \psi(1, x^5) = d\omega_1(1, x^5) = [x^5), \omega_1(1)].$$

Since $[x^5), L] \in \mathcal{L}_3$, it is not possible. So, $[\psi] \neq 0$, if $n_x > 1$.

The $p^{n_x+n_y}$-dimensional vector space $O_2(n_x, n_y)$ under the multiplication

$$[u, v]_\varepsilon = \partial_x(u)\partial_y(v) - \partial_y(u)\partial_x(v) + \varepsilon(\partial_x^2(u)\partial_x^3(v)$$

$$-\partial_x^2(v)\partial_x^3(u) + 2\Delta_1(u)\partial_x^5(v) - 2\Delta_1(v)\partial_x^5(u))$$

is the Lie algebra, which we denoted by $L(\varepsilon)$.

Theorem 1. *Let $p = 5, n_x > 1, n_y > 0$. The algebra $L(\varepsilon)$ gives us Gerstenhaber deformation of the Hamiltonian algebra $H_1(n_x, n_y)$ by local deformation $\psi_1 + 2\psi_2$. The Melikyan algebra $L(n_x - 1, n_y)$*[20] *is isomorphic to the algebra obtained from $L(\varepsilon)$ by specialisation $\varepsilon = 1$.*

Let us give a initial part of the grading in $L(\varepsilon)$:

$$L(\varepsilon)_{-2} = \langle x^{[0]} := \varepsilon^{-1}1\rangle \simeq K(-2),$$

$$L(\varepsilon)_{-1} = \langle x^{[0]} := x, x^{[1]} := -\varepsilon^{-1}x^{(2)}, x^{[2]} := \varepsilon^{-2}x^{(3)}, x^{[3]}$$

$$:= -\varepsilon^{-3}(x^{(4)} - \varepsilon y)\rangle \simeq \bar{U}(1),$$

$$L(\varepsilon)_0 = \langle e_{[-1]} := x^{(4)}, e_{[0]} := -xy, e_{[1]} := \varepsilon^{-1}x^{(2)}y, e_{[2]} := -\varepsilon^{-2}x^{(3)}y,$$

$$e_{[3]} := \varepsilon^{-3}(x^{(4)}y - \varepsilon y^{(2)})\rangle + \langle x^{[0]} := x^{(5)}y + \varepsilon xy\rangle \simeq \bar{U}(1) \oplus K,$$

$$L(\varepsilon)_1 = \langle x^{[0]} := -x^{(4)}y, x^{[1]} := xy^{(2)}, x^{[2]} := -\varepsilon^{-1}x^{(2)}y^{(2)},$$

$$x^{[3]} := \varepsilon^{-2}x^{(3)}y^{(2)}, x^{[4]} := -\varepsilon^{-3}(x^{(4)}y^{(2)} - \varepsilon y^{(3)})\rangle$$

$$+\langle x^{[0]} := \varepsilon^2 x^{(6)}, x^{[1]} := -\varepsilon x^{(7)}, x^{[2]} := x^{(8)}, x^{[3]} := x^{(5)} + \varepsilon xy^{(2)}\rangle$$

$$\simeq U(3) \oplus \bar{U}(1).$$

Here $U(t)$ is W_1-module endowed on divided power algebra $U = O_1(1)$ by formula

$$(u\partial)_t(v) = u\partial(v) + t\partial(u)v.$$

Instead of standard notation for basic vectors $x^{(i)}$ (or x^i) of divided power algebra, for elements of modules $U(t)$ we use notation $x^{[i]}$ (in other case may be mixing elements of modules and elements of Hamiltonian algebra as both are defined over just the same vector space of divided power series). Recall that $U(1)$ is reducible and has a submodule of codimension 1:

$$\bar{U}(1) =< x^{[i]} : 0 \le i \le p-2 > .$$

Recall also, that $U(-1)$ is isomorphic to the adjoint module, and we denote its basic vectors by $e_{[i]} := x^{[i+1]}\partial$. We denote by $K(t)$ 1-dimensional $W_1 \oplus < c >$-module with basic vector 1, such that $W_1 1 = 0, (c)_t 1 = t \cdot 1$. Notice that $L(\varepsilon)_0$ is isomorphic to direct sum of Witt algebra W_1 and 1-dimensional center $< c >$.

Similar realisations of some Skryabin, Kuznetzov and Ermolaev (see [16] and further references there) algebras as a Gerstenhaber deformations also can be done.

6. Defining relations and nonsplit extensions. Finite-dimensional simple Lie algebras are called classical and have a lot of applications in mathematics and physics. In nonclassical case, i.e. in case of infinite-dimensional Lie algebras over a field of characteristic 0 and finite-dimensional of characteristic $p > 0$ there appear new four classes of simple Lie algebras. These algebras are called Cartan Type Lie algebras, since they correspond to four classes of vector fields Lie algebras. Recall their definitions in characteristic 0. General type W_n can be defined as a Lie algebra of Laurent poynomials in n valuables, other three types can be defined as a subalgebras of general type saving some differential forms: special type $S_n \subseteq W_{n+1}$ save volume form $dx_1 \wedge \cdots \wedge dx_n$, Hamiltonian type $H_n \subseteq W_{2n}$ save form $\omega_H = \sum_{i=1}^n dx_i \wedge dx_{n+i}$ and contact type $K_{n+1} \subseteq W_{2n+1}$ save contact form $dx_{2n+1} + \omega_H$. An essential part of Serre defining relations $((ad\, e_i)^{1-a_{ji}} e_j = 0,\ i \ne j)$ corresponds to defining relations of graded nilpotent subalgebras, i.e. to 2-homology groups of subalgebras generated by positive roots. Similar things are true for nonclassical Lie algebras. Main part of defining relations of Cartan Type Lie algebra L corresponds to second homology group of maximal graded nilpotent subalgebra $\mathcal{L}_1$. In classical case, these subalgebras are symmetric. They are not so ladge and cycles are uniquely defined by Cartan matrix. Cartan type Lie algebras have no good root systems, and maximal subalgebras are very large. We have calculated $H_2 := H_2(\mathcal{L}_1, \mathbf{C})$ for all four Cartan series.

