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Minimal Identities for Right-Symmetric Algebras

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An algebra A with multiplication $A \times A \to A, (a,b) \mapsto a \circ b$, is called right-symmetric, if $a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b$, for any $a, b, c \in A$. The multiplication of right-symmetric Witt algebras $W_n = \{u\partial_i : u \in U, U = \mathcal{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, or $= \mathcal{K}[x_1, \ldots, x_n], i = 1, \ldots, n\}$, $p = 0$, or $W_n(\mathbf{m}) = \{u\partial_i : u \in U, U = O_n(\mathbf{m})\}$, $p > 0$, are given by $u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i$. An analogue of the Amitsur–Levitzki theorem for right-symmetric Witt algebras is established. Right-symmetric Witt algebras of rank n satisfy the standard right-symmetric identity of degree $2n+1$: $\sum_{\sigma \in \mathrm{Sym}_{2n}} \mathrm{sign}(\sigma) a_{\sigma(1)} \circ (a_{\sigma(2)} \circ \cdots \circ (a_{\sigma(2n)} \circ a_{2n+1}) \cdots) = 0$. The minimal degree for left polynomial identities of $W_n^{\mathrm{r\,sym}}, W_n^{+\mathrm{r\,sym}}$, $p = 0$, is $2n+1$. All left polynomial identities of right-symmetric Witt algebras of minimal degree follow from the left standard right-symmetric identity $s_{2n}^{\mathrm{r\,sym}} = 0$, if $p \neq 2$. © 2000 Academic Press

1. INTRODUCTION

Let s_k^{ass} be the standard skew-symmetric associative polynomial in k variables,

$$s_k^{ass}(t_1, \ldots, t_k) = \sum_{\sigma \in \mathrm{Sym}_k} \mathrm{sign}(\sigma)\, t_{\sigma(1)} \cdots t_{\sigma(k)},$$

where Sym_k is the permutation group in k elements and $\mathrm{sign}(\sigma)$ is the sign of a permutation σ. According to the Amitsur–Levitzki theorem [1] the matrix algebra Mat_n satisfies the standard polynomial identity of degree $2n$,

$$\sum_{\sigma \in \mathrm{Sym}_{2n}} \mathrm{sign}\,\sigma\, a_{\sigma(1)} \circ \cdots \circ a_{\sigma(2n)} = 0,$$

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where $a \circ b$ is the usual matrix multiplication. Moreover, Mat_n has no polynomial identity of degree less that $2n$. For details on polynomial identities of associative algebras see for example, [12].

The algebra $W_1 = \{e_i : e_i \circ e_j = (i+1)e_{i+j}, i, j \in \mathbf{Z}\}$ is right-symmetric. Since its Lie algebra is isomorphic to the Witt algebra $W_1 = \{e_i : [e_i, e_j] = (j-i)e_{i+j}\}$ we call it a right-symmetric Witt algebra of rank 1 and denote it by $W_1^{\mathrm{r\,sym}}$. This algebra satisfies *a right-symmetric identity*

$$a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b$$

and *a left-commutativity identity*

$$a \circ (b \circ c) = b \circ (a \circ c). \tag{1}$$

Such algebras are called *Novikov* [9–11].

There is a generalization of the Witt algebra to the many variables case. Let U be an associative commutative algebra with a set of commuting derivations $\mathscr{D} = \{\partial_i : i = 1, \ldots, n\}$. For any $u \in U$, an endomorphism $u\partial_i : U \to U$, such that $(u\partial_i)(v) = u\partial_i(v)$, is a derivation of U. Denote by $U\mathscr{D}$ a space of derivations $\sum_{i=1}^n u_i\partial_i$. Endow this space by multiplication

$$u\partial_i \circ v\partial_j := v\partial_j(u)\partial_i.$$

We obtain a right-symmetric algebra $U\mathscr{D}$. This algebra is called *a right-symmetric Witt algebra generated by U and $\mathscr{D}$.*

In our paper, U is $\mathscr{K}[x_1, \ldots, x_n]$, or Laurent polynomial algebra $\mathscr{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, or a divided power algebra $O_n(\mathbf{m}) = \{x^\alpha : x^\alpha x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}\}$, if the characteristic of $\mathscr{K}$ is $p > 0$. As a Lie algebra the Witt algebra of rank n is defined as a Lie algebra of derivations of U. The multiplication $u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i$ satisfies the right-symmetry identity. Obtained right-symmetric Witt algebras of rank n are denoted by $W_n^{\mathrm{r\,sym}}$ or $W_n^{+\mathrm{r\,sym}}$ or $W_n(\mathbf{m})^{\mathrm{r\,sym}}$, depending on $U = \mathscr{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, or $\mathscr{K}[x_1, \ldots, x_n]$, or $O_n(\mathbf{m})$. It is easy to observe that the right-symmetric Witt algebras of rank n do *not* satisfy the left-commutativity identity if $n > 1$.

We are interested in the analogues of the left-symmetric identities for the case of many variables. We suggest two ways to solve this problem.

In the first way we endow the vector space of the Witt algebra with two multiplications: the multiplication $(a, b) \mapsto a \circ b$, mentioned above, and the second multiplication defined by $u\partial_i * v\partial_j := \partial_i(u)v\partial_j$. We obtain an algebra with the following identities:

$$a \circ (b \circ c) - (a \circ b) \circ c - a \circ (c \circ b) + (a \circ c) \circ b = 0,$$

$$a * (b * c) - b * (a * c) = 0, \tag{2}$$

$$a \circ (b * c) - b * (a \circ c) = 0,$$

$$(a * b - b * a - a \circ b + b \circ a) * c = 0,$$

$$(a \circ b - b \circ a) * c + a * (c \circ b) - (a * c) \circ b - b * (c \circ a) + (b * c) \circ a = 0.$$

Note that (2), for the multiplication $*$, is similar to the identity (1) for the multiplication $\circ$. In the case of $n = 1$, these multiplications coincide and all identities are reduced to two: right-symmetric identity and Novikov identity. So, an algebra A with multiplications $\circ$ and $*$ can be considered as a Novikov algebra in the general case.

The second way concerns identities of right-symmetric Witt algebras. In this way we save right-symmetric multiplication $(a, b) \mapsto a \circ b$, and try to construct an identity for its left multiplication operators. For $a \in A$, denote by r_a and l_a operators of right and left multiplications on $A : br_a = b \circ a,\ bl_a = a \circ b$ (arguments are on the left side). In terms of right and left multiplication operators the right-symmetry identity is equivalent to the following conditions:

$$[r_a, r_b] = r_{[a,b]}, \qquad [r_a, l_b] = l_a l_b - l_{b\circ a}, \qquad \forall a, b \in A.$$

Let $s_k^{\mathrm{r\,sym}}$ be the following skew-symmetric non-associative left polynomial in $k + 1$ variables:

$$s_k^{\mathrm{r\,sym}}(t_1, \ldots, t_k, t_{k+1}) = \sum_{\sigma\in \mathrm{Sym}_k} \mathrm{sign}(\sigma)\, t_{\sigma(1)} \circ (t_{\sigma(2)} \circ \cdots \circ (t_{\sigma(k)} \circ t_{k+1}) \cdots).$$

Call it the *standard polynomial* of degree $k + 1$ (more exact names like “standard left polynomial for right-symmetric algebras” or “standard right-symmetric left polynomial” are too long and not convenient).

Note that $s_k^{\mathrm{r\,sym}}$ is skew-symmetric in the first k variables $t_1, \ldots, t_k$. Let

$$s_k^k := s_k^{\mathrm{r\,sym}},$$

$$s_k^r(t_1, \ldots, t_k) := s_k^{\mathrm{r\,sym}}(t_1, \ldots, \hat{t}_r, \ldots, t_k, t_r), \qquad 0 < r < k,$$

$$s_k^{(l,r)}(t_1, \ldots, t_k) := s_k^{\mathrm{r\,sym}}(t_1, \ldots, \hat{t}_l, \ldots, t_k, t_r), \qquad l \neq r.$$

If $s_k^{\mathrm{r\,sym}} = 0$ is identity for right-symmetric algebra A, then $s_k^l = 0$, $s_k^{(l,r)} = 0$, are also identities for A.

Our main result is the following. In the case of right-symmetric Witt algebras of any rank n for left multiplication operators the *standard associative polynomial identity of degree* $2n$ holds,

$$\sum_{\sigma\in \mathrm{Sym}_{2n}} \mathrm{sign}\,\sigma\, l_{a_{\sigma(1)}} \cdots l_{a_{\sigma(2n)}} = 0,$$

or in terms of the multiplication $\circ$, the following *left polynomial identity of degree* $2n+1$ is valid:

$$\sum_{\sigma\in\mathrm{Sym}_{2n}} \operatorname{sign}\sigma\, a_{\sigma(2n)}\circ(a_{\sigma(2n-1)}\circ\cdots\circ(a_{\sigma(2)}\circ(a_{\sigma(1)}\circ a_0))\cdots)=0.$$

We prove that in the space of left polynomial identities the degree $2n+1$ for right-symmetric Witt algebras of rank n is *minimal*. We also prove that any left polynomial identity of right-symmetric Witt algebras of rank n of minimal degree *follows from standard polynomial*, if $p\neq 2$. More exactly, the space of polynomial identities of minimal degree of right-symmetric Witt algebras W_n, $p=0$, $(W_n(\mathbf{m}), p>2)$ with N variables is $N\binom{N}{2n}$-dimensional. If $N=2n+1$, this space is $(2n+1)^2$-dimensional and has a basis consisisting of polynomials $s^l_{2n}, s^{l,r}_{2n}$, $l,r=1,2,\dots,2n+1$, $l\neq r$.

The main tool used here is a functor E constructed in [4]. It allows one to extend module structures, invariants, and identities of matrix algebras to module structures, invariants, and identities of Witt algebras. It should be mentioned that Witt algebras are considered not only in the Lie sense, but also in sense of right-symmetric algebras. In particular, the Novikov identity (1) that coincides with the identity $s_2^{\mathrm{r\,sym}}(a_1,a_2,a_3)=0$ is an extension of the Amitsur–Levitski identity for Mat_1; i.e., it is a prolongation of the commutativity condition of the main field. The generalized Novikov identity is an extension of the Amitsur–Levitski identity from matrix algebras to the right-symmetric Witt algebras (see Lemma 4.2). So, we can consider right-symmetric algebras that satisfy the standard right-symmetric identity,

$$\sum_{\sigma\in\mathrm{Sym}2n} \operatorname{sign}\sigma\, a_{\sigma(1)}\circ a_{\sigma(2)}\circ\cdots a_{\sigma(2n)}\circ a_{2n+1}=0, \tag{3}$$

as a generalization of Novikov algebras. This class of algebras includes Witt algebras in the case of many variables.

The identity (1) is true for any associative commutative algebra U and for a set $\mathcal{D}$ with one derivation ∂_1. The identity (3) holds for any associative commutative algebra U with a set of n commuting derivations $\mathcal{D}=\{\partial_i, i=1,\dots,n\}$. Moreover, it holds for some right-symmetric deformations of Witt algebras.

