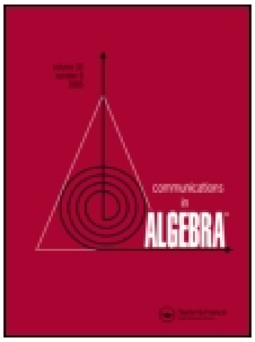

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FACTOR-COMPLEX FOR LEIBNIZ COHOMOLOGY

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Dedicated to the memory of A. I. Kostrikin.

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ABSTRACT

A complex quasi-isomorphic to the factor of Leibniz complex over Chevalley-Eilenberg complex for Lie algebras of characteristic 0 is constructed.

1. INTRODUCTION

The ground field k is supposed to be of characteristic 0. Let L be a Lie algebra considered as a Leibniz algebra and M a symmetric L-module. Let $C^*(L,M) = Hom(\wedge^* L, M)$ be Chevalley-Eilenberg cochain complex [1] and $T^*(L,M) = Hom(L^{\otimes}, M)$ Leibniz cochain complex [3]. There exists an imbedding of cochain complexes [3]

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$C^*(L, M) \to T^*(L, M)$.

Let $H_{rel}^{*-2}(L, M)$ be the cohomology group that makes exact the cohomological sequence

$$\cdots \to H^{k+2}(L, M) \to HL^{k+2}(L, M) \to$$
$$H_{rel}^{k}(L, M) \to H^{k+3}(L, M) \to \cdots$$

In [4] a spectral sequence for calculating of $H_{rel}^*(L, M)$ is constructed.

The aim of our paper is to construct a cochain complex $C_{rel}^*(L, M)$ that makes exact the following sequence of cochain complexes

$$0 \to C^*(L, M) \to T^*(L, M) \to C_{rel}^{*-2}(L, M) \to 0.$$

The cohomology group of the complex $(C_{rel}^*(L, M), D)$ will be isomorphic to $H_{rel}^*(L, M)$.

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2. LEIBNIZ ALGEBRAS, MODULES AND COHOMOLOGY

An algebra L is called *(left)Leibniz* [3], if the following idenity holds

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]].$$

In particular, any Lie algebra is a Leibniz algebra.

A vector space M with bilinear maps $L \times M \to M, (x, m) \mapsto [x, m]$ and $M \times L \to M, (m, x) \mapsto [m, x]$ is called as a *module* over Leibniz algebra L, if

$$[[x, y], m] = [x, [y, m]] - [y, [x, m]],$$
$$[[x, m], y] = [x, [m, y]] - [m, [x, y]],$$
$$[[m, x], y] = [m, [x, y]] - [x, [m, y]],$$

for any $x, y \in L, m \in M$. For L-module M the subspace

$$M^{ann} := \{a(x, m) := [x, m] + [m, x], \quad x \in L, m \in M\}$$

has a module structure over L:

$$[x, a(y, m)] = a([x, y], m) + a(y, [xm]), \quad [a(y, m), x] = 0,$$

for any $x, y \in L$, $m \in M$. A module M is called *symmetric*, if $M^{ann} = 0$ and *antisymmetric*, if $[m, x] = 0$, for any $x \in L, m \in M$.

Let be given L-module M. Denote by $T^k(L, M), k > 0$, the space of multilinear maps $\psi : L \times \cdots \times L \to M$ with k-arguments,

$$T^0(L, M) = M, \quad T^k(L, M) = 0, \quad k < 0$$

and $T^*(L, M) = \oplus_k T^k(L, M)$. Let

$$d : T^*(L, M) \to T^{*+1}(L, M),$$

be the Leibniz coboundary operator

$$\begin{aligned} d\psi(x_1, \ldots, x_{k+1}) &= \sum_{i<j} \psi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, [x_i, x_j], x_{j+1}, \ldots, x_{k+1}) \\ &\quad + \sum_{i=1}^{k} (-1)^{i+1} [x_i \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{k+1})] \\ &\quad + (-1)^{k+1} [\psi(x_1, \ldots, x_k), x_{k+1}]. \end{aligned}$$

Let

$$HL^*(L, M) = \oplus_k HL^k(L, M), \quad HL^k(L, M) = ZL^k(L, M)/BL^k(L, M)$$

be the Leibniz cohomology groups.