Let $\mathcal{L}_1 = \oplus_{i \ge 1} L_i$ be graded nilpotent Lie algebra. Then the first homology group $H_1(\mathcal{L}_1, k) \cong \mathcal{L}_1/[\mathcal{L}_1, \mathcal{L}_1]$ can be interpreted as a space of generators of $\mathcal{L}_1$ and the second homology group $H_2(\mathcal{L}_1, k)$ is interpreted as a group of defining relations [10]. More exactly, if elements $E_1, \ldots, E_k$ can not be presented as commutators and their factor-classes in $\mathcal{L}_1/[\mathcal{L}_1, \mathcal{L}_1]$ constitute basis, then $E_1, \ldots, E_k$ are generators of $\mathcal{L}_1$. Any element of $\mathcal{L}_1$ is a linear combination of commutators of $E_1, \ldots, E_k$. If $H_2(\mathcal{L}_1, k)$ is r-dimensional and classes of cycles $\sum_i E_i \wedge f_i(s),\ s = 1, \ldots, r$ constitute a basis, where $f_i(s)$ are elements of $\mathcal{L}_1$ presented as a linear combination of generators $E_1, \ldots, E_k$, then r relations

$$[E_i, f_i(s)] = 0, \quad s = 1, \ldots, r,$$

gives us defining relations of $\mathcal{L}_1$.

For finite-dimensional simple classical Lie algebras defining relations are known as Serre defining relations. These relations can be devided into three parts. Relations of nilpotent subalgebra $\mathcal{L}_-$, generated by elements of negative roots

$$(\mathrm{ad}\, f_i)^{-1-a_{j,i}} f_j = 0, \tag{7}$$

and relations of nilpotent subalgebra $\mathcal{L}_+$ generated by elements of positive roots

$$(\mathrm{ad}\, e_i)^{-1-a_{j,i}} e_j = 0, \tag{8}$$

and compatibility conditions

$$[e_i, f_j] = \delta_{i,j} h_i,$$

$$[h_i, e_j] = a_{j,i} e_j, [h_i, f_j] = -a_{j,i} f_j.$$

Here $A = (a_{i,j})$ is Cartan matrix. In homological terms relations (7) correspond to cycles $f_i \wedge (\mathrm{ad}\, f_i)^{-a_{j,i}} f_j$ of the group $H_2(\mathcal{L}_-, k)$, and relations (8) correspond to cycles $e_i \wedge (\mathrm{ad}\, e_i)^{-a_{j,i}} e_j$ of the group $H_2(\mathcal{L}_+, k)$.

Similar constructions take place for nonclassical Lie algebras too. For example, let $L = \oplus_{i \geq -2} L_i$ be a simple Lie algebra one of Cartan Types $W_n(\mathbf{m}), S_n(\mathbf{m}), H_n(\mathbf{m}), K_n(\mathbf{m})$. It has grading

$$L = \mathcal{L}_- \oplus L_0 \oplus \mathcal{L}_+,$$

where

$$\mathcal{L}_- = \oplus_{i<0} L_i, \;\; \mathcal{L}_+ = \oplus_{i>0} L_i,$$

and L_0 are classical simple Lie algebras sl_n for $S_n(\mathbf{m})$, sp_n for $H_n(\mathbf{m})$, and split central extensions gl_n for $W_n(\mathbf{m})$, $sp_n \oplus k$ for $K_n(\mathbf{m})$.

In contrast to classical case these gradings are not symmetric. A subalgebra $\mathcal{L}_-$ is "small": namely, it is an abelian algebra of dimension $\sim n$, isomorphic to L_{-1} for $L \neq K_n(\mathbf{m})$, and Heisenberg algebra $L_{-2} \oplus L_{-1}$ for $L = K_n(\mathbf{m})$. On the contrary, a positive part is "large": $\mathcal{L}_+$ (we will denote it $\mathcal{L}_1$) is nilpotent algebra of dimension $\sim p^n$.

Calculations of homology groups of $\mathcal{L}_-$ are easy. For any Cartant types with the exception of contact, $\mathcal{L}_- = L_{-1}$, and

$$H_2(\mathcal{L}_-, k) \cong \wedge^2 L_{-1}.$$

For contact type $\mathcal{L}_- = L_{-2} \oplus L_{-1}$, and

$$H_2(\mathcal{L}_-, k) \cong \wedge^2 \mathcal{L}_{-1}/k.$$

The calculation of $H_2(\mathcal{L}_+, k)$ is interesting also from the other point of view. According the Levi-Mal'cev theorem, any solvable extension of semisimple finite-dimensional algebra is split. In the case of positive characteristic it is not the case. Any modular Lie algebra has at least one nonsplit extension and the number of nonequivalent nonsplit extensions by irreducible modules is finite [6]. The question on describing nonsplit extensions of Cartan Type Lie algebras by irreducible modules can be reduced to the question on caclulating second (co)homology groups $H_2(\mathcal{L}_+, k)$. So, roughly speaking problems on description of defining relations and nonsplit extensions for Cartan Type Lie algebras are equivalent.

In our talk we give examples of defining relations descriptions for nilpotent subalgebras in cases of small charactersitics (classical Lie algebras), large characteristics (Zassenhaus algebra) and very large characteristic (Cartan Type Lie algebras).