Regarding deformations of right-symmetric algebras and the description of local deformations of $A=W_n^{\mathrm{r\,sym}}, W_n^{+\mathrm{r\,sym}}$, if $p=0$, or $W_n(\mathbf{m})$, if $p>0$ (see [6, 7]), as an example let us give some right-symmetric deformations of $A=W_1=\{a=u\partial : u\in\mathcal{K}[x^{\pm1}]\}$.

The space of local deformations of A is four-dimensional and generated by classes of the following right-symmetric 2-cocycles:

$$\psi^1(u\partial, v\partial)=x^{-1}uv\partial,$$

$$\psi^2(u\partial, v\partial)=x^{-1}\partial(u)v\partial,$$

$$\psi^3(u\partial, v\partial) = (u - x\partial(u))\partial(v)\partial,$$
$$\psi^4(u\partial, v\partial) = \partial(u)\partial(v)\partial.$$

If $\Psi_1 = \sum_{k=1}^4 \varepsilon_k \psi^k$ is a 4-parametrical local deformation of A, is it possible to construct prolongations $\Psi_l = \sum_{|\mathbf{i}|=l} \epsilon^{\mathbf{i}} \psi^{\mathbf{i}}$, where $\mathbf{i} = (i_1, i_2, i_3, i_4)$, $|\mathbf{i}| = i_1 + i_2 + i_3 + i_4$? In other words, is it possible to find $\psi^{\mathbf{i}} \in C^2_{\mathrm{r\,sym}}(A, A)$ such that a new multiplication,

$$a \circ_\varepsilon b = a \circ b + \sum_l \Psi_l,$$

will be a right-symmetric multiplication over a field $\mathscr{K}((\varepsilon))$? The answer is: Ψ_1 can be prolongated to a global deformation if and only if $\varepsilon_1\varepsilon_4 + \varepsilon_2\varepsilon_3 = 0$. We give prolongation formulas for some special cases.

The local deformation $\varepsilon_1\psi^1 + \varepsilon_2\psi^2$ of W_1 has a trivial prolongation,

$$(u\partial, v\partial) \mapsto \partial(u)v + \varepsilon_1 x^{-1} uv + \varepsilon_2 x^{-1}\partial(u)v,$$

and is a right-symmetric multiplication. This algebra was obtained by Osborn [11]. Notice that cocycles ψ^3, ψ^4 do not satisfy left-commutativity identity. They are not Novikov cocycles [2]. Each of these cocycles has the following prolongations:

$$(a, b) \mapsto \partial(a)b + \varepsilon_3[x\partial, a]\left(\sum_i \varepsilon_3^i(-1)^i x^i \partial^i/\{(\varepsilon_3 + 1)\cdots(i\varepsilon_3 + 1)\}\right)\partial(b), \tag{4}$$

$$(a, b) \mapsto \partial(a)\sum_i \varepsilon_4^i \partial^i(b).$$

In the case of (4) we should change expressions like $(i\varepsilon_3 + 1)^{-1}$ to the formal series $1 - i\varepsilon_3 + i^2\varepsilon_3^2 - i^3\varepsilon_3^3 + \cdots$. Then we obtain a formal power series

$$\begin{aligned}(a, b) \quad \mapsto \quad & \partial(a)b + \varepsilon_3[x\partial, a]\partial(b) \\ & -\varepsilon_3^2[x\partial, a](x\partial^2 + x^2\partial^3)(b) \\ & +\varepsilon_3^3[x\partial, a](x\partial^2 + 3x^2\partial^3 + x^3\partial^4)(b) \\ & -\varepsilon_3^4[x\partial, a](x\partial^2 + 7x^2\partial^3 + 6x^3\partial^4 + x^4\partial^5)(b) \\ & +\varepsilon_3^5[x\partial, a](x\partial^2 + 63x^2\partial^3 + 25x^3\partial^4 + 10x^4\partial^5 \\ & \qquad + x^5\partial^6)(b) + \cdots .\end{aligned}$$

This is one of the prolongations of the local deformation $\varepsilon_3[x\partial, a]\partial(b)$. It would be interesting to construct prolongation formulas for a linear combination of cocycles and find polynomial identities of the obtained algebras.

It is also interesting to find right polynomial identities of right-symmetric algebras. Right multiplication operators satisfy Lie algebraic conditions and in this case one can expect Lie algebraic difficulties (see [13, 14]). Let us mention that an identity of degree 5 for the Lie algebra W_1^+ is true also for right multiplication operators:

$$\sum_{\sigma\in \mathrm{Sym}_4} \operatorname{sign}\sigma\, r_{a_{\sigma(1)}} r_{a_{\sigma(2)}} r_{a_{\sigma(3)}} r_{a_{\sigma(4)}} = 0.$$

Moreover, for right-symmetric Witt algebra $U\mathcal{D}$, $\mathcal{D} = \{\partial\}$, the following right polynomial identity of degree 3 holds:

$$\sum_{\sigma\in \mathrm{Sym}_3} \operatorname{sign}\sigma (a_{\sigma(1)} \circ a_{\sigma(2)}) \circ a_{\sigma(3)} = 0.$$

This identity follows from the Novikov identity (1).

For a right-symmetric algebra $U\mathcal{D}$, where $\mathcal{D} = \{\partial_1, \partial_2\}$, the following right polynomial identity of degree 7 is true:

$$\sum_{\sigma\in \mathrm{Sym}_7} \operatorname{sign}\sigma (((((a_{\sigma(1)} \circ a_{\sigma(2)}) \circ a_{\sigma(3)}) \circ a_{\sigma(4)}) \circ a_{\sigma(5)}) \circ a_{\sigma(6)}) \circ a_{\sigma(7)} = 0.$$

We have checked it in Mathematica, which takes about 2 hours. Via Maple we have checked that the minimal degree of a right polynomial identity for $\mathcal{D} = \{\partial_1, \partial_2, \partial_3\}$ is 14. I am deeply grateful to C. Löfwall who wrote me a program in Mathematica. It allows me to prove that the minimal right polynomial identity for $\mathcal{D} = \{\partial_1, \partial_2, \partial_3, \partial_4\}$ has degree 23.

Conjecture. The minimal degree of polynomial identities for right multiplications of right-symmetric Witt algebra W_n, $n > 1$, is $n^2 + 2n - 1$.

We do not know if is it possible to obtain minimal right polynomial identity from the left standard identity. As we mentioned above, the answer to this question in the case of $n = 1$ is positive.

Polynomials corresponding to pre-minimal identities also have interesting properties. For example, the standard polynomial for right multiplication operators,

$$s_k(a_1, \dots, a_k) = \sum_{\sigma\in \mathrm{Sym}_k} \operatorname{sign}(\sigma)\, r_{a_{\sigma(1)}} \cdots r_{a_{\sigma(k)}},$$

for $k = n^2 + 2n - 2$ induces k-ary operation in a space of derivations. For any derivations $D_1, \dots, D_{n^2+2n-2}$ in n variables the operator $s_{n^2+2n-2}(D_1, \dots, D_{n^2+2n-2})$ is also a derivation. In particular, a subspace of first order differential operators $\{\sum_{i=1}^n u_i\partial_i : i = 1, \dots, n\}$ of an algebra of differential operators has $(n^2 + 2n - 2)$-ary operation $s^{ass}_{n^2+2n-2}$. Here $u_1, \dots, u_n$ belong to any associative commutative algebra U with commuting derivations $\partial_1, \dots, \partial_n$. It is well known that the space of derivations in general is not close under the composition operation $(D_1, D_2) \mapsto D_1 D_2$, but it is close under the commutator,

$$s_2(D_1, D_2) = D_1 D_2 - D_2 D_1 = D_2 \circ D_1 - D_1 \circ D_2 \in \operatorname{Der} U,$$

for any $D_1, D_2 \in \operatorname{Der} U$. So, $s^{ass}_{n^2+2n-2}$ can be considered as a nontrivial generalized commutator on the space of first order differential operators in n variables. In this sense a minimal right polynomial identity for right-symmetric algebras can be interpretered as a Jacobi identity for generalized commutators. The number $n^2 + 2n - 2$ here cannot be improved: $s_k = 0$ is an identity for Witt algebras, if $k > n^2 + 2n - 2$, and a differential operator $s_k(D_1, \ldots, D_k)$ can have an order of more than 1 for some differential operators of first order $D_1, \ldots, D_k$, if $k < n^2 + 2n - 2$.

These statements require long computer calculations. For example, for constructing the generalized commutator s_{13} for $\mathcal{D} = \{\partial_1, \partial_2, \partial_3\}$ we spent about half a year. I am deeply grateful to the Mittag-Leffler Institut, and to Proffessor J. Mickelsson and K. O. Widman, the organizers of the program "Topology and geometry of quantum fields," for the beautiful possibility to do computer calculations.

A complete text about right polynomial identities and generalized commutators will be published elsewhere.

2. RIGHT-SYMMETRIC ALGEBRAS

Let A be an algebra with multiplication $A \times A \to A$, $(a, b) \mapsto a \circ b$. Let $(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$ be an associator of elements $a, b, c \in A$. Associative algebras are defined by condition $(a, b, c) = 0$, for any $a, b, c \in A$. Right-symmetric algebras are defined by identity

$$(a, b, c) = (a, c, b),$$

i.e., by identity

$$a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b.$$

The left-symmetric identity is

$$(a, b, c) = (b, a, c).$$

There is a one-to-one correspondence between right-symmetric and left-symmetric algebras. Namely, if $(a, b) \to a \circ b$ is right(left)-symmetric, then a new multiplication, $(a, b) \mapsto b \circ a$, is left(right)-symmetric. In our paper left-symmetric algebras are not considered. Right-symmetric algebras are sometimes called Vinber or Vinberg–Kozsul algebras [15, 8]. Right-symmetric algebras are Lie-admissible, i.e., under the commutator $[a, b] = a \circ b - b \circ a$, we obtain a Lie algebra.

Any associative algebra is right-symmetric. In a such cases, we will use notations like A^{ass}, if we consider A as an associative algebra, and $A^{\mathrm{r\,sym}}$, if we consider A as a right-symmetric algebra. Similarly, for a right-symmetric algebra A the notation $A^{\mathrm{r\,sym}}$ will mean that we use only right-symmetric structure on A, and A^{lie} stands for a Lie algebra structure under the commutator $(a, b) \mapsto [a, b]$.

In terms of operators of right multiplication r_a and left multiplication l_a of the algebra A,

$$ar_b := a \circ b, \qquad al_b := b \circ a,$$

right-symmetry identities are equivalent to the following conditions:

$$[r_a, r_b] - r_{[a,b]} = 0,$$
$$[r_a, l_b] - l_a l_b + l_{b\circ a} = 0.$$

Let $A_r = \{a_r : a \in A\}$ and $A_l = \{a_l : a \in A\}$ be two copies of A, and let $\mathcal{A} = A_r \oplus A_l$ be their direct sum. Let $T(\mathcal{A}) = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{A} \otimes \mathcal{A} \oplus \cdots$ be the tensor algebra of $\mathcal{A}$. *A universal enveloping algebra* of A, denoted by $U(A)$, is defined as a factor-algebra of $T(A_r \oplus A_l)$ over an ideal generated by $[a_r, b_r] - [a, b]_r, [a_r, b_l] - a_l b_l + (b \circ a)_l$. Denote elements of $U(A)$ corresponding to a_r, a_l by r_a, l_a.