If L is a Lie algebra and M is an L-module (symmetric module in the category of Leibniz algebras) one can consider also Chevalley-Eilenberg cochain complex $C^*(L, M) = \oplus C^k(L, M)$ and in this case the operator d coincides with the usual coboundary operator. Recall that $C^k(L, M) = Hom(\wedge^k(L), M)$, if $k \geq 0$, and $C^k(L, M) = 0$, if $k < 0$. Let $H^*(L, M) = \oplus_k H^k(L, M)$ be the Chevalley-Eilenberg cohomology for Lie algebra [1].

For Lie algebra L and symmetric L-module M there exists natural imbedding of cochain complexes $C^*(L, M) \to T^*(L, M)$. Therefore appears $H^*_{rel}(L, M) = \oplus_k H^k_{rel}(L, M)$, the cohomology group that makes exact the following sequence

$$\cdots H^k(L, M) \to HL^k(L, M) \to H^{k-2}_{rel}(L, M) \to H^{k+1}(L, M) \to \cdots.$$

In our paper we construct a cochain complex $C^*_{rel}(L, M)$ with cohomology group isomorphic to $H^*_{rel}(L, M)$. In fact we construct the pre-simplicial complex $C^*_{rel}(L, M)$ such that the following sequence will be exact not only as a sequence of cochain complexes

$$0 \to C^*(L, M) \to T^*(L, M) \to C^{*-2}_{rel}(L, M) \to 0.$$

It will be exact as a sequence of pre-simplicial complexes also.

3. CONSTRUCTION OF THE COMPLEX $C^*_{rel}(L, M)$

Let $\mathbf{Z}_+ = \{1, 2, 3, \ldots\}$. For any $i \in \mathbf{Z}_+$ define the operator $a_i : T^k(L, M) \to T^k(L, M)$ by

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$$a_i\psi(x_1,\dots,x_i,x_{i+1},\dots,x_k) \\ = (\psi(x_1,\dots,x_i,x_{i+1},\dots,x_k)+\psi(x_1,\dots,x_{i+1},x_i,\dots,x_k))/2,$$

if $1\le i\le k$, and $a_i\psi=0$, if $i>k$. Define also an operator $J(i): T^k(L,M)\to T^k(L,M)$ by

$$J(i)\psi(x_1,\dots,x_{k+2}) \\ = \psi(x_1,\dots,x_i,x_{i+1},x_{i+2},\dots,x_k)+\psi(x_1,\dots,x_{i+1},x_{i+2},x_i,\dots,x_k) \\ + \psi(x_1,\dots,x_{i+2},x_i,x_{i+1},\dots,x_k),$$

if $i\le k-2$ and $J(i)\psi=0$ if $i>k-2$.

Consider in the space of multilinear maps $T^{k+2}(L,M^{\otimes k+1})$ the subspace denoted by $C^k_{rel}(L,M)$ and defined by the following way. If $\Psi=(\psi_1,\dots,\psi_{k+1})\in C^k_{rel}(L,M), k>0$, where $\psi_i\in T^{k+2}(L,M)$, then

$$a_i\psi_i=\psi_i,\quad i=1,\dots,k+1,$$
$$a_i\psi_j=a_j\psi_i,\quad |i-j|>1,$$
$$J(i)\psi_i=J(i)\psi_{i+1},\quad i=1,\dots,k.$$

Let $C^0_{rel}(L,M)=S^2(L,M)$ be a space of symmetric bilinear maps with coefficients in L-module M. For $k<0$ we set $C^k_{rel}(L)=0$.

So, $C^k_{rel}(L,M), k\ge 0$, consists of multilinear maps Ψ with $k+2$ arguments and with coefficients in $M^{\oplus k+1}$. Each i-th coordinate ψ_i of Ψ is a multilinear map with $k+2$ arguments and with coefficients in M. Each coordinate ψ_i is symmetric in i and $i+1$-th arguments and any consequative coordinates ψ_i and ψ_{i+1} satisfy the conditions

$$\psi_i(x_1,\dots,x_i,x_{i+1},x_{i+2},\dots,x_{k+2}) \\ + \psi_i(x_1,\dots,x_{i+1},x_{i+2},x_i,\dots,x_{k+2}) \\ + \psi_i(x_1,\dots,x_{i+2},x_i,x_{i+1},\dots,x_{k+2}) \\ = \psi_{i+1}(x_1,\dots,x_i,x_{i+1},x_{i+2},\dots,x_{k+2}) \\ + \psi_{i+1}(x_1,\dots,x_{i+1},x_{i+2},x_i,\dots,x_{k+2}) \\ + \psi_{i+1}(x_1,\dots,x_{i+2},x_i,x_{i+1},\dots,x_{k+2}),$$

for any $x_1,\dots,x_{k+2}\in L$.