From these examples we can see that for nonclassical Lie algebras in general $H_2(\mathcal{L}_+, k)$ is very big. Their dimensions are about constant multiple of n^6. But as a modules over 0-components L_0, they have a quite transparent structure.

7. Defining relations and nonsplit extensions. Defining relations for Classical Lie algebras. Let $\mathbf{a} = (a_{i,j})$ be a Cartan matrix of rank n. By classical simple Lie algebras we understand simple Lie algebras over an algebraically closed field of characteristic $p > 0$ of the following types: $A_n(n > 1), B_n(n > 1, p > 2), C_n(n > 2, p > 2), D_n(n > 3), G_2(p > 3), F_4(p > 2), E_6, E_7, E_8$.

Recall that for small characteristics isomorphisms between different types are possible. For example, G_2 in the cases $p = 2, 3$ have ideals such that corresponding simple factors are isomorphic to algebras of types A_2 and A_3. Algebras A_{np-1} have 1-dimensional centers with simple factors. Algebras D_{2k+1}, $E_7\,(p = 2)$ and $E_6\,(p = 3$ have also 1-dimensional ideals with simple factors. For $p = 2$ algebras of types C_n have ideal D_n and F_4 has ideal D_4.

The following result for the case of large and very large characteristic $p > 3$ follows from results of J.-P. Serre, G. Seligman, R.V. Moody, S. Berman. In the case of small characteristics it was obtained by A.S. Dzhumadil'daev and S. Ibraev.

Theorem 2. *Let L be a simple classical Lie algebra, $p > 0$, and $\mathcal{L}_+$ be its nilpotent subalgebra generated by positive root vectors. If $p > 3$, then $\mathcal{L}_+$ can be defined by Serre relations $(\operatorname{ad} e_i)^{1-a_{j,i}} e_j = 0$, as in the case of zero characteristic. If $p = 2, 3$, then defining relations of $\mathcal{L}_+$ consist of Serre defining relations and additional relations. The list of additional relations is given in the Table 1.*

8. Defining relations and nonsplit extensions. Defining relations of $\mathcal{L}_+$ for Zassenhaus algebra. Recall that Zassenhaus algebra $L = W_1(m)$ is p^m-dimensional and graded

$$L = \oplus_{i=-1}^{p^m-2} L_i, \quad [L_i, L_j] \subseteq L_{i+j},$$

with 1-dimensional homogeneous components $L_i = < e_i >$. The multiplication on the basis $\{e_i : -1 \le i \le p^m - 2\}$ can be given by

$$[e_i, e_j] = N_{i,j} e_{i+j},$$

where

$$N_{i,j} = \binom{i+j+1}{j} - \binom{i+j+1}{i}.$$

Let

$$\mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots \supset \mathcal{L}_{p^m-2} \supset 0,$$

$$\mathcal{L}_i = \oplus_{j\geq i} L_j,$$

be the standard filtration. Then

$$\mathcal{L}_1/[\mathcal{L}_1,\mathcal{L}_1] \cong L_{-1}\oplus L_{-2}\oplus_{k=1}^{m-1} L_{p^k-1}.$$

In other words the first homology group $H_1(\mathcal{L}_1,k)$ is $(m+1)$-dimensional and has a basis $e_1,e_2,e_{p-1},\ldots,e_{p^{m-1}-1}$. In particular, this means that any element e_i for $i\neq 1,2,p^k-1$ can be presented as a linear combination of above mentioned elements or their commutators. Fix for any e_i some of such presentation. Change all elements e_i in formal way by E_i.

Theorem 3. *For $p>7$, the nilpotent subalgebra $\mathcal{L}_+$ of the Zassenhaus algebra $W_1(m)$ is $(m+1)$-generated. It is generated by the elements E_1,E_2,E_{p^k-1}, where $0<k<m$, and the following defining relations:*

$$5[E_1,E_4]-9[E_2,E_3]=0,$$

$$[E_2,E_5]-2[E_3,E_4]=0,$$

$$(\delta_{l,k}+\delta_{s,k}-1)[E_{p^k-2},E_{p^l+p^s}]-2[E_{p^k-1},E_{p^l+p^s-1}]=0, 0<k\leq l\leq s<m,$$

$$2[E_{p^k-1},E_{p^l-1}]-[E_1,E_{p^k+p^l-3}]=0, 0<k<l<m,$$

$$[E_1,E_{p-2}]=0,$$

$$[E_{p^{k-1}-1},E_{p^k-p^{k-1}}]=0, k>1,$$

$$[E_2,E_{p^m-2}]=0,$$

$$[E_1,E_{p^k}]+[E_2,E_{p^k-1}]=0,$$

$$(1+\delta_{k,l})[E_2,E_{p^k+p^l-1}]+[E_{p^k-1},E_{p^l+2}]=0, 0<k\leq l<m,$$

$$[E_2,E_{p^k}]-2[E_3,E_{p^k-1}]=0,$$

$$3[E_2,E_{p^k+1}]+2[E_3,E_{p^k}]=0.$$

Here $\delta_{i,j}=\delta(i=i)$ is the Kroneker symbol and elements E_i for $i\neq 1,2,p^k-1$ are defined as a linear combination of commutators of elements $E_1,E_2,E_{p^k-1}, 1<k<m$ just like in Zassenhaus algebra. For example,

$$E_3 := [E_1,E_2]/2,\ E_4 := [E_1,E_3]/5,\ E_5 := [E_1,E_4]/9,$$

$$E_6 := [E_1,E_5]/14,\ E_{p^k} := -[E_1,E_{p^k-1}].$$

Under above mentioned relations, definitions of E_i do not depend of the chois of such presentations.