A vector space M with two actions of A,

$$(M, A) \to M, \qquad (m, a) \mapsto ma$$

(left action) and

$$(A, M) \to M, (a, m) \mapsto am$$

(right action), is called a *module* over a right-symmetric algebra A, if

$$(ma)b - m(a \circ b) = (mb)a - m(b \circ a),$$
$$(am)b - a(mb) = (a \circ b)m - a(bm),$$

for any $a, b, c \in A$, and $m \in M$. Notice that the right action induces a right Lie-module structure on M,

$$m[a, b] = (ma)b - (mb)a, \qquad a, b \in A, \quad m \in M.$$

A module with trivial left action, $am = 0$, for any $a \in A, m \in M$, is called *right antisymmetric*.

Let B be a subalgebra of A. Let

$$Z_A^{l.ass}(B) = \{a \in A : (a, b_1, b_2) = 0, \ \forall b_1, b_2 \in B\},$$
$$Z_A^{r.ass}(B) = \{a \in A : (b_1, b_2, a) = 0, \ \forall b_1, b_2 \in B\},$$

be *left* and *right associative centralizers of B in A*. Let

$$N_A^{l.ass}(B) = \{a \in A : (a, b_1, b_2) \in B, \forall b_1, b_2 \in B\},$$
$$N_A^{r.ass}(B) = \{a \in A : (b_1, b_2, a) \in B, \forall b_1, b_2 \in B\},$$

be the *left* and *right normalizers of B in A*. It is clear that

$$Z_A^{l.ass}(B) \subseteq N_A^{l.ass}(B),$$
$$Z_A^{r.ass}(B) \subseteq N_A^{r.ass}(B).$$

Let

$$Z_A^{\text{left}}(B) = \{a \in A : a \circ b = 0, \ \forall b \in B\},$$
$$Z_A^{\text{right}}(B) = \{a \in A : b \circ a = 0, \ \forall b \in B\},$$

be *left* and *right centralizers of B in A* and

$$N_A^{\text{left}}(B) = \{a \in A : a \circ b \in B, \ \forall b \in B\},$$
$$N_A^{\text{right}}(B) = \{a \in A : b \circ a \in B, \ \forall b \in B\},$$

be *left* and *right normalizers of B in A.* We have

$$Z_A^{\text{left}}(B) \subseteq Z_A^{l.ass}(B),$$
$$Z_A^{\text{right}}(B) \subseteq Z_A^{r.ass}(B),$$
$$N_A^{\text{left}}(B) \subseteq N_A^{l.ass}(B),$$
$$N_A^{\text{right}}(B) \subseteq N_A^{r.ass}(B).$$

For the left cases and if $A = B$ we reduce these notations: $Z(A) = Z_A^{\text{left}}(A)$, $N(A) = N_A^{\text{left}}(A)$, $Z^{\text{right}}(A) = Z_A^{\text{right}}(A)$, $N^{\text{right}}(A) = N_A^{\text{right}}(A)$, $Z^{l.ass}(A) = Z_A^{l.ass}(A), N^{l.ass}(A) = N_A^{l.ass}(A)$, $Z^{r.ass}(A) = Z_A^{r.ass}(A)$, $N^{r.ass}(A) = N_A^{r.ass}(A)$.

We call $Z(A)$ and $Z^{\text{right}}(A)$ the *left* and *right centers* of A. We also call $Z^{l.ass}(A)$ and $Z^{r.ass}(A)$ *left* and *right associative centers* of A.

Left (right) associative centers are closed under multiplication $\circ$. To see this let us consider for simplicity the case of left associative centers. Suppose that $X, Y \in Z^{l.ass}(A)$. Then according to the right-symmetric identity

$$\begin{aligned}(X \circ Y) \circ (a \circ b) &= X \circ (Y \circ (a \circ b)) = X \circ ((Y \circ a) \circ b) \\ &= (X \circ (Y \circ a)) \circ b = ((X \circ Y) \circ a) \circ b.\end{aligned}$$

So, $X \circ Y \in Z^{l.ass}(A)$, and $Z^{\text{left}}(A)$ is a subalgebra of A.

Notice that $Z(A)$ and $N(Z(A))$ are also subalgebras of A:

$$(z_1 \circ z_2) \circ a = z_1 \circ (z_2 \circ a) - (z_1, a, z_2) = 0,$$
$$((n_1 \circ n_2) \circ z) \circ a = (n_1 \circ (n_2 \circ z)) \circ a - (n_1, z, n_2) \circ a = 0,$$

for any $a \in A, z_1, z_2 \in Z(A), n_1, n_2 \in N(Z(A))$. The same is true for $Z^{\text{right}}(A)$ and $N^{\text{right}}(Z^{\text{right}}(A))$.

PROPOSITION 2.1. *If $z \in Z(A)$, then r_z is a derivation of A.*

Proof. Since $z \circ b = 0$, we have $a \circ (z \circ b) = 0$. According to the right-symmetric identity

$$(a \circ b) \circ z = a \circ (b \circ z) + (a \circ z) \circ b,$$

for any $a, b \in A$. ∎

PROPOSITION 2.2. *For any $z \in Z(A)$, $a \in N(Z(A))$, and for any $b \in A$,*

$$a \circ (b \circ z) = (a \circ b) \circ z.$$

Proof. Let $z \in Z(A)$. Then $z \circ b = 0$, and $a \circ (z \circ b) = 0$. Let $a \in N(Z(A))$. Then $(a \circ z) \circ b = 0$. So,

$$a \circ (b \circ z) - (a \circ b) \circ z = a \circ (z \circ b) - (a \circ z) \circ b = 0.$$

∎

COROLLARY 2.3. *For $N = N(Z(A))$,*

$$Z^{\text{left}}(A) \subseteq Z^{r.ass}(N).$$

Proof. Evident.

COROLLARY 2.4. *For any $a_1, \ldots, a_{n-1} \in N(Z(A))$, and $a_n \in A$, $z \in Z(A)$, the following relation holds*:

$$a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n \circ z = (a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n) \circ z.$$

Proof. For $n = 2$, the statement follows from Proposition 2.2. Suppose that this is also true for $n - 1$. Then by Proposition 2.2

$$\begin{aligned} &a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ (a_n \circ z)) \cdots) \\ &\quad = a_1 \circ \{(a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots) \circ z\} \\ &\quad = \{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ z. \end{aligned}$$

∎

PROPOSITION 2.5. *Let U be a right antisymmetric A-module and let*

$$A \cup U \to A$$

be a pairing of A-modules,

$$\begin{aligned} a \circ (b \cup u) &= (a \circ b) \cup u, \\ (a \cup u) \circ b &= (a \circ b) \cup u + a \circ (u \circ b), \end{aligned} \tag{5}$$

for any $a, b \in A$, $u \in U$ (about cup-products see [6]*). Suppose that any element of A can be presented by a cup-product as $z \cup u$, for some $u \in U$ and $z \in Z(A)$. Then for any $a_1, \ldots, a_{n-1} \in N(Z(A))$, and $a_n \in A$,*

$$a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n \circ a = (a_1 \circ a_2 \circ \cdots a_{n-1} \circ a_n) \circ a.$$

Proof. Let $a = z \cup u$. Then by (5), and Corollary 2.4,

$$\begin{aligned} & a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ (a_n \circ a)) \cdots) \\ &\quad = a_1 \circ \{(a_2 \circ \cdots (a_{n-1} \circ ((a_n \circ z) \cup u))\} \\ &\quad = a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ (a_n \circ z)) \cdots) \cup u \\ &\quad = \{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ z) \cup u \\ &\quad = \{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ (z \cup u) \\ &\quad = \{a_1 \circ (a_2 \circ \cdots (a_{n-1} \circ a_n) \cdots)\} \circ a. \end{aligned}$$

∎

PROPOSITION 2.6. $Z^{l.ass}(A) \subseteq N(Z(A))$.

Proof. Let $a \in Z^{l.ass}(A)$. Then for any $z \in Z(A)$, and any $b \in A$,

$$(a \circ z) \circ b = a \circ (z \circ b) - (a, b, z) = 0.$$

∎

EXAMPLE 1. Any associative algebra is right-symmetric. As associative algebras the matrix algebra Mat_n give us examples of right-symmetric algebras.

EXAMPLE 2. Let U be an associative algebra with commuting derivations $\mathscr{D} = \{\partial_i, i = 1, \ldots, n\}$. Then an algebra of derivations $UD = \{u\partial_i : u \in U, \partial_i \in \mathscr{D}\}$ with multiplication $u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i$ is right-symmetric. Since the Lie algebras corresponding to $U\mathscr{D}$ are Witt algebras, i.e.,

$$[u\partial_i, v\partial_j] = -u\partial_i \circ v\partial_j + v\partial_j \circ u\partial_i = u\partial_i(v)\partial_j - v\partial_j(u)\partial_i,$$

we call such algebras right-symmetric Witt algebras.

Let Γ_n be a set of n-typles $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_i are integers. Let Γ_n^+ be its subset consisting of such α that $\alpha_i \geq 0$, $i = 1, \dots, n$. In the case of $p = \operatorname{char} \mathcal{K} > 0$, we consider a subset $\Gamma_n(\mathbf{m}) = \{\alpha : 0 \leq \alpha_i < p^{m_i}, i = 1, \dots, n\}$, where $\mathbf{m} = (m_1, \dots, m_n)$, $m_i > 0$, $m_i \in \mathbf{Z}$, $i = 1, \dots, n$.

For $\operatorname{char} \mathcal{K} = 0$ suppose that

$$U = \mathcal{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \{x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i} : \alpha \in \Gamma_n\}$$

is an algebra of Laurent polynomials and

$$U^+ = \mathcal{K}[x_1, \dots, x_n] = \{x^\alpha : \alpha \in \Gamma_n^+\},$$

its subalgebra of polynomials.

Let

$$O_n(\mathbf{m}) = \{x^{(\alpha)} = \prod_i x_i^{(\alpha_i)} : \alpha \in \Gamma_n(\mathbf{m}), i = 1, \dots, n\}$$

be a divided power algebra if $\operatorname{char} \mathcal{K} = p > 0$. Recall that $O_n(\mathbf{m})$ is p^m-dimensional and the muliplication is given by

$$x^{(\alpha)} x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha+\beta)},$$

where $m = \sum_i m_i$, and

$$\binom{\alpha + \beta}{\alpha} = \prod_i \binom{\alpha_i + \beta_i}{\alpha_i}, \qquad \binom{n}{l} = \frac{n!}{l!(n-l)!}, \ n, l \in \mathbf{Z}_+.$$

Let $\epsilon_i = (0, \dots, \underset{i}{1}, \dots, 0)$. Define ∂_i as a derivation of U,

$$\partial_i(x^\alpha) = \alpha_i x^{\alpha - \epsilon_i}, \qquad p = 0,$$
$$\partial_i(x^{(\alpha)}) = x^{(\alpha - \epsilon_i)}, \qquad p > 0.$$

Denote the right-symmetric algebras $U\mathcal{D}$, $U^+\mathcal{D}$ for $U = \mathcal{K}[x^{\pm 1}, \dots, x_n^{\pm 1}]$ as $W_n^{\mathrm{r\,sym}}$ and $W_n^{+\mathrm{r\,sym}}$. Similarly, denote the right-symmetric algebra $O_n(\mathbf{m})\mathcal{D}$ as $W_n(\mathbf{m})^{\mathrm{r\,sym}}$.