Let $C^*_{rel}(L,M)=\oplus_k C^k_{rel}(L,M)$. For any $i\in \mathbf{Z}_+$ construct the operator

$$D_i: C^k_{rel}(L,M)\to C^{k+1}_{rel}(L,M),$$

by the following way. Set

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$$(D_i\Psi)_l(x_1,\ldots,x_{k+3})$$
$$= -\sum_{j=i+1}^{k+3}\psi_{l-\delta(i<l)}(x_1,\ldots,\hat{x}_i,\ldots,\underset{j-1}{[x_i,x_j]},\ldots,x_{k+3})$$
$$+ [x_i,\psi_{l-\delta(i<l)}(x_1,\ldots,\hat{x}_i,\ldots,x_{k+3})],$$

if $1 \leq l \leq k+2, l \neq i, i-1$. Set $(D_j\Psi)_l = a_l d_j \psi$, if $l=j$ or $l=j-1$, where $\pi\psi = \Psi, \psi \in T^{k+2}(L,M)$. In other cases, $(D_i\Psi)_l = 0$. Here $\hat{x}$ means that the element x is omitted, $\delta(i<l)$ is 1, if $i<l$, and 0 in other case. In the construction of $(D_j\Psi)_l, l=j,j-1$, we use ψ such that $\Psi = \pi\psi$. The correctness of this definition follows from lemma 1 and from the fact that $a_l\phi = 0$, for any $\phi \in C^{k+2}(L,M)$.

Let

$$\pi : T^{k+2}(L,M) \to C^k_{rel}(L,M)$$

be the operator defined by $\pi\psi = (a_1\psi,\ldots,a_{k+1}\psi)$. It is easy to check that this definition is correct:

$$\psi \in T^{k+2}(L,M) \Rightarrow \pi\psi \in C^k_{rel}(L,M).$$

It is also easy to see that

$$\pi\psi = 0, \quad \psi \in T^{k+2}(L,M) \Rightarrow \psi \in C^{k+2}(L,M).$$

We will prove that in case of characteristic 0, the map π is surjection (lemma 1).

Theorem 1. *Let L be a Lie algebra over a field of characteristic 0 and M be a symmetric L-module. If $i<j$, then $D_jD_i = D_iD_{j-1}$. In particular, the operator*

$$D : C^*_{rel}(L,M) \to C^{*+1}_{rel}(L,M),$$

given by $D = \sum_i (-1)^{i+1} D_i$ is a coboundary operator: $D^2 = 0$. The cohomology group of the cochain complex $(C^*_{rel}(L,M), D)$ is isomorphic to $H^*_{rel}(L,M)$.

From lemma 2 ($a_i(d_i - d_{i+1}) = 0$) will follow the following expression for D

$$(D\Psi)_l(x_1,\ldots,x_{k+3})$$
$$= \sum_{i<j\leq k+3, i\neq l,l+1} (-1)^i \psi_{l-\delta(i<l)}(x_1,\ldots,\hat{x}_i,\ldots,\underset{j-1}{[x_i,x_j]},\ldots,x_{k+3})$$
$$+ \sum_{i\neq l,l+1} (-1)^{i+1}[x_i,\psi_{l-\delta(i<l)}(x_1,\ldots,\hat{x}_i,\ldots,x_{k+3})], \quad 1 \leq l \leq k+2.$$

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4. THE OPERATOR π IS SURJECTIVE

Lemma 1. *For any $k \geq 2$ the following sequence is exact*

$$0 \to C^k(L,M) \to T^k(L,M) \xrightarrow{\pi} C^{k-2}_{rel}(L,M) \to 0$$

Proof. Let $t_i = (i, i+1)$ be the transposition. Then $a_i = 1/2(1+t_i) : T^k(L,M) \to T^k(L,M)$ be the symmetriser on i-th and $i+1$-th places and $C^k_{rel}(L,M)$ consists of $\Psi = (\psi_1, \dots, \psi_{k+1}), \psi_i \in T^{k+2}\ (L,M)$, such that

$$a_i\psi_i = \psi_i, \quad i = 1, \dots, k+1,$$
$$a_j\psi_i = a_i\psi_j, \quad i,j = 1, \dots, k+1,$$
$$a_i a_{i+1}\psi_i - 4\psi_i = a_{i+1}a_i\psi_{i+1} - 4\psi_{i+1}, \quad i = 1, \dots, k.$$

Now we shall prove that for any $\Psi \in C^{k-2}_{rel}(L,M)$ one can construct $\psi \in T^k(L,M)$, such that $\pi\psi = \Psi$.