For $p = 0$ this result follows from results of Gelfand-Fuks [12]. For $p = 0$ it was proved also by Fialovski [11] and Ufnarovski [27]. For $p > 7$ this result in cohomological terms is given in [4].

What about small characteristics? For $p = 3, 4, 5, 7$ the above table will be changed not so much, but in the case of $p = 2$ the picture is quite different (see Table 2). Notice that in case of $p = 2$ weights are presented as a sum of powers of 2 and such presentations are not unique. Therefore repeatings of some cocycles in some weights are possible.

9. Defining relations and nonsplit extensions. Defining relations of $\mathcal{L}_+$ for nonclassical Lie algebras (the case of very large characteristic). As it was mentioned earlier, one of differences between simple Lie algebras of classical and nonclassical types is the following one. Nilpotent subalgebras of classical Lie algebras are symmetric and not so big. In nonclassical case these subalgebras are very big and in general not symmetric. Since $\mathcal{L}_-$ is abelian or isomorphic to Heisenberg algebra and calculations of homology groups for such algebras are very easy, we shall exclude the case $\mathcal{L}_+$ from the consideration.

As we mentioned above a modular result in the case of very large characteristic is a part of characteristic 0 results. That is why appearence of characteristic 0 result in our talk will not be surprising. What kind of characteristic p cocycles can appear in addition to cocycles, obtained from characteristic 0, one can see, for example, in [9].

In this section we give description of defining relations of nilpotent subalgebra $\mathcal{L}_1$ for Cartan Type Lie algebra $L = W_n, S_n (n > 2), H_n, K_{n+1}$ in characteristic 0. Since $H_2(\mathcal{L}_+, k) \cong H^2(\mathcal{L}_+, k)$, this result in cohomological terms follows from results of [7]. Below we give descriptions of $H_2(\mathcal{L}_+, k)$ made by A.S. Dzhumadil'daev and R. Kerimbaev. Independently second homology group was calculated by D.Leites and E.Poletaeva [19]. In some particular cases they also were obtained in [14, 10]. As was mentioned above the group $H_2(\mathcal{L}_+, k)$ can be interpretered as a group of defining relations of $\mathcal{L}_1$. We give exact constructions for basic cycles and give an algorithm how to construct defining relations using cycles.

For classical simple Lie algebra L_0 of rank n denote by $\pi_1, \ldots, \pi_n$ its fundamental weights and by $R(\pi)$ the irreducible module with weight π. Recall that $L_0 \cong sl_{n+1}$ for $L = S_n$ and $L_0 \cong sp_n$ for $L = H_n$. For $L = W_n$, when $L_0 \cong gl_n$, denote by $R(\pi, \lambda)$ the irreducible gl_n -module with weight π (over sl_n) and conformal weight λ:

$$(x_i\partial_j)_\lambda m = (x_i\partial_j)m + (1 - \lambda)\delta(i = j)m, \quad m \in R(\pi, \lambda), \lambda \in \mathbf{C}.$$

Similar definition for $L = K_{n+1}, L_0 \cong sp_n \oplus \mathbf{C}$, means that

$$(u)_\lambda m = (u)m + (\lambda - 2)\partial_0(u)m, \ u \in L_0, \ m \in R(\pi, \lambda), \lambda \in \mathbf{C}.$$

Below $\delta = 0$, if $n = 3$, and $\delta = 1$, if $n > 3$ and $\mathcal{L}_+ = \mathcal{L}_1, H_2(\mathcal{L}_1) = H_2(\mathcal{L}_1, \mathbf{C})$.

Theorem 4. *Let $L = W_n$. If $n > 2$, then*

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1),$$

and as gl_n -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong \delta R(2\pi_2 + \pi_{n-2},\, 2) \oplus R(4\pi_1 + \pi_{n-2},\, 2) \oplus R(2\pi_1 + \pi_2 + 2\pi_{n-1},\, 3)$$

$$\oplus R(3\pi_1 + \pi_{n-1},\, 2) \oplus R(\pi_1 + \pi_2 + \pi_{n-1},\, 2)$$

$$\oplus R(\pi_1 + \pi_2 + \pi_{n-1},\, 2) \oplus R(\pi_2,\, 1) \oplus R(\pi_2,\, 1).$$

If $n = 2$, then

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1) \oplus H_2^{(3)}(\mathcal{L}_1),$$

and as gl_2 -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong R(3\pi_1 + \pi_{n-1},\, 2) \oplus R(\pi_2,\, 1) \oplus R(\pi_2,\, 1),$$

$$H_2^{(3)}(\mathcal{L}_1) \cong R(7\pi_1,\, 3) \oplus R(5\pi_1,\, 2) \oplus R(\pi_1,\, 0).$$

If $n = 1$, then

$$H_2(\mathcal{L}_1) \cong H_2^{(7)}(\mathcal{L}_1) \oplus H_2^{(5)}(\mathcal{L}_1).$$

Corollary 1. *For $L = W_n$,*

$$\begin{aligned}\dim H_2(\mathcal{L}_\infty) \quad &= n^2(n+1)(3n^3 + 3n^2 - 4n - 14)/24 \sim n^6/8,\ \textit{if}\ n > 2,\\ &= 25,\ \textit{if}\ n = 2,\\ &= 2,\ \textit{if}\ n = 1.\end{aligned}$$

Theorem 5. *Let $L = S_n, n > 2$. Then*

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1),$$

and as sl_n -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong \delta R(2\pi_2 + \pi_{n-2}) \oplus R(4\pi_1 + \pi_{n-2})$$

$$\oplus R(2\pi_1 + \pi_2 + 2\pi_{n-1}) \oplus R(\pi_1 + \pi_2 + \pi_{n-1}) \oplus R(\pi_2).$$