As in the case of Lie algebras, $A = W_n^{\mathrm{r\,sym}}$ has a grading

$$A = \oplus_k A_k, \qquad A_k \circ A_l \in A_{k+l}, \qquad k, l \in \mathbf{Z},$$
$$A_k = \{x^\alpha \partial_i : |x^\alpha| = |\alpha| = \sum_{i=1}^{n} \alpha_i = k + 1\}.$$

This grading induces gradings on $W_n^{+\mathrm{r\,sym}}$ and $W_n^{\mathrm{r\,sym}}(\mathbf{m})$.

EXAMPLE 3. Let A be an associative algebra, $C^*(A, A) = \oplus_k C^k(A, A)$, and $C^k(A, A) = \{\psi : A \times \cdots \times A \to A\}$ be a space of polylinear maps with k-arguments, if $k > 0$, $C^0(A, A) = A$, and $C^k(A, A) = 0$, if $k < 0$. Endow $C^*(A, A)$ by a "shuffle product" multiplication

$$C^*(A, A) \times C^*(A, A) \to C^{*-1}(A, A).$$

Corresponding to $\psi \in C^{k+1}(A, A)$, $\phi \in C^{l+1}(A, A)$, their shuffle product $\psi \circ \phi \in C^{k+l+1}(A, A)$, $k, l \geq 0$ is given by

$$\begin{aligned}\psi \circ \phi(a_1, \dots, a_{k+l+1}) \\ &= \sum \psi(\phi(a_{\sigma(1)}, \dots, a_{\sigma(k+1)}), a_{\sigma(k+2)}, \dots, a_{\sigma(k+l+1)}),\end{aligned}$$

where the summation is over permutations $\sigma \in \mathrm{Sym}_{k+l+1}$, such that $\sigma(1) < \cdots < \sigma(k+1)$, $\sigma(k+2) < \cdots < \sigma(k+l+1)$.

Let $\varepsilon_k \in C^{k+1}(A, A)$, $k \geq 0$, be a standard skew-symmetric polynomial with shifted index: $\varepsilon_k = s^{ass}_{k+1}$.

Then [3]

$$\varepsilon_i \circ \varepsilon_{2k} = (i+1)\varepsilon_{2k+i},$$

$$\varepsilon_{2k+1} \circ \varepsilon_{2l+1} = 0,$$

$$\varepsilon_{2k} \circ \varepsilon_{2l+1} = \varepsilon_{2k+2l+1},$$

for any $k, l, i \geq 0$.

Therefore, the algebra of standard polynomials under the shuffle product is isomorphic to the right-symmetric algebra $A_0 \oplus A_1$, such that

$$A_0 = \{e_i : i \geq 0\}, \qquad A_0 \circ A_0 \subseteq A_0,$$

$$e_i \circ e_j = (i + 1/2)e_{i+j}, \qquad 0 \leq i, j,$$

$$A_1 = \{x^{j+1} : j \geq 0\}, \qquad A_1 \circ A_1 = 0,$$

$$A_0 \circ A_1 \subseteq A_1, \qquad A_1 \circ A_0 \subseteq A_1,$$

$$x^{i+1} \circ x^{j+1} = 0, \qquad e_i \circ x^{j+1} = (1/2)x^{i+j+1}, \qquad i, j \geq 0.$$

This isomorphism is given by

$$e_i \mapsto \varepsilon_{2i}/2, \qquad x^{j+1} \mapsto \varepsilon_{2j+1},$$

where $i, j = 0, 1, 2, \dots$.

This algebra has also a multiplication $\cup : C^*(A, A) \otimes C^*(A, A) \to C^*(A, A)$, which called a cup-product,

$$\begin{aligned}\psi \cup \phi(a_1, \dots, a_{k+l}) \\ &= \sum_{\substack{\sigma \in \mathrm{Sym}_{k+l}, \sigma(1) < \cdots < \sigma(k), \\ \sigma(k+1) < \cdots < \sigma(k+l)}} \psi(a_{\sigma(1)}, \dots, a_{\sigma(k)})\phi(a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)}).\end{aligned}$$

Then

$$\varepsilon_i \cup \varepsilon_j = \varepsilon_{i+j+1},$$

for any $i, j \geq 0$. In particular, a subalgebra generated by ε_i, is a commutative associative algebra under a cup-product. These multiplications satisfy the conditions

$$(\varepsilon_i \cup \varepsilon_j) \circ \varepsilon_k = (-1)^{k(j-1)}(\varepsilon_i \circ \varepsilon_k) \cup \varepsilon_j + \varepsilon_i \cup (\varepsilon_j \circ \varepsilon_k),$$
$$(a \cup b) \cup c = a \cup (b \cup c),$$
$$a \cup b = b \cup a,$$

where $b \in C^k(A, A)$, $c \in C^j(A, A)$. So, an algebra of standard polynomials has the structure of a Poissson–Novikov algebras in the sense of [5].

3. RIGHT-SYMMETRIC IDENTITIES

Right-symmetric algebras are not associative. They are not even power-associative. In defining polynomials for right-symmetric algebras one should fix the positions of the brackets. Further, expressions like $t_1 \circ t_2 \circ \cdots \circ t_{k-1} \circ t_k$ will mean a right normed element $t_1 \circ (t_2 \circ (\cdots \circ (t_{k-1} \circ t_k) \cdots))$. For any finite sequence of integers $\mathbf{i} = (i_1, i_2, \ldots, i_k)$, with $i_l = 1, \ldots, N$, and $l = 1, \ldots, k$, where k is any integer, set $t^{\mathbf{i}} = t_{i_1} \circ \ldots \circ t_{i_k}$. Elements of the form $\mathbf{T} = \lambda_{\mathbf{i}} t^{\mathbf{i}}$, where $\lambda_{\mathbf{i}} \in \mathcal{K}$, are called *(left) monomials* with N variables and t_{i_k} is called as the *head* of the monomial $\mathbf{T}$. If $\lambda_{\mathbf{i}} = 0$, then the monomial $t^{\mathbf{i}}$ is called trivial. If we would like to pay attention to monomials with $\lambda_{\mathbf{i}} \neq 0$, then we call $\lambda_{\mathbf{i}} t^{\mathbf{i}}$ a nontrivial monomial. The sum of monomials is called a (left) polynomial. For a polynomial

$$f = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} t^{\mathbf{i}} \in R_N^{\text{left}},$$

we will say that *f has a monomial $\lambda_{\mathbf{i}} t^{\mathbf{i}}$*, if $\lambda_{\mathbf{i}} \neq 0$. If a (left) polynomial g is a sum of some monomials of f, then g is called a *part* of f.

A space of (left) polynomials in variables $t_1, \ldots, t_N$ is denoted by $R^{\text{left}}[t_1, \ldots, t_N]$ or simply R_N^{left}. Suppose that $f \in R_N^{\text{left}}$, i.e., $f = f(t_1, \ldots, t_N)$, and $f \notin R_{N-1}^{\text{left}}$. In such cases we will say that f depends on N variables.

For a monomial $t^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \ldots, i_k)$, we denote by $\deg_{t_r}(t^{\mathbf{i}})$ the number of indices $i_1, \ldots, i_k$ that are equal to r. This number is called the *t_r-degree* of $t^{\mathbf{i}}$. If $\deg_{t_r} t^{\mathbf{i}} > 0$, and $\lambda_{\mathbf{i}} \neq 0$, then t_r is said to be *an essential* variable for $\lambda_{\mathbf{i}} t^{\mathbf{i}}$ and $\lambda_{\mathbf{i}} t^{\mathbf{i}}$ is then called r-essential. Let R be a subset of the set $\{1, \ldots, N\}$. A nontrivial monomial $\lambda_{\mathbf{i}} t^{\mathbf{i}}$ is called *R-essential*, if it is

r-essential, for any $r \in R$. A left polynomial $f \in R_N^{\text{left}}$ is called *R-essential* if all its nontrivial monomials are R-essential. If $R = \{1, \ldots, N\}$, then an R-essential polynomial is called *essential*. If $R = \{1, \ldots, r-1, r+1, \ldots, N\}$, then an R-essential polynomial is called *$\hat{r}$-essential.*

The maximal t_r-degree of monomials of f is called the *t_r-degree* of f and is denoted by $\deg_{t_r} f$. Let $\tau_r(f)$ be a sum of monomials with t_r-degree $\deg_{t_r} f$. Call it the *t_r-leading part* of f.

The *degree* of the nontrivial monomial $\lambda_{\mathbf{i}} t_{i_1} \circ \cdots \circ t_{i_k}$ is by definition k

$$\deg t^{\mathbf{i}} = |\mathbf{i}| = k.$$

The maximal degree of the nontrivial monomials of f is called the *degree* of the polynomial f,

$$\deg f = \max\{|\mathbf{i}| : \lambda_{\mathbf{i}} \neq 0\}.$$

Let

$$\Delta_{t_r, x, y}: R_N^{\text{left}}[t_1, \ldots, t_N] \to R_{N+1}^{\text{left}}[t_1, \ldots, t_{r-1}, x, y, t_{r+1}, \ldots, t_N]$$

be *a linearization operator* defined by the rule

$$\begin{aligned}\Delta_{t_r, x, y} & f(\ldots, t_{r-1}, x, y, t_{r+1}, \ldots) \\ &= f(\ldots, t_{r-1}, x+y, t_{r+1}, \ldots) - f(\ldots, t_{r-1}, x, t_{r+1}, \ldots) \\ &\quad - f(\ldots, t_{r-1}, y, t_{r+1}, \ldots).\end{aligned}$$

Observe that

$$\Delta_{t_r, x, y} f = 0,$$

if and only if f is linear in the rth variable or f does not depend on t_r, i.e., $\deg_{t_r} f \leq 1$. So, we have the following proposition.

PROPOSITION 3.1. *If*

$$\Delta_{t_r, t_{r,1}, t_{r,2}} \cdots \Delta_{t_r, t_r, t_{r,l-1}} \Delta_{t_r, t_r, t_{r,l}} f = 0,$$

then

$$\tau_r(f) = 0,$$

in other words

$$\deg_{t_r} f < l.$$

PROPOSITION 3.2. *Let* $f \in R_N^{\text{left}}[t_1, \ldots, t_N]$, *and let*

$$g = \Delta_{t_r,\, t_{r,1},\, t_{r,2}} \Delta_{t_r,\, t_r,\, t_{r,3}} \cdots \Delta_{t_r,\, t_r,\, t_{r,l}} f.$$

Then

$$g(\ldots, t_{r-1}, t_{r,1}, \ldots, t_{r,l}, t_{r+1}, \ldots) = g(\ldots, t_{r-1}, t_{r,\sigma(1)}, \ldots, t_{r,\sigma(l)}, t_{r+1}, \ldots),$$

for any $\sigma \in \mathrm{Sym}_l$.