In order to prove this statement we shall construct a section

$$s_k : C^{k-2}_{rel}(L,M) \to T^k(L,M)$$

such that

$$\pi s_k = Id_{C^{k-2}_{rel}(L,M)}, \quad s_k\pi = Id_{T^k(L,M)} - Alt_k,$$

where Alt_k is the projector on the subspace $C^k(L,M)$. The symmetric group S_k acts on the space $T^k(L,M)$ by

$$\sigma^{-1}(\psi)(x_1, \dots, x_k)$$
$$= \psi(x_{\sigma(1)}, \dots, x_{\sigma(k)}), \quad \sigma \in S_k,\ \psi \in T^k(L,M),\ x_i \in L.$$

In particular, we can write the projector Alt_k as

$$Alt_k = \frac{1}{k!}\sum_{\sigma \in S_k} sign(\sigma)\,\sigma. \tag{1}$$

Since the group S_k is generated by the elements $t_i, i = 1, \dots, k-1$ with defining relations

$$t_i^2 = 1, \quad i = 1, \dots, k-1,$$
$$t_it_j = t_jt_i, \quad |i-j| \geq 2,$$
$$t_it_{i+1}t_i = t_{i+1}t_it_{i+1}, \quad i = 1, \dots, k-2,$$

the group algebra $k[S_k]$ is generated (as a unitary algebra) by the elements $a_i, i = 1, \dots, k-1$ with defining relations

$$a_i^2 = a_i, \quad i = 1, \dots, k-1, \tag{2}$$

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$$a_j a_i = a_i a_j, \quad |i-j| \geq 2, \tag{3}$$

$$a_i a_{i+1} a_i - 4a_i = a_{i+1} a_i a_{i+1} - 4a_{i+1}, \quad i = 1, \ldots, k-2. \tag{4}$$

All the elements a_i are orthogonal to the projector Alt_k :

$$a_i Alt_k = \frac{1}{2}(1 + t_i) Alt_k = \frac{1}{2}(Alt_k - Alt_k) = 0.$$

Hence the direct summand $(1 - Alt_k)k[S_k]$ of the group algebra $k[S_k]$ coincides with the algebra A generated (as a unitary algebra) by the elements $a_i, i = 1, \ldots, n-1$ with defining relations (2), (3), (4). This subalgebra, of course, has its own unite element $(1 - Alt_k)$, which can be written as a polynomial $F_{k-1}(a_1, \ldots, a_{k-1})$ without constant term in the non-commuting variables $a_1, \ldots, a_{k-1}$.

Let us note that the components of any vector $\Psi \in C^{k-2}_{rel}(L, M)$ generate the A-module N, consisting of linear combinations of the products $a_{i_1} \ldots a_{i_l} \psi_i$ and the map $\rho : A \to N$, sending $a_{i_1} \ldots a_{i_l} a_i$ into $a_{i_1} \ldots a_{i_l} \psi_i$, is an A-linear surjection.

Having the polynomial $F_{k-1}(a_1, \ldots, a_{k-1})$ we can write the section s_k as

$$s_k(\psi_1, \ldots, \psi_{k-1}) = \rho(F_{k-1}(a_1, \ldots, a_{k-1})).$$

The property $a_k s_k = Id_{C^{k-2}_{rel}(L,M)}$ follows from the fact that ρ is a A-linear and $F_{k-1}(a_1, \ldots, a_{k-1})$ is the identity in A,

$$a_i \rho(F_{k-1}(a_1, \ldots, a_{k-1})) = \rho(a_i F_{k-1}(a_1, \ldots, a_{k-1})) = \rho(a_i) = \psi_i.$$

The second property $s_k a_k = Id_{T^k(L,M)} - Alt_k$ follows from the relation $1 - Alt_k = F_{k-1}(a_1, \ldots, a_{k-1})$,

$$\psi = Alt_k(\psi) + F_{k-1}(a_1, \ldots, a_{k-1})(\psi).$$

In what follows we construct the polynomial F_k. We shall start with the decomposition $S_{k+1} = S_k \cup S_k t_k S_k$ of symmetric group S_{k+1} acting on $k+1$ letters into the disjoint union of double cosets by the symmetric subgroup $S_k \subset S_{k+1}$ acting trivially on the last letter.