Corollary 2. For $L = S_n, n > 2$,

$$\dim H_2(\mathcal{L}_1) \quad = (n+1)n(n-1)(3n^3 + 6n^2 - 10n - 24)/24 \sim n^6/8.$$

Theorem 6. *Let $L = H_n$. If $n > 1$, then*

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1) \oplus H_2^{(3)}(\mathcal{L}_1),$$

and as sp_n -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong R(4\pi_1 + \pi_2) \oplus R(3\pi_2) \oplus R(2\pi_2) \oplus R(\pi_2) \oplus R(0),$$

$$H_2^{(3)}(\mathcal{L}_1) \cong R(\pi_1).$$

If $n = 1$, then

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1) \oplus H_2^{(3)}(\mathcal{L}_1) \oplus H_2^{(4)}(\mathcal{L}_1),$$

and as sl_2 -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong R(0),$$

$$H_2^{(3)}(\mathcal{L}_1) \cong R(7\pi_1) \oplus R(\pi_1),$$

$$H_2^{(4)}(\mathcal{L}_1) \cong R(2\pi_1).$$

Corollary 3. *For $L = H_n$,*

$$\begin{aligned} dim H_2(\mathcal{L}_1) &= (2n-1)n(8n^4 + 28n^3 + 34n^2 + 5n - 21)/18 \\ &\sim 8n^6/9, \ if\, n > 1, \\ &= 14, \ if\, n = 1. \end{aligned}$$

Theorem 7. *Let $L = K_{n+1}$. If $n > 1$, then*

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1),$$

and as $sp_n \oplus \mathbf{C}$ -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong R(4\pi_1 + \pi_2, 0) \oplus R(3\pi_2, -6) \oplus R(2\pi_2, -4) \oplus R(\pi_2, -2) \oplus$$

$$R(0, -2) \oplus R(2\pi_1 + \pi_2, -2) \oplus R(4\pi_1, 2),$$

If $n = 1$, then

$$H_2(\mathcal{L}_1) \cong H_2^{(2)}(\mathcal{L}_1) \oplus H_2^{(3)}(\mathcal{L}_1),$$

and as $sl_2 \oplus \mathbf{C}$ -module,

$$H_2^{(2)}(\mathcal{L}_1) \cong R(4\pi_1, 2) \oplus R(0, -2),$$

$$H_2^{(3)}(\mathcal{L}_1) \cong R(7\pi_1,\, 0) \oplus R(5\pi_1,\, 2) \oplus R(3\pi_1,\, 0) \oplus R(\pi_1,\, -2) \oplus R(\pi_1,\, -2),$$

Corollary 4. *For* $L = K_{n+1}$,

$$\begin{aligned}\dim H_2(\mathcal{L}_1) \quad & = (2n+1)(8n^5 + 20n^4 + 34n^3 + 7n^2 - 15n - 18)/18 \\ & \sim 8n^6/9,\ \textit{if}\, n > 1, \\ & = 28,\ \textit{if}\, n = 1.\end{aligned}$$

Let us give an algorithm to write defining relations of $\mathcal{L}_1$ for Cartan Type Lie algebra L. For simplicity we exclude the case $L = W_1$ from the consideration.

i) Take as generators of $\mathcal{L}_1$ elements $\mathbf{e}$ that corresponds to basic vectors e of L_1. Any element of L_k can be represented as a linear combination of commutators of L_1. Write, instead of elements of L_k that appear below, their expressions by L_1.

ii) Defining relations correspond to irreducible components of the decomposition of the L_0 -module $H_2(\mathcal{L}_1)$. Let $R(\pi)$ or $R(\pi,\, \lambda)$ be one of such components. Take its heighest vector and generate all other elements of the considered module. These elements should be in $\wedge^2\mathcal{L}_1$.

iii)Change all exterior products $\wedge$ in these elements by commutators $[\ ,\ ]$, change all basic elements e by $\mathbf{e}$ and put obtained expressions equal to 0. These are defining relations of $\mathcal{L}_1$.

Example. $L = H_1$. Its nilpotent subalgebra $\mathcal{L}_1$ generates by 4 elements $\mathbf{x}^3, \mathbf{x}^2\mathbf{y}, \mathbf{x}\mathbf{y}^2, \mathbf{y}^3$ and 14 defining relations: (below we use the notation $[a,b,c,d] := [a,[b,[c,d]]]$)

$$[\mathbf{x^3}, \mathbf{y^3}] - 3[\mathbf{xy^2}, \mathbf{x^2y}] = 0;$$

$$[\mathbf{x^3}, [\mathbf{x^3}, \mathbf{x^2y}]] = 0,$$

$$[\mathbf{x^3}, [\mathbf{x^3}, \mathbf{xy^2}]] + 2[\mathbf{x^2y}, [\mathbf{x^3}, \mathbf{x^2y}]] = 0,$$

$$3[\mathbf{xy^2}, [\mathbf{x^3}, \mathbf{x^2y}]] + 6[\mathbf{x^2y}, [\mathbf{x^3}, \mathbf{xy^2}]] + 2[\mathbf{x^3}, [\mathbf{x^3}, \mathbf{y^3}]] = 0,$$

$$[\mathbf{y^3}, [\mathbf{x^3}, \mathbf{x^2y}]] + 6[\mathbf{xy^2}, [\mathbf{x^3}, \mathbf{xy^2}]] + 6[\mathbf{x^2y}, [\mathbf{x^3}, \mathbf{y^3}]] + 2[\mathbf{x^3}, [\mathbf{x^2y}, \mathbf{y^3}]] = 0,$$