Proof. We will argue by induction on l. If $l = 1$, the statement is trivial. Suppose that for $l - 1$ Proposition 3.2 is true and prove it for l. Let

$$h = \Delta_{t_r,\, t_r,\, t_{r,3}} \cdots \Delta_{t_r,\, t_r,\, t_{r,l}} f.$$

Then

$$g = \Delta_{t_r,\, t_{r,1},\, t_{r,2}} h.$$

For $i = 2$, we have

$$\begin{aligned} &g(\ldots, t_{r-1}, t_{r,1}, t_{r,2}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\quad = h(\ldots, t_{r-1}, t_{r,1} + t_{r,2}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\qquad - h(\ldots, t_{r-1}, t_{r,1}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\qquad - h(\ldots, t_{r-1}, t_{r,2}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\quad = h(\ldots, t_{r-1}, t_{r,2} + t_{r,1}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\qquad - h(\ldots, t_{r-1}, t_{r,2}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\qquad - h(\ldots, t_{r-1}, t_{r,1}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\quad = g(\ldots, t_{r-1}, t_{r,2}, t_{r,1}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots). \end{aligned}$$

So, if $l = 2$, the statement is established.

Consider the case $l > 2$. By the inductive hypothesis, g is symmetric in the variables $t_{r,3}, \ldots, t_{r,l}$. In order to simplify notations, set $x = t_{r,1}$, $y = t_{r,2}$, $t_{r,3}$, and instead of expressions like

$$g(\ldots, t_{r-1}, t_{r,1}, t_{r,2}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots),$$

write more simply $G(x, y, z)$. It is enough to prove that

$$\begin{aligned} &g(\ldots, t_{r-1}, t_{r,1}, t_{r,2}, t_{r,3}, \ldots, t_{r,l}, t_{r+1}, \ldots) \\ &\quad = g(\ldots, t_{r-1}, t_{r,1}, t_{r,3}, t_{r,2}, \ldots, t_{r,l}, t_{r+1}, \ldots), \end{aligned}$$

or in our short notations,

$$G(x, y, z) = G(x, z, y). \tag{6}$$

Let

$$q = \Delta_{t_r, t_r, t_{r,4}} \cdots \Delta_{t_r, t_r, t_{r,l}} f,$$

if $l > 3$, and

$$q = f,$$

if $l = 3$. Then

$$h = \Delta_{t_r, t_r, t_{r,3}} q.$$

We have

$$\begin{aligned} G(x, y, z) &= H(x + y, z) - H(x, z) - H(y, z) \\ &= Q(x + y + z) - Q(x + y) - Q(z) - Q(x + z) + Q(x) + Q(z) \\ &\quad -Q(y + z) + Q(y) + Q(z) \\ &= Q(x + y + z) - Q(x + y) - Q(x + z) - Q(y + z) \\ &\quad +Q(x) + Q(y) + Q(z) \end{aligned}$$

and

$$\begin{aligned} G(x, z, y) &= H(x + z, y) - H(x, y) - H(z, y) \\ &= Q(x + z + y) - Q(x + z) - Q(y) \\ &\quad -Q(x + y) + Q(x) + Q(y) \\ &\quad -Q(z + y) + Q(z) + Q(y) \\ &= Q(x + y + z) - Q(x + z) - Q(x + y) - Q(y + z) \\ &\quad +Q(x) + Q(y) + Q(z). \end{aligned}$$

So, (6) is established and therefore Proposition 3.2 is proved completely. ∎

Let $f \in R_N^{\text{left}}$. We will say that $f = 0$ is *a right-symmetric (left) identity* for the right-symmetric algebra A, if $f(a_1, \ldots, a_N) = 0$, for any $a_1, \ldots, a_N \in A$. Let $\rho = (i_1, \ldots, i_N)$, where $i_1, \ldots, i_N = 1, \ldots, N$ (indexes i_k and i_l may be equal for some k, l). Let f^ρ be a polynomial obtained from f by substitution $t_k \mapsto t_{i_k}, k = 1, \ldots, N$,

$$f^\rho(t_1, \ldots, t_N) = f(t_{i_1}, \ldots, t_{i_N}).$$

Notice that f^ρ is *a left* polynomial and if $f = 0$ is a polynomial identity on A, then $f^\rho = 0$ is also a left polynomial identity on A. In particular, if $s_k^{\text{r sym}} = 0$ is a right-symmetric left identity for A, then $s_k^l = 0$ and $s_k^{(l,r)} = 0$ are also right-symmetric left identities, where $l, r = 1, \ldots, N, l \neq r$.

Suppose that A is a graded right-symmetric algebra,

$$A = \oplus_i A_i, \qquad A_i \circ A_j \subseteq A_{i+j}.$$

Elements of A_i are called homogeneous elements of weight i. The notation $|a| = i$ will mean that a is a homogeneous element of A and $a \in A_i$. The element obtained from $t^{\mathbf{i}}$ by substituting $t_i := a_i$ we denote by $a^{\mathbf{i}}$. For a graded right-symmetric algebra A and for any monomial $t^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \ldots, i_k)$, it is evident that

$$a^{\mathbf{i}} \in A_{|a_{i_1}|+\cdots+|a_{i_k}|}. \tag{7}$$

In our paper we deal only with left polynomial identities.

The following left polynomial of R^{left}_{k+1} is called the *left standard polynomial* of degree $k+1$ for right-symetric algebras:

$$s_k^{\text{r sym}}(t_1, \ldots, t_{k+1}) = \sum_{\sigma \in \text{Sym}_k} \text{sign}\, \sigma t_{\sigma(1)} \circ \cdots \circ t_{\sigma(k)} \circ t_{k+1}.$$

Note that it can be considered as an associative standard polynomial of degree k for left multiplications in the universal enveloping algebra of right-symmetric algebras [6]. If

$$s_k^{ass}(l_{t_1}, \ldots, l_{t_k}) = \sum_{\sigma \in \text{Sym}_k} \text{sign}\, \sigma\, l_{t_{\sigma(1)}} \cdots l_{t_{\sigma(k)}},$$

then

$$s_k^{\text{r sym}}(t_1, \ldots, t_k, t_{k+1}) = (t_{k+1}) s_k^{ass}(l_{t_1}, \ldots, l_{t_k}).$$

Denote by $\mathcal{N}$ the set of indices $\{1, \ldots, N\}$. Let $I = \{h_1, \ldots, h_k\} \subseteq \mathcal{N}$. If I has $2n+1$ elements, one can consider standard left polynomials $s^r_{2n,I}, s^{(l,r)}_{2n,I} \in R^{\text{left}}_N$, having variables, indexed by elements of I,

$$s^r_{2n,I}(t_{h_1}, \ldots, t_{h_{2n+1}}) = \sum_{\sigma \in \text{Sym}_{2n}} \text{sign}\, \sigma\, t_{h_{\sigma(1)}} \circ \cdots \circ \hat{t}_{h_r} \cdots \circ t_{h_{\sigma(2n+1)}} \circ t_{h_r},$$

$$s^{(l,r)}_{2n,I}(t_{h_1}, \ldots, t_{h_{2n+1}}) = \sum_{\sigma \in \text{Sym}_{2n}} \text{sign}\, \sigma\, t_{h_{\sigma(1)}} \circ \cdots \hat{t}_{h_l} \cdots \circ t_{h_{\sigma(2n+1)}} \circ t_{h_r}.$$

If I has $2n$ elements and $r \in \mathcal{N}$, then one can consider the polynomial $s_{2n,I,r}$ in variables $t_{h_1}, \ldots, t_{h_{2n}}, t_r$, defined as follows:

$$s^r_{2n,I}(t_{h_1}, \ldots, t_{h_{2n}}, t_r) = \sum_{\sigma \in \text{Sym}_{2n}} \text{sign}\, \sigma\, t_{h_{\sigma(1)}} \circ \cdots \circ t_{h_{\sigma(2n)}} \circ t_r.$$

If $r \notin I$, and $J = I \cup \{r\}$, then $s_{2n,I,r}$ is a left polynomial with $2n+1$ variables indexed by a set $J = I \cup \{r\}$ and

$$s_{2n,I,r} = s^r_{2n,J}.$$

If $r \in I$, then $s_{2n,I,r}$ is a left polynomial in $2n$ variables indexed by I and $\deg_{t_r} s_{2n,I,r} = 2$, and

$$s_{2n,I,r} = s^{(l,r)}_{2n,J},$$

for any $l \notin I$, and $J = I \cup \{l\}$.

Note that the left polynomials $s_{2n,I,r}$, for $2n$ elements subset $I \subseteq \mathcal{N}$ and $r \in \mathcal{N}$ are defined uniquely by the pair (I, r). If $I' = \{i'_1, \ldots, i'_{2n}\}$ is another subset of the set $\mathcal{N}$ with $2n$ elements, $r' \in \mathcal{N}$, then

$$s^r_{2n,I}(t_{i'_1}, \ldots, t_{i'_{2n}}, t_{r'}) = 0,$$

for $(I, r) \neq (I', r')$.

The following is our main result.

THEOREM 3.3. *Let A be one of the following right-symmetric algebras $W_n^{\mathrm{r\,sym}}(p = 0)$, $W_n^{+\mathrm{r\,sym}}(p = 0)$, $W_n^{\mathrm{r\,sym}}(\mathbf{m})(p > 0)$.*

(i) *A satisfies the right-symmetric standard identity of degree $2n + 1$,*

$$\sum_{\sigma \in \mathrm{Sym}2n} \mathrm{sign}\,\sigma\, a_{\sigma(1)} \circ a_{\sigma(2)} \circ \cdots a_{\sigma(2n)} \circ a_{2n+1} = 0,$$

for any $a_1, \ldots, a_{2n+1} \in A$. In particular, the left polynomials s^l_{2n}, $s^{(l,r)}_{2n}$, $l, r = 1, \ldots, 2n + 1$, $l \neq r$, also give right-symmetric identities for A.

(ii) *A has no nontrivial left polynomial identity of degree less than $2n + 1$.*

(iii) $(p \neq 2)$. *The space of minimal left polynomial identities with N variables is $N\binom{N}{2n}$-dimensional and left polynomials $s_{2n,I,r}$, where I runs through all subspaces of $\mathcal{N}$ with $2n$ elements and $r \in \mathcal{N}$, generate a basis. In particular, the space of minimal polynomial identities of $2n + 1$ variables is $(2n + 1)^2$-dimensional and has a basis consisting of left polynomials s^l_{2n}, $s^{(l,r)}_{2n}$, $l, r = 1, \ldots, 2n + 1$, $l \neq r$.*

Proof of theorem 3.3 will be given in Section 4.