We shall use it to express the projector Alt_{k+1} via Alt_k. By the equation (1)

$$\begin{aligned} Alt_{k+1} &= \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} sign(\sigma)\, \sigma \\ &= \frac{1}{k+1} \left(\frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma)\, \sigma + \frac{1}{k!} \sum_{\sigma \in S_k t_k S_k} sign(\sigma)\, \sigma \right). \end{aligned} \tag{5}$$

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From the other side

$$Alt_k\, t_k\, Alt_k = \frac{1}{(k!)^2} \sum_{\sigma,\tau \in S_k} sign(\sigma\tau)\, \sigma t_k \tau$$

which equals

$$-\frac{(k-1)!}{(k!)^2} \sum_{\sigma \in S_k t_k S_k} sign(\sigma)\, \sigma$$

since the centralizer $C_{S_k}(t_k)$ of t_k in S_k is equal to S_{k-1}. So, for $q_k = 1 - a_k$, we have

$$\begin{aligned} Alt_k\, q_k\, Alt_k &= \frac{1}{2}(Alt_k - Alt_k\, t_k\, Alt_k) \\ &= \frac{1}{2}(Alt_k - \frac{1}{k(k!)} \sum_{\sigma \in S_k t_k S_k} sign(\sigma)\, \sigma) \end{aligned}$$

and conversely

$$\frac{1}{k!} \sum_{\sigma \in S_k t_k S_k} sign(\sigma)\, \sigma = 2kAlt_k\, q_k Alt_k - kAlt_k.$$

We can substitute this into the expression (5) for Alt_{k+1} and obtain

$$\begin{aligned} Alt_{k+1} &= \frac{1}{k+1}(Alt_k + 2kAlt_k\, q_k\, Alt_k - kAlt_k) \\ &= \frac{2k}{k+1} Alt_k\, q_k\, Alt_k - \frac{k-1}{k+1} Alt_k. \end{aligned} \tag{6}$$

Now we can use this to write the projector Alt_k as a non-commutative polynomial in $a_1, \ldots, a_{k-1}$. We can do it inductively. The base of induction is provided by the formula $Alt_2 = q_1 = 1 - a_1$. Suppose that for any $l < k$ we have the polynomial $G_l(x_1, \ldots, x_l)$ in non-commuting variables $x_1, \ldots, x_l$ such that $G_l(0, \ldots, 0) = 1$ and $Alt_l = G_{l-1}(a_1, \ldots, a_{l-1})$. From (6) we obtain

$$\begin{aligned} &G_k(x_1, \ldots, x_k) \\ &\quad = \frac{2k}{k+1} G_{k-1}(x_1, \ldots, x_{k-1})(1 - x_k) G_{k-1}(x_1, \ldots, x_{k-1}) \\ &\quad\quad - \frac{k-1}{k+1} G_{k-1}(x_1, \ldots, x_{k-1}) \end{aligned} \tag{7}$$

We have

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$$G_k(0,\dots,0) = \frac{2k}{k+1}G_{k-1}(0,\dots,0)^2 - \frac{k-1}{k+1}G_{k-1}(0,\dots,0)$$
$$= \frac{2k}{k+1} - \frac{k-1}{k+1} = 1.$$

Now $F_k = 1 - G_k$ provides the expression of identity element of the algebra A as a polynomial in a_i.

For example,

$$G_1(a_1) = 1 - a_1,$$
$$G_2(a_1,a_2) = \frac{4}{3}(1-a_1)(1-a_2)(1-a_1) - \frac{1}{3}(1-a_1)$$
$$= 1 - \frac{1}{3}a_1 - \frac{4}{3}a_2 + \frac{4}{3}a_1a_2 + \frac{4}{3}a_2a_1 - \frac{4}{3}a_1a_2a_1$$

and the sections

$$s_2(\psi_1) = \psi_1,$$
$$s_3(\psi_1,\psi_2) = \frac{1}{3}\psi_1 + \frac{4}{3}\psi_2 - \frac{4}{3}a_1\psi_2 - \frac{4}{3}a_2\psi_1 + \frac{4}{3}a_1a_2\psi_1.$$

5. PRE-SIMPLICIAL STRUCTURE ON $C^*_{rel}(L,M)$

For $i \in \mathbf{Z}$ define operators

$$d_i : T^*(L,M) \to T^{*+1}(L,M)$$

on $\psi \in T^k(L,M)$ by the rules

$$d_i\psi(x_1,\dots,x_{k+1})$$
$$= [x_i, \psi(x_1,\dots,\hat{x}_i,\dots,x_{k+1})]$$
$$- \sum_{i<j} \psi(x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,[x_i,x_j],x_{j+1},\dots,x_{k+1}), \quad 1 \le i \le k,$$
$$d_{k+1}\psi(x_1,\dots,x_{k+1}) = -[\psi(x_1,\dots,x_k),x_{k+1}],$$
$$d_i\psi(x_1,\dots,x_{k+1}) = 0, \quad k+1 < i$$