$$2[\mathbf{y^3}, [\mathbf{x^3}, \mathbf{xy^2}]] + 6[\mathbf{xy^2}, [\mathbf{x^3}, \mathbf{y^3}]] + 6[\mathbf{x^2y}, [\mathbf{x^2y}, \mathbf{y^3}]] + [\mathbf{x^3}, [\mathbf{xy^2}, \mathbf{y^3}]] = 0,$$

$$2[\mathbf{y^3}, [\mathbf{x^3}, \mathbf{y^3}]] + 6[\mathbf{xy^2}, [\mathbf{x^2y}, \mathbf{y^3}]] + 3[\mathbf{x^2y}, [\mathbf{xy^2}, \mathbf{y^3}]] = 0,$$

$$2[\mathbf{y^3},[\mathbf{x^2y},\mathbf{y^3}]]+3[\mathbf{xy^2},[\mathbf{xy^2},\mathbf{y^3}]]=0,$$

$$[\mathbf{y^3},[\mathbf{y^3},\mathbf{xy^2}]]=0;$$

$$[\mathbf{x^3},[\mathbf{x^2y},\mathbf{y^3}]]-2[\mathbf{x^2y},[\mathbf{x^3},\mathbf{y^3}]]+3[\mathbf{xy^2},[\mathbf{x^3},\mathbf{xy^2}]]-2[\mathbf{y^3},[\mathbf{x^3},\mathbf{x^2y}]]=0,$$

$$-2[\mathbf{x^3},[\mathbf{xy^2},\mathbf{y^3}]]+3[\mathbf{x^2y},[\mathbf{x^2y},\mathbf{y^3}]]-2[\mathbf{xy^2},[\mathbf{x^3},\mathbf{y^3}]]+[\mathbf{y^3},[\mathbf{x^3},\mathbf{xy^2}]]=0;$$

$$[\mathbf{x^3},\mathbf{x^3},\mathbf{xy^2},\mathbf{y^3}]-2[\mathbf{x^2y},\mathbf{x^3},\mathbf{x^2y},\mathbf{y^3}]$$

$$+2[\mathbf{xy^2},\mathbf{x^3},\mathbf{x^3},\mathbf{y^3}]-2[\mathbf{y^3},\mathbf{x^3},\mathbf{x^3},\mathbf{xy^2}]=0,$$

$$9[\mathbf{x^3},\mathbf{x^2y},\mathbf{xy^2},\mathbf{y^3}]-18[\mathbf{x^2y},\mathbf{x^3},\mathbf{xy^2},\mathbf{y^3}]$$

$$+12[\mathbf{xy^2},\mathbf{x^3},\mathbf{x^2y},\mathbf{y^3}]-4[\mathbf{y^3},\mathbf{x^3},\mathbf{x^3},\mathbf{y^3}]=0,$$

$$2[\mathbf{x^3},\mathbf{y^3},\mathbf{y^3},\mathbf{x^2y^2}]-2[\mathbf{x^2y},\mathbf{y^3},\mathbf{y^3},\mathbf{x^3}]$$

$$-2[\mathbf{xy^2},\mathbf{y^3},\mathbf{x^3},\mathbf{xy^2}]-[\mathbf{y^3},\mathbf{y^3},\mathbf{x^3},\mathbf{x^2y}]=0.$$

As examples, in the tables 3 and 4 we give constructions of heighest weight vectors for irreducible components of L_0-modules of $H_2(\mathcal{L}_1,k)$ for $L=H_n$ and $L=K_{n+1}$.

Table 1. Additional relations for nilpotent subalgebra of classical Lie algebras.

types	p	additional relations	dim
$A_n, n>2$	2	$[e_{i-1}, e_i, e_{i+1}, e_i], i = 2, \dots, n-1$	$n-2$
$B_n, n>2$	3	$[e_{n-2}, e_{n-1}, e_n, e_n, e_{n-1}, e_n]$	1
$C_n, n>2$	3	$[e_{n-2}, e_{n-1}, e_n, e_{n-1}, e_{n-1}]$	1
$D_n, n>3$	2	$[e_{i-1}, e_i, e_{i+1}, e_i], i = 2, \dots, n-2$ $[e_{n-3}, e_{n-2}, e_n, e_{n-2}]$ $[e_{n-2}, e_{n-1}, e_n, e_{n-2}]$ $[e_{n-3}, e_{n-2}, e_{n-1}, e_n, e_{n-2}, e_n]$ $[e_{n-3}, e_{n-2}, e_{n-1}, e_n, e_{n-2}, e_{n-1}]$ $[e_i, e_{i+1}, \dots, e_n, e_{n-3}, e_{n-2}, \dots, e_i], i = 1, \dots, n-3$	$2n-2$
F_4	3	$[e_3, e_4, e_2, e_3, e_3]$ $[e_2, e_3, e_3, e_1, e_2, e_3]$ $[e_3, e_4, e_2, e_3, e_1, e_2, e_3, e_3]$ $[e_3, e_4, e_2, e_3, e_4, e_1, e_2, e_3, e_4]$ $[e_3, e_4, e_2, e_3, e_4, e_1, e_2, e_3, e_3, e_2, e_3]$	5
E_n $n = 6, 7, 8$	2	$[e_1, e_3, e_4, e_3]$ $[e_2, e_4, e_3, e_4]$ $[e_2, e_4, e_5, e_4]$ $[e_3, e_4, e_5, e_4]$ $[e_4, e_5, e_6, e_5]$ $\delta(n=7)[e_5, e_6, e_7, e_6]$ $\delta(n=8)[e_6, e_7, e_8, e_7]$ $[e_2, e_4, e_3, e_5, e_4, e_2]$ $[e_2, e_4, e_3, e_5, e_4, e_3]$ $[e_2, e_4, e_3, e_5, e_4, e_5]$ $[e_1, e_3, e_4, e_2, e_5, e_4, e_3, e_1]$ $[e_2, e_4, e_3, e_5, e_4, e_6, e_5, e_6]$ $\delta(n=7)[e_2, e_4, e_3, e_5, e_4, e_6, e_5, e_7, e_6, e_7]$ $\delta(n=8)[e_2, e_4, e_3, e_5, e_4, e_6, e_5, e_7, e_6, e_8, e_7, e_8]$	$2(n-1)$