EXAMPLE. *$W_1^{\mathrm{r\,sym}}$ has no polynomial identity of degree 2. Any left polynomial f of degree 3 such that $f = 0$ is the identity on $W_1^{\mathrm{r\,sym}}$ is a linear combination of the following nine left polynomials*:

$$s^1_2(t_1, t_2, t_3) = t_2 \circ (t_3 \circ t_1) - t_3 \circ (t_2 \circ t_1),$$

$$s^2_2(t_1, t_2, t_3) = t_1 \circ (t_3 \circ t_2) - t_3 \circ (t_1 \circ t_2),$$

$$s^3_2(t_1, t_2, t_3) = t_1 \circ (t_2 \circ t_3) - t_2 \circ (t_1 \circ t_3),$$

$$s^{(1,2)}_2(t_1, t_2, t_3) = t_2 \circ (t_3 \circ t_2) - t_3 \circ (t_2 \circ t_2),$$

$$s_2^{(1,3)}(t_1,t_2,t_3) = t_2\circ(t_3\circ t_3) - t_3\circ(t_2\circ t_3),$$
$$s_2^{(2,1)}(t_1,t_2,t_3) = t_1\circ(t_3\circ t_1) - t_3\circ(t_1\circ t_1),$$
$$s_2^{(2,3)}(t_1,t_2,t_3) = t_1\circ(t_3\circ t_3) - t_3\circ(t_1\circ t_3),$$
$$s_2^{(3,1)}(t_1,t_2,t_3) = t_1\circ(t_2\circ t_1) - t_2\circ(t_1\circ t_1),$$
$$s_2^{(3,2)}(t_1,t_2,t_3) = t_1\circ(t_2\circ t_2) - t_2\circ(t_1\circ t_2).$$

4. IDENTITIES OF RIGHT-SYMMETRIC WITT ALGEBRAS

If otherwise is not stated, in this section A will denote $W_n^{\mathrm{r\,sym}}$, $W_n^{+\mathrm{r\,sym}}$, or $W_n^{\mathrm{r\,sym}}(\mathbf{m})$, $p>0$ and A^+ will denote $W_n^{+\mathrm{r\,sym}}$. Let $U=\mathscr{K}[x_1^{\pm1},\dots,x_n^{\pm1}]$, $\mathscr{K}[x_1,\dots,x_n]$, or $O_n(\mathbf{m})$, if $A=W_n, W_n^+$, or $W_n(\mathbf{m})$. Let

$$A_{-1}^+ = \{\partial_i : i=1,\dots,n\},$$
$$A_0^+ = \{x_i\partial_j : i,j,=1,\dots,n\}.$$

LEMMA 4.1 [6].

$$Z(A) = A_{-1}^+,$$
$$N(Z(A)) = A_{-1}^+\oplus A_0^+.$$

The subalgebra A_0^+ *is associative and isomorphic to* Mat_n. *In particular,* $N(Z(A))$ *has a subalgebra isomorphic to* Mat_n.

LEMMA 4.2. *For any* $u_k\partial_{i_1},\dots,u_k\partial_{i_k}, u\partial_r\in A$, *the following formula holds*:

$$s_k(u_1\partial_{i_1},\dots,u_k\partial_{i_k},u\partial_r)$$
$$= \textstyle\sum_{j_1,\dots,j_k=1}^n \partial_{j_1}(u_1)\cdots\partial_{j_k}(u_k)u\partial_r(s_k^{ass}(x_{j_1}\partial_{i_1},\dots,x_{j_k}\partial_{i_k})).$$

Proof. Let L be W_n as a Lie algebra. L acts on associative commutative algebra U as a derivation algebra:

$$l(uv) = l(u)v + ul(v), \qquad \forall l\in L, u,v\in U.$$

Morover, the adjoint L-module L has an additional structure of a module over U,

$$u\,(v\partial_i) = uv\partial_i,$$

such that

$$l(ul_1) = l(u)l_1 + l(ul_1), \qquad \forall l,l_1\in L, u\in U.$$

In the terminology of [4], L is an (L, U)-module with base L^+_{-1}. Let $C^k(L, L)$ be the space of polylinear maps with k arguments $L \times \cdots \times L \to L$ and let $C^k(L, L)^{L^+_{-1}}$ be the subspace of $L^+_{-1}-$invariants, i.e., the space of polylinear maps $\psi : L \times \cdots \times L \to L$, such that

$$\delta\psi(l_1, \dots, l_k) = \sum_{i=1}^{k} \psi(l_1, \dots, l_{i-1}, [\delta, l_i], l_{i+1}, \dots, l_k),$$

for any $l_1, \dots, l_k \in L$, $\delta \in L^+_{-1}$.

For an (L, U)-module M denote by $\bar{M}$ its base: $\bar{M} = \{m \in M : \delta(m) = 0, \forall\delta \in L^+_{-1}\}$. Recall that $\bar{M}$ has an $\mathscr{L}^+_0$-module structure induced by the projection to $\bar{M}$:

$$l(\bar{m}) = pr_{\bar{M}} l(m).$$

According to the results of [4], any cochain $\psi \in C^{k+1}(L, L)^{L^+_{-1}}$ can be reconstructed by its base $\bar{\psi} \in C^{k+1}(L^+, L^+_{-1})^{L^+_0}$, i.e., by the cochain

$$\bar{\psi}: L^+ \times \cdots \times L^+ \to L^+_{-1},$$

$$\bar{\psi}(l_1, \dots, l_{k+1}) = pr_{L^+_{-1}} \psi(l_1, \dots, l_{k+1}),$$

according to the following rule:

$$\psi(l_1, \dots, l_{k+1}) = \sum_{a_1, \dots, a_{k+1}} E_{a_1}(l_1) \cdots E_{a_{k+1}}(l_{k+1}) \bar{\psi}(a_1, \dots, a_{k+1}). \tag{8}$$

Here $a_1, \dots, a_{k+1}$ runs through all basic elements of L^+_0, if M is an (L, U)-module with height 1, i.e., $\mathscr{L}^+_1 \bar{M} = 0$ and linear maps $E_a: L \to U$ are defined by

$$E_{x^\alpha \partial_i}(v\partial_j) = \delta_{i,j} \partial^\alpha(v)/\alpha!.$$

Recall that E_a are defined uniquely by the condition

$$\delta E_a(l) = E_a([\delta, l]), \qquad \forall\delta \in L^+_{-1}, l \in L.$$

By Lemma 4.1,

$$s_k^{\text{r sym}} \in C^{k+1}(L, L)^{L^+_{-1}}.$$

Moreover,

$$s_k^{\text{r sym}} \in C^{k+1}(L, L)^{L^+_0}.$$

Note that $C^{k+1}(L^+, L^+)^{L^+_{-1}}$ is generated by L^+_0-invariants in $C^{k+1}(L^+, L^+_{-1})$ and $C^{k+1}(L^+, L^+_{-1})^{L^+_0}$ is generated by polylinear maps

$$L^+_{i_1} \times \cdots \times L^+_{i_k} \to L^+_{-1},$$

such that $i_1 + \dots + i_{k+1} = -1$.

By the grading property of right-symmetric algebra A, observe that the necessary conditions for homogeneous elements $a_1, \dots, a_k, a_{k+1} \in A^+$, to satisfy the condition

$$s_k^{\text{r sym}}(a_1, \dots, a_k, a_{k+1}) \in L_{-1}^+,$$

are

$$|a_{k+1}| = -1, \qquad |a_1| + \cdots + |a_k| = 0.$$

If $|a_i| = -1$, for some $i \le k$, then

$$s_k^{\text{r sym}}(a_1, \dots, a_k, a_{k+1}) = 0,$$

since $a_i \in Z(A)$ (see Lemma 4.1). So, $\bar{s}_k^{\text{r sym}}(a_1, \dots, a_{k+1}) = 0$, if the following conditions are not satisfied: $a_1, \dots, a_k \in L_0^+$, $a_{k+1} \in L_{-1}^+$. Therefore, $\bar{s}_k^{\text{r sym}}$ can be considered as a cochain of $C^{k+1}(L_0^+, L_{-1}^+)$, defined by the rule

$$\bar{s}_k^{\text{r sym}}(a_1, \dots, a_k, \partial_r) = \partial_r\Big(\sum_{\sigma \in \mathrm{Sym}_k} \operatorname{sign} \sigma \, a_{\sigma(1)} \circ \cdots \circ a_{\sigma(k)}\Big). \tag{9}$$

According to Lemma 4.1, for any $a_1, \dots, a_k \in L_0^+ = A_0^+ = \{x_i \partial_j : i, j = 1, \dots, n\}$, the right multiplications in $a_{\sigma(1)} \circ \cdots \circ a_{\sigma(k)}$ can be changed by associative muliplications corresponding to the usual multiplication of matrices. So, the right hand of (9) is equal to $s_k^{ass}(a_1, \dots, a_k)$.

It remains to take $\psi = s_k^{\text{r sym}}$ and use (8) and (9). ∎

COROLLARY 4.3. *If $k = 2n$, then $s_{2n} = 0$ is an identity for A.*

Proof. By the Amitsur–Levitski theorem,

$$s_k^{ass}(x_{j_1}\partial_{i_1}, \dots, x_{j_{2n}}\partial_{i_{2n}}) = 0,$$

for any $i_1, j_1, \dots, i_{2n}, j_{2n} = 1, \dots, n$. ∎

For a left polynomial $f = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} t^{\mathbf{i}} \in R_N^{\text{left}}$ define $\pi_r(f) \in R_N^{\text{left}}$ as follows:

$$\pi_r(f) = \sum_{i_1, \dots, i_k \neq r} \lambda_{i_1, \dots, i_k, r} t_{i_1} \circ \cdots \circ t_{i_k} \circ t_r.$$

Denote by $\bar{\pi}_r : R_N^{\text{left}} \to R_N^{\text{left}}$ an endomorphism of R_N^{left}, which corresponds to $f \in R_N^{\text{left}}$ a sum of its monomials that do not depend on t_r :

$$\bar{\pi}_r\Big(\sum_{i_1, \dots, i_k} \lambda_{i_1, \dots, i_k} t_{i_1} \circ \cdots \circ t_{i_k}\Big) = \sum_{i_1, \dots, i_k \neq r} \lambda_{i_1, \dots, i_k} t_{i_1} \circ \cdots t_{i_k}.$$

LEMMA 4.4. *Let $f \in R_{2n+1}^{\text{left}}$ and $f = 0$ be an identity for A. Then*

(i) *$\pi_r(f) = 0$ and $\bar{\pi}_r(f) = 0$ are also identities for A.*

(ii) *$\pi_r(f)$ is a scalar multiple of s_{2n}^r.*

In particular, the space of multilinear left polynomials, which are identities for A, is $(2n+1)$-dimensional and has a basis consisiting of left polynomials $s^r_{2n}, r = 1, \ldots, 2n+1$.

Proof. Notice that $a_{i_1} \circ \cdots \circ a_{i_k} = 0$, if one of the elements $a_{i_1}, \ldots, a_{i_{k-1}}$ belongs to $Z(A)$.