It is shown in [2] that $d_jd_i = d_id_{j-1}$, if $j > i$. In other words,

$$(T^*(L,M), \{d_i, i \in \mathbf{Z}\})$$

is a pre-simplicial complex. The coboundary condition $d^2 = 0$ for the Leibniz coboundary operator is an easy corollary of the condition $d_jd_i = d_id_{j-1}, j > i$. In particular, for symmetric L-module M by the following rules

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$$
\begin{aligned}
&d_i\psi(x_1,\ldots,x_{k+1})\\
&\quad=[x_i,\psi(x_1,\ldots,\hat{x}_i,\ldots,x_{k+1})]\\
&\qquad-\sum_{i<j}\psi(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,[x_i,x_j],x_{j+1},\ldots,x_{k+1}),\quad 1\le i\le k+1,\\
&d_i\psi(x_1,\ldots,x_{k+1})=0,\quad k+1<i
\end{aligned}
$$

the cochain complex $T^*(L,M)$ can be endowed by a structure of pre-simplicial complex.

Lemma 2.

$$
\begin{aligned}
&a_id_j=d_ja_i,\quad \textit{if } j-i>1,\quad a_i(d_i-d_{i+1})=0,\\
&\quad a_id_j=d_ja_{i-1},\quad \textit{if } j-i<0.
\end{aligned}
$$

Proof. Let $\psi\in T^{k+2}(L,M)$. If $i>k+2$ or $j>k+3$ our statements are trivial. If $k+3\ge j>i+1$, Then

$$
\begin{aligned}
&t_id_j\psi(x_1,\ldots,x_{k+3})\\
&\quad=d_j\psi(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_{k+3})\\
&\quad=[x_j,\psi(x_1,\ldots,x_{i-1},x_{i+1},x_i,\ldots,\hat{x}_j,\ldots,x_{k+3})\\
&\qquad-\sum_{s=j+1}^{k+3}\psi(x_1,\ldots,x_{i-1},x_{i+1},x_i,\ldots,\hat{x}_j,\ldots,\underset{s}{[x_j,x_s]},\ldots,x_{k+3})\\
&\quad=[x_j,t_i\psi(x_1,\ldots,\hat{x}_j,\ldots,x_{k+3})\\
&\qquad-\sum_{s=j+1}^{k+3}t_i\psi(x_1,\ldots,\hat{x}_j,\ldots,\underset{s}{[x_j,x_s]},\ldots,x_{k+3})\\
&\quad=d_jt_i\psi(x_1,\ldots,x_{k+3}).
\end{aligned}
$$

Therefore, if $j-i>1$, then $a_id_j=d_ja_i$.

If $k+3\ge j=i+1$, then

$$
\begin{aligned}
&t_id_j\psi(x_1,\ldots,x_{k+3})\\
&\quad=d_j\psi(x_1,\ldots,x_{i-1},x_{i+1},\underset{j}{x_i},x_{i+2}\ldots,x_{k+3})\\
&\quad=[x_i,\psi(x_1,\ldots,x_{i-1},x_{i+1},x_{i+2},\ldots,x_{k+3})\\
&\qquad-\sum_{s=i+2}^{k+3}\psi(x_1,\ldots,x_{i-1},x_{i+1},x_{i+2},\ldots,\underset{s}{[x_i,x_s]},\ldots,x_{k+3})\\
&\quad=d_i\psi(x_1,\ldots,x_{k+3})+\psi(x_1,\ldots,x_{i-1},[x_i,x_{i+1}],x_{i+2},\ldots,x_{k+3}).
\end{aligned}
$$

If $k+3>j=i$, then

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$$\begin{aligned}
&t_i d_j \psi(x_1, \ldots, x_{k+3}) \\
&\quad = d_j \psi(x_1, \ldots, x_{i-1}, \underset{j}{x_{i+1}}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\quad = [x_{i+1}, \psi(x_1, \ldots, x_{i-1}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\qquad - \psi(x_1, \ldots, x_{i-1}, [x_{i+1}, x_i], x_{i+2}, \ldots, x_{k+3}) \\
&\qquad - \sum_{s=i+2}^{k+3} \psi(x_1, \ldots, x_{i-1}, x_i, x_{i+2}, \ldots, [\underset{s}{x_{i+1}}, x_s], \ldots, x_{k+3}) \\
&\quad = d_{i+1}\psi(x_1, \ldots, x_{k+3}) - \psi(x_1, \ldots, x_{i-1}, [x_{i+1}, x_i], x_{i+2}, \ldots, x_{k+3}).
\end{aligned}$$

Therefore, for any $k+2 \geq i$,

$$t_i(d_{i+1} - d_i)\psi(x_1, \ldots, x_{k+3}) = -(d_{i+1} - d_i)\psi(x_1, \ldots, x_{k+3}).$$

This means that $a_i(d_i - d_{i+1}) = 0$.