Table 2. Basic cocycles of $H^2_a(\mathcal{L}_1(m), k)$, where $\mathcal{L}_1(m)$ is nilpotent subalgebra of Zassenhaus algebra $W_1(m)$ for $p = 2$.

a	$2-$cocycles for $H^2_a(\mathcal{L}_1(m), k)$	dim
2^k-4 $3 \le k$	$f_{2^{k-2}-1} \wedge f_{3(2^{k-2}-1)} + f_{2^{k-2}} \wedge f_{3\,2^{k-2}-4} + \cdots$ $+f_{2^{k-1}-3} \wedge f_{2^{k-1}-1}$	$1-\delta(k=3)$
2^k-3 $3 \le k$	$f_{2^{k-1}-2} \wedge f_{2^{k-1}-1}$ $f_{2^{k-2}-1} \wedge f_{2^{k-2}+2^{k-1}-2}$	$2-\delta(k=3)$
2^k-2 $2 \le k$	$f_1 \wedge f_{2^k-3} + f_2 \wedge f_{2^k-4} + \cdots + f_{2^{k-1}-2} \wedge f_{2^{k-1}}$	$1-\delta(k=2)$
2^k-1 $2 \le k$	$f_1 \wedge f_{2^k-2}$ $f_{2^{k-1}-1} \wedge f_{2^{k-1}}$	$2-\delta(k=2)$
2^k $2 \le k$	$f_2 \wedge f_{2^k-2}$ $f_3 \wedge f_{2^k-3} + f_4 \wedge f_{2^k-4} + \cdots + f_{2^{k-1}-2} \wedge f_{2^{k-1}+2}$	$2-\delta(k \le 3)$ $-\delta(k \le 2)$
2^k+1 $2 \le k$	$f_2 \wedge f_{2^k-1}$ $f_3 \wedge f_{2^k-2}$ $f_{2^{k-1}-1} \wedge f_{2^{k-1}+2}$	$3-\delta(k \le 3)$ $-\delta(k \le 2)$
2^k+2 $2 \le k$	$f_1 \wedge f_{2^k+1} + f_2 \wedge f_{2^k}$	1
2^k+3 $3 \le k$	$f_1 \wedge f_{2^k+2} + f_3 \wedge f_{2^k}$	1
2^k+4 $4 \le k$ $4 < k$	$f_3 \wedge f_{2^k+1} + f_4 \wedge f_{2^k} + f_5 \wedge f_{2^k-1}$ $f_6 \wedge f_{2^k-2}$ $f_7 \wedge f_{2^k-3} + f_8 \wedge f_{2^k-4} + \cdots + f_{2^{k-1}-2} \wedge f_{2^{k-1}+6}$	$2+\delta(k>4)$
2^k+5 $3 \le k$	$f_3 \wedge f_{2^k+2}$ $f_6 \wedge f_{2^k-1}$ $f_7 \wedge f_{2^k-2}$ $f_{2^{k-1}-1} \wedge f_{2^{k-1}+6}$	$4-\delta(k \le 4)$ $-\delta(k \le 3)$
2^k+2^l-4 $3 \le k < l$	$f_{2^k-2} \wedge f_{2^l-2}$, $f_{2^k-1} \wedge f_{2^l-3} + f_{2^k} \wedge f_{2^l-4}$ $+ \cdots + f_{2^{l-1}-2} \wedge f_{2^k+2^{l-1}-2}$ $f_{2^{k-1}-1} \wedge f_{2^l+2^{k-1}-3} + \cdots + f_{2^k-3} \wedge f_{2^l-1}$	$3-\delta(l=k+1)$

Table 2 (Continuation).