Take $a_r = z_r \in Z(A)$. Thus for any $a_i \in A$, we have

$$\begin{aligned} 0 &= f(a_1, \ldots, a_N) \\ &\Rightarrow \quad \pi_r(f)(a_1, \ldots, a_N) + \bar{\pi}_r(f)(a_1, \ldots, a_N) = 0. \end{aligned} \tag{10}$$

By Corollary 2.4, for any $a_i \in N(Z(A)), i \neq r, a_r \in Z(A)$,

$$\pi_r(f)(a_1, \ldots, a_N) = F_r(a_1, \ldots, \hat{a}_r, \ldots, a_N) \circ z_r,$$

for some $F_r \in R^{\text{left}}_{N-l}$ which does not depend on t_r. Namely,

$$\begin{aligned} &F_r(t_1, \ldots, \hat{t}_r, \ldots, t_N) \\ &\quad = \textstyle\sum_{k \leq 2n+1} \sum_{i_1, \ldots, i_k \neq r} \lambda_{i_1, \ldots, i_k, r} t_{i_1} \circ \cdots \circ t_{i_k}, \end{aligned}$$

if

$$\pi_r(f)(t_1, \ldots, t_N) = \sum_{k \leq 2n+1} \sum_{i_1, \ldots, i_k \neq r} \lambda_{i_1, \ldots, i_k, r} t_{i_1} \circ \cdots \circ t_{i_k} \circ t_r.$$

Thus, condition (10) can be written in the following way:

$$F_r(a_1, \ldots, \hat{a}_r, \ldots, a_N) \circ z_r + \bar{\pi}_r(f)(a_1, \ldots, a_N) = 0. \tag{11}$$

The first summand of the left hand of (11) depends linearly on z_r and the second does not depend on z_r. Therefore,

$$F_r(a_1, \ldots, \hat{a}_r, \ldots, a_N) \circ z_r = 0, \tag{12}$$

$$\bar{\pi}_r(f)(a_1, \ldots, a_N) = 0, \tag{13}$$

for any $z_r \in Z(A)$ and $a_1, \ldots, \hat{a}_r, \ldots, a_N \in N(Z(A))$.

From (13) it follows that $\bar{\pi}_r(f) = 0$ is an identity for A.

Note that $\deg \pi_r(f) \le 2n+1$, and $\deg F_r \le 2n$. Since the centralizer of $Z(A)$ coincides with $Z(A)$ (see Lemma 4.1) and $Z(A) \subset A_{-1}$, from (7) and (12) we obtain that

$$F_r(a_1, \ldots, \hat{a}_j, \ldots, a_N) = 0$$

are identities for $N(Z(A))$. In particular, they are identities for $A_0^+ \subset N(Z(A))$ (recall that $N(Z(A)) = A_{-1}^+ \oplus A_0^+$). We have mentioned that $A_0^+ = \{x_i\partial_j : i, j = 1, \ldots, n\} \cong \mathrm{Mat}_n$.

So, $F_r = 0$, are identities on Mat_n with no more than $2n$ variables of degree at most $2n$. By the Amitsur–Levitzki theorem, F_r is a scalar multiple of standard polynomial in variables $1, \ldots, \hat{j}, \ldots, 2n+1$. Therefore

$$\pi_r f(t_1, \ldots, t_{2n+1}) = \mu_r \sum_{\sigma \in \mathrm{Sym}_{1,\ldots,\hat{j},\ldots,2n+1}} \operatorname{sign}\sigma\, t_{\sigma(1)} \circ \cdots \hat{t}_{\sigma(r)} \cdots \circ t_{\sigma(2n+1)} \circ t_r,$$

for some $\mu_r \in \mathcal{K}$. In particular, $\pi_r f$ is multilinear.

If f is multilinear, then $f = \sum_{r=1}^{2n+1} \pi_r f$. Thus, for multilinear f,

$$f = \sum_{r=1}^{2n+1} \pi_r(f) = \sum_{r=1}^{2n+1} \mu_r s_{2n}^r.$$

The linear independence of the polynomials s_{2n}^r, $r = 1, \ldots, 2n+1$, is evident. ∎

Let f be some left polynomial. We will say that f has l-symmetric variables $t_{i_1}, \ldots, t_{i_l}$, if

$$\begin{aligned} f(t_1, \ldots, t_{i_1-1}, t_{i_1}, t_{i_1+1}, \ldots, t_{i_l-1}, t_{i_l}, t_{i_l+1}, \ldots) \\ = f(t_1, \ldots, t_{i_1-1}, t_{i_{\sigma(1)}}, t_{i_1+1}, \ldots, t_{i_l-1}, t_{i_{\sigma(l)}}, t_{i_l+1}, \ldots), \end{aligned}$$

for any permutation $\sigma \in \mathrm{Sym}_l$. Similarly, f has (l,k)-symmetric variables $t_{i_1}, \ldots, t_{i_l}$, and $t_{j_1}, \ldots, t_{j_k}$, if the sets of indices $\{i_1, \ldots, i_l\}$ and $\{j_1, \ldots, j_k\}$ have no common elements and

$$\begin{aligned} f(\ldots, t_{i_1}, \ldots, t_{i_l}, \ldots, t_{j_1}, \ldots, t_{j_k}, \ldots) \\ = f(\ldots, t_{i_{\alpha(1)}} \ldots, t_{i_{\alpha(l)}}, \ldots, t_{j_{\beta(1)}}, \ldots, t_{\beta(k)}, \ldots), \end{aligned}$$

for any permutations $\alpha \in \mathrm{Sym}_l$ and $\beta \in \mathrm{Sym}_k$. Here we suppose that $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_k$, but the ordering of the joint set $\{i_1, \ldots i_l, j_1, \ldots, j_k\}$ may be mixed. It can be, for example, $j_1 < i_1 < j_2 < i_2 < \cdots$. So, a more correct writing of the (l,k)-symmetry condition in this case should be

$$\begin{aligned} f(\ldots, f_{j_1}, \ldots, t_{i_1}, \ldots, t_{j_2}, \ldots, t_{i_2}, \ldots) \\ = f(\ldots, t_{j_{\beta(1)}}, \ldots, t_{i_{\alpha(1)}}, \ldots, t_{j_{\beta(2)}}, \ldots, t_{i_{\alpha(2)}}, \ldots). \end{aligned}$$

COROLLARY 4.5. $(p \neq 2)$. *Let f be nontrivial multilinear left polynomial with degree $2n+1$ and let $f = 0$ be an identity for A. Then f has no 3-symmetric or $(2,2)$-symmetric variables.*

Proof. Since s^l_{2n} is skew-symmetric for all variables, except the last one, we have

$$s^l_{2n}(t_1, \ldots, t_{2n+1}) = 0,$$

if $t_i = t_j = t_k$, for some i, j, k, such that $(i-j)(i-k)(j-k) \neq 0$ or $t_i = t_j, t_k = t_r$, for some $(i, j) \neq (k, r)$, such that $(i-j)(k-r) \neq 0$. So, any nontrivial linear combination of s^l_{2n} has no 3-symmetric variables. Therefore, by Lemma 4.4, f has no 3-symmetric variables.

Suppose that $f = \sum_{l=1}^{2n+1} \lambda_l s^l_{2n}$ has 2-symmetric variables, say $t_i, t_j, i < j$. Then

$$\begin{aligned} f(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_{2n+1}) \\ = \lambda_i s^i_{2n}(t_1, \ldots, \hat{t}_i, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_{2n+1}, t) \\ + \lambda_j s^j_{2n}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, \hat{t}_j, \ldots, t_{2n+1}, t) \\ = (\lambda_i s^{(i,j)}_{2n} + \lambda_j s^{(j,i)}_{2n})(t_1, \ldots, t_{2n+1}), \end{aligned}$$

where t is substituted for t_i and for t_j. Note that

$$\begin{aligned} s^{(i,j)}_{2n}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_{2n+1}) \\ = -(-1)^{i+j} s^{(j,i)}_{2n}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_{2n+1}). \end{aligned}$$

So,

$$\begin{aligned} f(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_{2n+1}) \\ = (\lambda_i - (-1)^{i+j}\lambda_j) s^{(i,j)}_{2n}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_{2n+1}). \end{aligned}$$

As we mentioned above, s^i_{2n} has no 3-symmetric variables, so $s^{(i,j)}_{2n}$ has no 2-symmetric variables, except t_j, t_{2n+1}. Therefore, f has no $(2,2)$-symmetric variables. ∎

LEMMA 4.6. $(p \neq 2)$. *Let $f \in R^{\mathrm{left}}_{2n+1}$ of degree $2n+1$ and let $f = 0$ be an identity on $W^{\mathrm{r\,sym}}_n$. Then*

$$f = \sum_{r:\deg_{t_r} f = 2} \tau_{t_r}(f) + \sum_i \pi_i(f),$$

where $\tau_{t_r}(f)$ is a linear combination of left polynomials $s^{(l,r)}_{2n}$, $l \neq r, l, r = 1, \ldots, 2n+1$, if $\deg_{t_r} f = 2$ and $\pi_i(f)$ is a scalar multiple of s^i_{2n}.

Proof. Prove that

$$\tau_{t_r} f = 0,$$

if $deg_{t_r} f > 2$.

Suppose that $\deg_{t_r} f = l_r > 2$, for some r. Make t_r-linearization for f with linearization variables $t_{r,1}, \ldots, t_{r,l_r}$. By Proposition 3.2 the polynomial $R = \Delta_{t_r, t_{r,1}, t_{r,2}} \Delta_{t_r, t_r, t_{r,3}} \cdots \Delta_{t_r, t_r, t_{r,l_r}} f$ has l_r-symmetric variables $t_{r,1}, \ldots, t_{r,l_r}$ and $G = 0$ is the identity for W_n. If R is multilinear, then by Corollary 4.5 we have $R = 0$ and by Proposition 3.1 $\deg_{x_r} f < l_r$, a contradiction.

If R is not multilinear, then there exists another variable (say t_h) such that $\deg_{t_h} G = l_h > 1$. Make x_h-linearization of G. Let

$$H = \Delta_{t_h, t_{h,1}, t_{h,2}} \Delta_{t_h, t_h, t_{h,3}} \cdots \Delta_{t_h, t_h, t_{h,l_h}} G.$$

Note that H does not change the l_r-symmetricity property for the variables $t_{r,1}, \ldots, t_{r,l_r}$. In particular, H has a 3-symmetric variable $t_{r,1}$, $t_{r,2}$, $t_{r,3}$. If H is multilinear, we obtain from Corollary 4.5 that $H = 0$. By Proposition 3.1 this contradicts the condition $\deg_{t_h} G = l_h > 1$.

So, repeating such arguments gives us that $\operatorname{def}_{t_i} f < 3$ for all $i = 1, \ldots, 2n+1$.

Prove now that $\tau_{t_r} f$ has no 2-symmetric variable t_q, $q \neq r$, if $\deg_{t_r} f = 2$, for some r. Suppose that it is not true, say $\deg_{t_q}(\tau_{t_r} f) = l_q > 1$. Then by Proposition 3.2

$$Q = \Delta_{t_q, t_{q,1}, t_{q,2}} \Delta_{t_q, t_q, t_{q,3}} \cdots \Delta_{t_q, t_q, t_{q,l_q}} \Delta_{t_r, t_{r,1}, t_{r,2}} f$$

gives identity $Q = 0$ for A and Q has $(2,2)$-symmetric variables $t_{q,1}, t_{q,2}$ and $t_{r,1}, t_{r,2}$. If Q is multilinear, then by Corollary 4.5, $Q = 0$. If Q is not multilinear, linearization of nonlinear variables does not change the $(2,2)$-linearilty property for $t_{q,1}, t_{q,2}$ and $t_{r,1}, t_{r,2}$. Repeating this procedure and using Proposition 3.1 and Corollary 4.5 gives us a contradiction. Thus f has no $(2,2)$-symmetric variables.