If $k+3 > i > j$, then

$$\begin{aligned}
&t_i d_j \psi(x_1, \ldots, x_{k+3}) \\
&\quad = d_j \psi(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\quad = [x_j, \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{x_{i+1}}, x_i, \ldots, x_{k+3}) \\
&\qquad - \sum_{j+1 \leq s < i} \psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{s-1}{[x_j, x_s]}, \ldots, x_{i-1}, \underset{i-1}{x_{i+1}}, x_i, \ldots, x_{k+3}) \\
&\qquad - \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{[x_j, x_{i+1}]}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\qquad - \psi(x_1, \ldots, \hat{x^j}, \ldots, x_{i-1}, x_{i+1}, \underset{i}{[x_j, x_i]}, x_{i+2}, \ldots, x_{k+3}) \\
&\qquad - \sum_{i+2 \leq s \leq k+3} \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{x_{i+1}}, x_i, \ldots, \underset{s-1}{[x_j, x_s]}, \ldots, x_{k+3}) \\
&\quad = [x_j, t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{x_i}, x_{i+1}, \ldots, x_{k+3}) \\
&\qquad - \sum_{j+1 \leq s < i} t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{s-1}{[x_j, x_s]}, \ldots, x_{i-1}, \underset{i-1}{x_i}, x_{i+1}, \ldots, x_{k+3}) \\
&\qquad - \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{[x_j, x_{i+1}]}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\qquad - \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, x_{i+1}, \underset{i}{[x_j, x_i]}, x_{i+2}, \ldots, x_{k+3}) \\
&\qquad - \sum_{i+2 \leq s \leq k+3} t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{x_i}, x_{i+1}, \ldots, \underset{s-1}{[x_j, x_s]}, \ldots, x_{k+3}) \\
&\quad = [x_j, t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{k+3}) - \\
&\qquad\qquad \sum_{j+1 \leq s \leq k+3, s \neq i, i+1} t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{s-1}{[x_j, x_s]}, \ldots, x_{k+3})
\end{aligned}$$

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$$
\begin{aligned}
&- \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{[x_j, x_{i+1}]}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\quad - \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, x_{i+1}, \underset{i}{[x_j, x_i]}, x_{i+2}, \ldots, x_{k+3}) \\
&= [x_j, t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, x_{k+3}) \\
&\quad - \sum_{j+1 \le s \le k+3} t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{s-1}{[x_j, x_s]}, \ldots, x_{k+3}) \\
&\quad + t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{i-1}{[x_j, x_i]}, \ldots x_{k+3}) \\
&\quad + t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{i}{[x_j, x_{i+1}]}, \ldots, x_{k+3}) \\
&\quad - \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, \underset{i-1}{[x_j, x_{i+1}]}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&\quad - \psi(x_1, \ldots, \hat{x}_j, \ldots, x_{i-1}, x_{i+1}, \underset{i}{[x_j, x_i]}, x_{i+2}, \ldots, x_{k+3}) \\
&= d_j t_{i-1}\psi(x_1, \ldots, x_{k+3}) \\
&\quad + t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{i-1}{[x_j, x_i]}, \ldots, x_{k+3}) \\
&\quad - \psi(x_1, \ldots, x_{i-1}, x_{i+1}, \underset{i}{[x_j, x_i]}, x_{i+2}, \ldots, x_{k+3}) \\
&\quad + t_{i-1}\psi(x_1, \ldots, \hat{x}_j, \ldots, \underset{i}{[x_j, x_{i+1}]}, \ldots, x_{k+3}) \\
&\quad - \psi(x_1, \ldots, x_{i-1}, \underset{i-1}{[x_j, x_{i+1}]}, x_i, x_{i+2}, \ldots, x_{k+3}) \\
&= d_j t_{i-1}\psi(x_1, \ldots, x_{k+3}).
\end{aligned}
$$

Therefore, if $j - i < 0$, then $a_i d_j = d_j a_{i-1}$.