$2^k + 2^l - 3$ $3 \leq k \leq l$	$f_{2^k-2} \wedge f_{2^l-1}$ $f_{2^k-1} \wedge f_{2^l-2}$ $f_{2^{k-1}-1} \wedge f_{2^{k-1}+2^l-2}$ $f_{2^{l-1}-1} \wedge f_{2^{l-1}+2^k-2}$	$4 - 2\delta(l = k)$ $-\delta(l = k + 1)$
$2^k + 2^l - 2$ $3 \leq k < l$	$f_1 \wedge f_{2^k+2^l-3} + f_2 \wedge f_{2^k+2^l-4} + \cdots + f_{2^k-2} \wedge f_{2^l}$	1
$2^k + 2^l - 1$ $3 \leq k < l$ v	$f_1 \wedge f_{2^k+2^l-2} + f_{2^k-1} \wedge f_{2^l}$	1
$2^k + 2^l$ $3 \leq k < l$	$f_1 \wedge f_{2^k+2^l-1} + f_{2^k} \wedge f_{2^l}$ $f_3 \wedge f_{2^k+2^l-3} + f_4 \wedge f_{2^k+2^l-4}$ $+ \cdots + f_{2^k-2} \wedge f_{2^l+2}$	2
$2^k + 2^l + 1$ $3 \leq k < l$	$f_3 \wedge f_{2^k+2^l-2} + f_{2^k-1} \wedge f_{2^l+2}$ $f_{2^k-1} \wedge f_{2^l+2} + f_{2^l-1} \wedge f_{2^k+2}$	2
$2^k + 2^l + 2^s - 4$ $3 \leq k < l < s$	$f_{2^k-1} \wedge f_{2^l+2^s-3} + f_{2^k} \wedge f_{2^l+2^s-4}$ $+ \cdots + f_{2^l-2} \wedge f_{2^s+2^k-2}$ $f_{2^k-1} \wedge f_{2^s+2^k-3} + f_{2^l} \wedge f_{2^s+2^k-4}$ $+ \cdots + f_{2^s-2} \wedge f_{2^k+2^l-2}$	2
$2^k + 2^l + 2^s - 3$ $3 \leq k < l < s$	$f_{2^l-1} \wedge f_{2^s+2^k-2} + f_{2^s-1} \wedge f_{2^k+2^l-2}$ $f_{2^k-1} \wedge f_{2^l+2^s-2} + f_{2^l-} \wedge f_{2^s+2^k-2}$	2
$2^k + 2^l + 2^s - 2$ $2 \leq k < l < s < m$	$f_1 \wedge f_{2^k+2^l+2^s-3} + \cdots + f_{2^k-1} \wedge f_{2^l+2^s-1}$ $+f_{2^l} \wedge f_{2^k+2^s-2} + \cdots + f_{2^k+2^l-2} \wedge f_{2^s}$	1
$2^k + 2^l + 2^s - 1$ $2 \leq k < l < m$	$f_1 \wedge f_{2^k+2^l+2^s-2} + f_{2^k} \wedge f_{2^l+2^s-1}$ $+f_{2^l} \wedge f_{2^s+2^k-1} + f_{2^s} \wedge f_{2^k+2^l-1}$	 1

Table 3. $L = H_n$**. Highest vectors for the irreducible components of** sp_n**-module** $H_2(\mathcal{L}_+, k)$**.**

π	*highest vectors of* $R(\pi)$
$4\pi_1+\pi_2, n>1,$	$x_1^3 \wedge x_1^2x_2 - x_1^2x_2 \wedge x_1^3,$
$3\pi_2, n>1,$	$x_1^3 \wedge x_2^3 - 3x_1^2x_2 \wedge x_1x_2^2 + 3x_1x_2^2 \wedge x_1^2x_2 - x_2^3 \wedge x_1^3,$
$2\pi_2, n>1,$	$\sum_{i=1}^n x_1^2x_i \wedge x_2^2x_{-i} - 2x_1x_2x_i \wedge x_1x_2x_{-i} - x_1^2x_{-i} \wedge x_2^2x_i,$
$\pi_2, n>1,$	$(x_1 \wedge x_2)(\sum_{i=1}^n x_i \wedge x_{-i})^2,$
$\pi_1,$	$(1 \otimes x_2)(\sum_{i=1}^n x_i \wedge x_{-i})^3;$
$0,$	$(\sum_i x_i \wedge x_{-i})^3,$
$7\pi_1, n=1,$	$x_1^3 \wedge x_1^4,$
$2\pi_1, n=1,$	$x_{-1}^3 \wedge x_1^5 - 3x_{-1}^2x_1 \wedge x_{-1}x_1^4 + 3x_{-1}x_1^2 \wedge x_{-1}^2x_1^3 - x_1^3 \wedge x_{-1}^3x_1^2$

Table 4. $L = K_{n+1}$**. Highest vectors for the irreducible components of** $sp_n \oplus k$**-module** $H_2(\mathcal{L}_+, k)$**.**

π	*highest vectors of* $R(\pi)$
$4\pi_1+\pi_2, n>1,$	$x_1^3 \wedge x_1^2x_2,$
$4\pi_1,$	$(n+2)x_1^3 \wedge x_0x_1 + \sum_{i=1}^n x_1^2x_i \wedge x_1^2x_{-i},$
$2\pi_1+\pi_2, n>1,$	$x_1^3 \wedge x_0x_2 - x_1^2x_2 \wedge x_1x_0,$
$3\pi_2, n>1,$	$x_1^3 \wedge x_2^3 - 3x_1^2x_2 \wedge x_1x_2^2,$
$2\pi_2, n>1,$	$\sum_{i=1}^n x_1^2x_i \wedge x_2^2x^{-i} - 2x_1x_2x_i \wedge x_1x_2x_{-i} + x_2^2x_i \wedge x_1^2x_{-i},$
$\pi_2, n>1,$	$x_0x_1 \wedge x_0x_2,$
$\pi_2, n>1,$	$(x_1 \wedge x_2)(\sum_{i=1}^n x_i \wedge x_{-i})^2,$
$0,$	$(\sum_{i=1}^n x_i \wedge x_{-i})^3,$
$(below\, n=1)$	
$7\pi_1,$	$x_1^3 \wedge x_1^4,$
$5\pi_1,$	$7x_1^3 \wedge x_0x_1^2 + x_1^3 \wedge x_{-1}x_1^3 - x_{-1}x_1^2 \wedge x_1^4,$
$3\pi_1,$	$x_1^3 \wedge x_0^2 + 2x_0x_1 \wedge x_0x_1^2,$
$\pi_1,$	$x_{-1}^3 \wedge x_1^4 - 3x_{-1}^2x_1 \wedge x_{-1}x_1^3 + 3x_{-1}x_1^2 \wedge x_{-1}^2x_1^2 - x_1^3 \wedge x_{-1}^3x_1,$
$\pi_1,$	$x_0x_1 \wedge x_0^2.$

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