Reformulate these results in terms of essential variables. To do this, let us make two remarks. If f is a left polynomial of degree N and the number of variables is N, then essential polynomials and multilinear polynomials are just the same. If $f \in R^{\text{left}}_{N-1}$ and $\deg f = N-1$, then $\hat{l}$-essentiality of f is equivalent to the following condition: $\deg_{t_r} f = 2$, for some $r \neq l$, and $\deg_{t_i} f = 1$, for all $i \neq l, r$, and $\deg_{t_l} f = 0$.

So, we obtain the following facts:

(i) Any nontrivial monomial of a polynomial f of degree $2n+1$ has at least $2n$ essential variables. If it is not true, say, a nontrivial monomial $\mathbf{T} = \lambda_{\mathbf{i}} t^{\mathbf{i}}$ of f has no more than $2n-1$ essential variables, then either $\deg_{t_r} \mathbf{T} > 2$, for some r, or $\deg_{t_r} \mathbf{T} = 2, \deg_{t_q} \mathbf{T} = 2$, for some $q \neq r$. It is impossible.

(ii) Any nontrivial monomial of f with $2n+1$ essential variables is multilinear monomial.

(iii) If $\mathbf{T} = \lambda_{\mathbf{i}} t^{\mathbf{i}}$ is a nontrivial monomial of f with $2n$ essential variables, say $t_1, \ldots, \hat{t}_l, \ldots, t_{2n+1}$, then $\mathbf{T}$ is $\hat{l}$-essential and, moreover, $\deg_{t_r} \mathbf{T} = 2$, for some $r \neq l$, and $\deg_{t_i} \mathbf{T} = 1$, for all $i \neq l, r$.

So, we have proved that f is a sum of essential monomials and $\hat{r}$-essential monomials, for $r = 1, \ldots, 2n+1$. By the two remarks given above and by Lemma 4.4, part (i), this means that

$$f = \sum_{r : \deg_{t_r} f = 2} \tau_{t_r}(f) + \sum_i \pi_i(f).$$

By Lemma 4.4, part (ii), any $\pi_i(f)$ is a scalar multiple of s_{2n}^i.

To end the proof of Lemma 4.6 we should prove that $\tau_{t_r}(f)$ is a linear combination of left polynomials $s_{2n}^{(l,r)}$, $l \neq r, l, r = 1, \ldots, 2n+1$, if $\deg_{t_r} f = 2$, and $\deg_{t_l} f = 0$.

Note that

$$\tau_r(f) = \sum_{l \neq r} \pi_l(\tau_r(f)).$$

Recall that $\pi_l(\tau_r(f))$ is a sum of monomials of $\tau_r(f)$ that do not depend on t_l. Prove that $\pi_l(\tau_r(f))$ is a scalar multiple of $s_{2n}^{(l,r)}$.

Suppose for simplicity of denotions that $l < r$. Let

$$\tilde{f}_l = \Delta_{t_r, t_l, t_r} \pi_l(\tau_r(f)).$$

Then $\tilde{f}_l$ is multilinear. By Lemma 4.4, $\tilde{f}_l$ is a linear combination of left polynomials s_{2n}^i, $i = 1, \ldots, 2n+1$,

$$\tilde{f}_l = \sum_{i=1}^{2n+1} \mu_i s_{2n}^i, \tag{14}$$

for some $\mu_i \in \mathcal{K}$. In particular,

$$\begin{aligned}
&\tilde{f}_l(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}) \\
&\quad = \textstyle\sum_{i=1}^{2n+1} \mu_i s_{2n}^i(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{r-1}, t, t, t_{r+1}, \ldots, t_{2n+1}).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\tilde{f}_l(t_1, \ldots, t_{r-1}, t, t, t_{r+1}, \ldots, t_{2n+1}) \\
&\quad = 2f(t_1, \ldots, t_{l-1}, t_{l+1}, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}),
\end{aligned}$$

and

$$s^i_{2n}(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}) = 0, \qquad i \neq l, r,$$

$$\begin{aligned} s^l_{2n}&(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}) \\ &= s^{(l,r)}_{2n}(t_1, \ldots, \hat{t}_l, \ldots, t_{r-1}, t, t_{r+1}, t_{2n+1}), \\ s^r_{2n}&(t_1, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}) \\ &= (-1)^{r-1-l} s^{(l,r)}_{2n}(t_1, \ldots, \hat{t}_l, \ldots, t_{r-1}, t, t_{r+1}, t_{2n+1}). \end{aligned}$$

So, from (14) it follows that

$$\begin{aligned} f&(t_1, \ldots, \hat{t}_l, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}) \\ &= (1/2)(\mu_l - (-1)^{r-l}\mu_r) s^{(l,r)}_{2n}(t_1, \ldots, \hat{t}_l, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_{2n+1}). \end{aligned}$$

In other words,

$$\pi_l(\tau_r(f)) = \gamma_{l,r}\, s^{(l,r)}_{2n},$$

for $\gamma_{l,r} = (1/2)(\mu_l - (-1)^{r-l}\mu_r) \in \mathcal{K}$. Lemma 4.6 is proved completely. ∎

LEMMA 4.7. *If* $f \in R^{\text{left}}_N$, $\deg f < 2n+1$, *and* $f = 0$ *is identity on* A, *then* $f = 0$ *as a left polynomial.*

Proof. If A has nontrivial left polynomial identity of degree d, then it has a multilinear nontrivial left polynomial identity of degree $\leq d$. The proof of this statement is based on the linearization method and it does not depend on the associativity of A. Suppose that a nontrivial multilinear polynomial $g \in R^{\text{left}}_d$ has degree $d \leq 2n$ and $g = 0$ is the identity for A. Then g has a monomial of the form $\lambda t_d t_{d-1} \ldots t_1$, $\lambda \neq 0$. As in the case of matrices, if $d < 2n+1$, we can take $a_1 = \partial_1, a_2 = x_1\partial_1, a_3 = x_1\partial_2, a_4 = x_2\partial_2, \ldots, a_d = x_m\partial_m$, if $d = 2m$, and $a_d = x_{m-1}\partial_m$, if $d = 2m-1$. Then

$$a_d \circ a_{d-1} \circ \cdots \circ a_1 = \partial_m,$$

and for any permutation $\sigma \in \mathrm{Sym}_d$, $\sigma \neq id$,

$$a_{\sigma(d)} \circ \ldots \circ a_{\sigma(1)} = 0.$$

Therefore,

$$g(a_1, \ldots, a_d) = \lambda\partial_m \neq 0,$$

a contradiction. Lemma 4.7 is proved. ∎

Let $J = \{h_1, \ldots, h_k\}$ be a subset of $\mathcal{N}$ with k elements. For $f \in R^{\text{left}}_N$, denote by f_J a sum of monomials of f of the form $\lambda_{i_1,\ldots,i_k} t_{i_1} \circ \cdots \circ t_{i_k}$, where $i_1, \ldots, i_k \in J$.

LEMMA 4.8. *The space of polynomials* $f \in R_N^{\text{left}}$ *such that* $\deg f = 2n+1$ *and* $f = 0$ *is an identity for* A *is* $N\binom{N}{2n}$*-dimensional. A basis of this space can be generated by left polynomials* $s_{2n,I,r}$*, where* I *runs through all subsets of* $\mathcal{N}$ *with* $2n$ *elements and* r *runs through all elements of the set* $\mathcal{N}$.

Proof. Since $s_{2n,I,r} = s_{2n,I',r'}$, if and only if $(I, r) = (I', r')$, the linear independence of the polynomials $s_{2n,I,r}$, $I \subseteq \mathcal{N}$, $|I| = 2n$, $r \in \mathcal{N}$, is evident.

Let $f \in R_N^{\text{left}}$ be a left polynomial of degree $2n+1$ and let $f = 0$ be an identity for A. Prove that f is a linear combination of left polynomials $s_{2n,I,r}$, where I runs all subsets of $\mathcal{N}$ with $2n$ elements and r runs all elements of $\mathcal{N}$.

Let I be any subset of $\mathcal{N}$. Take $a_j = 0$, if $j \notin I$. Then

$$f(a_1, \dots, a_N) = 0 \qquad \Rightarrow \qquad f_J(a_{i_1}, \dots, a_{i_{2n+1}}) = 0.$$

Thus, $f_I = 0$ is identity on A.

If $N < 2n$, then according to Lemma 4.7, $f = 0$, and Lemma 4.8 is true. If $N = 2n+1$, then Lemma 4.8 is true by Lemma 4.6.

Suppose that $N > 2n+1$. Since $\deg f = 2n+1$, any monomial of f has no more than $2n+1$ essential variables. So, any nontrivial monomial of f will be a part of f_I for some subset I of $\mathcal{N}$ with $2n+1$ elements. On the other hand, for any subset J of $\mathcal{N}$ with $2n+1$ elements, by Lemma 4.6 f_J is a linear combination of left polynomilas $s^r_{2n,J}$, and $s^{(l,r)}_{2n,J}$, where $l, r \in J$, $l \neq r$. So, any nontrivial monomial $\mathbf{T}$ of f has $2n$ or $2n+1$ essential variables.

Denote by g a sum of nontrivial monomials of f with $2n+1$ essential variables and by h a sum of nontrivial monomials of f with $2n$ essential variables. Then

$$f = g + h.$$

Let J be any subset of $\mathcal{N}$ with $2n+1$ elements. Then f_J is a part of g. By Lemma 4.8, f_J is a linear combination of left polynomials $s^r_{2n,J}$, where $r \in J$. Let J' be another subset of $\mathcal{N}$ with $2n+1$ elements and $r' \in J'$. Note that left polynomials $s^r_{2n,J}$ and $s^{r'}_{2n,J'}$ have common monomials if and only if $J = J'$ and $r = r'$. Thus, f_J and $f_{J'}$ have common monomials if and only if $(J, r) = (J', r')$. Therefore, for some $\mu_{J,r} \in \mathcal{K}$,

$$g = \sum_J f_J = \sum_J \sum_{r \in J} \mu_{J,r} s^r_{2n,J},$$

where J runs all subsets of $\mathcal{N}$ with $2n+1$ elements.

The case of h is considered similarly. Let I be any subset of $\mathcal{N}$ with $2n$ elements. Then f_I is a part of the left polynomial h and $f_I = 0$ is an identity for A. Therefore, by Lemma 4.8, f_I is a linear combination of polynomials $s_{2n,I,r}$, where $r \in I$. Let I' be another subset of $\mathcal{N}$ with $2n$ elements. Notice that $s_{2n,I,r}$ and $s_{2n,I',r'}$ have common nontrivial monomials if and only if

$I = I'$ and $r = r'$. Therefore, f_I and $f_{I'}$ have common nontrivial monomials if and only if $f_I = f_{I'}$. So, for some $\mu_{I,r} \in \mathcal{K}$,

$$h = \sum_I f_I = \sum_I \sum_{r \in I} \mu_{I,r} s_{2n,I,r},$$

where I runs all subsets of $\mathcal{N}$ with $2n$ elements.

Therefore, if $f \in R_N^{\text{left}}$, and $f = 0$ is an identity for A, then $f = g + h$ is a linear combination of left polnomials $s_{2n,I,r}$. ∎

Proof of Theorem 3.3.

(i) Corollary 4.3 of Lemma 4.2,

(ii) Lemma 4.7,

(iii) Lemma 4.6 and Lemma 4.8.

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