Lemma 3. *For any Lie algebra L, symmetric L-module M and $i, k \in \mathbf{Z}, i > 0, k \geq 0$ the following diagram is commutative*

$$
\begin{array}{ccc}
T^{k+2}(L, M) & \xrightarrow{\pi} & C^k_{rel}(L, M) \\
\downarrow d_j & & \downarrow D_j \\
T^{k+3}(L, M) & \xrightarrow{\pi} & C^{k+1}_{rel}(L, M)
\end{array}
$$

Proof. Let $\psi \in T^{k+2}(L, M)$. Below we use lemma 2. If $j > i + 1$, then, $(\pi d_j \psi)_i = a_i d_j \psi = d_j a_i \psi$. If $j = i + 1$, then $(\pi(d_j - d_{j-1})\psi)_i = a_i(d_j - d_{j-1})\psi = 0$. If $j < i$, then $(\pi d_j \psi)_i = a_i d_j \psi = d_j a_{i-1}\psi$. Thus,

$$
\begin{aligned}
\pi d_j \psi &= ((\pi d_j \psi)_1, \ldots, (\pi d_j \psi)_{k+2}) \\
&= (d_j a_1 \psi, \ldots, d_j a_{j-2}\psi, a_{j-1} d_j \psi, a_j d_j \psi, d_j a_j \psi, \ldots, d_j a_{k+1}\psi).
\end{aligned}
$$

On the other hand, if $1 \leq l \leq k+2, l \neq j, j-1$, then

$$
\begin{aligned}
&(D_j\pi\psi)_l(x_1,\ldots,x_{k+3})\\
&\quad = -\sum_{s=j+1}^{k+3} a_{l-\delta(j<l)}\psi(x_1,\ldots,\hat{x}_j,\ldots,\underset{s-1}{[x_j,x_s]},\ldots,x_{k+3})\\
&\qquad + [x_j, a_{l-\delta(j<l)}\psi(x_1,\ldots,\hat{x}_j,\ldots,x_{k+3})]\\
&\quad = d_j a_{l-\delta(j<l)}\psi(x_1,\ldots,x_{k+3}).
\end{aligned}
$$

In other words,

$$(D_j\pi\psi)_l = d_j a_{l-\delta(j<l)}, \quad 1 \leq l \leq k+2, \ \ l \neq j, j-1.$$

If $l = j$ or $l = j-1$, then by definition $(D_j\pi\psi)_l = a_l d_j \psi$. So, $\pi d_j \psi = D_j \pi\psi$, for any $\psi \in T^{k+2}(L,M)$.

Corollary 1. *The following diagram is commutative*

$$
\begin{array}{ccc}
T^{*+2}(L,M) & \xrightarrow{\pi} & C^*_{rel}(L,M)\\
\downarrow d_j & & \downarrow D\\
T^{*+3}(L,M) & \xrightarrow{\pi} & C^{*+1}_{rel}(L,M)
\end{array}
$$

Proof. Since $D = \sum_i (-1)^{i+1} D_i$ and $d = \sum_i (-1)^{i+1} d_i$,

$$\pi d = \sum_i (-1)^{i+1}\pi d_i = \sum_i (-1)^{i+1} D_i \pi = D\pi.$$

Corollary 2. *If $j > i$ then $D_j D_i = D_i D_{j-1}$. In particular, if $D = \sum_i (-1)^{i+1} D_i$, then $D^2 = 0$.*

Proof. By lemma 1 any $\Psi \in C^*_{rel}(L,M)$ can be presented as a homomorphic image of some $\psi \in T^{*+2}(L,M) \ : \ \Psi = \pi\psi$. Therefore

$$D_j D_i \Psi = D_j D_i \pi\psi = D_j(\pi d_i \psi) = \pi d_j d_i \psi,$$

and by [2], $\pi d_j d_i \psi = \pi d_i d_{j-1}\psi = D_i \pi d_{j-1}\psi = D_i D_{j-1}\pi\psi$. Thus, $D_j D_i \Psi = D_i D_{j-1}\Psi$, for any $i < j$.

6. PROOF OF THEOREM 1

By corollary 2 $(C^*_{rel}(L,M), \{D_i\})$ is a pre-simplicial complex. In particular, $(C^*_{rel}(L,M), D)$ is a cochain complex.

By lemma 1 and corollary 1 the following sequence of cochain complexes is exact

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$$0 \to C^*(L, M) \to T^*(L, M) \to C^{*-2}_{rel}(L, M) \to 0.$$

Therefore, $H^{*-2}_{rel}(L, M)$ is isomorphic to the cohomology group of the cochain complex $(C^{*-2}_{rel}(L, M), D)$.

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