COMMUNICATIONS IN ALGEBRA, 29(9), 3759–3771 (2001)

JACOBSON FORMULA FOR RIGHT-SYMMETRIC ALGEBRAS IN CHARACTERISTIC p

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Dedicated in memory of A. I. Kostrikin.

ABSTRACT

An algebra is called right-symmetric, if it satisfies the identity $a \circ (b \circ c - c \circ b) = (a \circ b) \circ c - (a \circ c) \circ b$. Right-symmetric algebras over a field of characteristic p are considered. A formula for the p-th power of a sum of two elements of right-symmetric algebras is established. The formula is similar to Jacobson formula for the p-th power of a sum of two elements of Lie algebras.

1. THE MAIN RESULT

An algebra A over a field k of characteristic $p \geq 0$ with multiplication $(a, b) \mapsto a \circ b$ is called *right-symmetric*, if takes place the following identity

$$a \circ (b \circ c) - (a \circ b) \circ c = a \circ (c \circ b) - (a \circ c) \circ b, \quad \forall a, b, c \in A.$$

Right-symmetric algebras was defined in [5], [4], [2]. In fact right-symmetric identity was appeared in considering rooted trees algebra about a hundred years before in [1].

Example. Any associative algebra is right-symmetric.

Example. Let $\mathbf{Z}$ be a ring of integers and $\mathbf{Z}_+$ be a subset of nonnegative integers. Let $\mathbf{m} = (m_1, \ldots, m_n), m_i > 0, m_i \in \mathbf{Z}$. Recall that the multiplication of divided power algebra

$$O_n(\mathbf{m}) = \{x^{(\alpha)} : \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbf{Z}_+, 0 \le \alpha_i < p^{m_i}\}$$

is given by

$$x^{(\alpha)}x^{(\beta)} = \prod_{i=1}^{n} \binom{\alpha_i + \beta_i}{\alpha_i} x^{(\alpha+\beta)}.$$

Let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (ith component is 1). The derivation

$$\partial_i : x^{(\alpha)} \mapsto x^{(\alpha-\epsilon_i)},$$

is called special. Let

$$W_n(\mathbf{m}) = \{x^{(\alpha)}\partial_i : u \in U, i = 1, \ldots, n\}$$

be a space of special derivations of divided power algebra $U = O_n(\mathbf{m})$. Endow $W_n(\mathbf{m})$ by multiplication

$$u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i.$$

This multiplication is right-symmetric. Its Lie algebra is called *Witt* algebra.

Any right-symmetric algebra is Lie-admissable: under commutator $[a, b] = a \circ b - b \circ a$ it can be endowed by a structure of Lie algebra. Denote a Lie algebra obtained from the right-symmetric algebra A by A^{lie}. So, in some sense right-symmetric algebras form a class of algebras between associative and Lie algebras. If L is a Lie algebra, for any $x \in L$, one can correspond derivation $ad\,x : A \to A,\ y \mapsto [x, y]$, called adjoint derivation. For $chap\,k = p > 0$, a Lie algebra L is restricted if and only if, $(ad\,x)^p$ is interior derivation for any $x \in L$. For any element $x \in L$ of a restricted Lie algebra L there exists some element denoted by $x^{[p]} \in L$, such that $(ad\,x)^p = ad\,x^{[p]}$. Then p-structure on L can be given by the map $x \mapsto x^{[p]}$.

Assume now that A be a right-symmetric and $U(A)$ be its universal multiplicative enveloping algebra. Recall that $U(A)$ can be defined as a factor algebra of tensor algebra $T^*(A_r + A_l)$ generated by elements r_a and l_a and relations

$$r_{\alpha a+\beta b} = \alpha r_a + \beta r_b, \quad l_{\alpha a+\beta b} = \alpha l_a + \beta l_b,$$
$$[r_a, r_b] = r_{[a,b]}, \tag{1}$$

$$[r_a, l_b] = l_a l_b - l_{b\circ a}, \tag{2}$$

where a runs elements of A. Here r_a corresponds to the right-multiplication operator

$$R_a : A \to A, \quad b \mapsto b \circ a,$$

and l_a corresponds to the left-multiplication operator

$$L_a : A \to A, \quad b \mapsto a \circ b.$$

Contrary to Lie case for right-symmetric algebras the multiplication operators R_a, L_a, and $ad\, a = R_a - L_a$ in general are not derivations.
Set $a^{\cdot k} = aR_a^{k-1}$, i.e.,

$$a^{\cdot k} = (\cdots((\underbrace{a \circ a) \circ a) \cdots) \circ a}_{k \text{ times}}$$

be kth degree of $a \in A$ in left-normed bracketing. Recall that a linear operator $D : A \to A$, is called derivation, if

$$D(a \circ b) = D(a) \circ b + a \circ D(b), \quad \forall a, b \in A.$$

For $a \in A$, define a linear operator d_a of the universal enveloping algebra $U(A)$ by

$$d_a = r_a^p - r_{a^{\cdot p}}.$$

Let $S(A)$ be a symmetric algebra of A, i.e., an algebra of polynomials on A. Denote by $d(A)$ a subalgebra of $U(A)$ generated by elements $d_a, a \in A$. Define on A a structure of right $U(A)$–module by

$$r_a \mapsto R_a, \quad l_a \mapsto L_a.$$

In particular, A has a structure of right module under $d(A)$. We will prove that $d(A)$ is a commutative subalgebra isomorphic to a symmetric algebra $S(A)$. The map $d : A \to End\, U(A), \quad a \mapsto d_a$, has the following properties.

Theorem 1.1. *For any $a \in A$, an operator $D_a := R_a^p - R_{a^{\cdot p}} \in End\, A$, is a derivation. For any $a, b \in A, \alpha \in k$,*

$$d_{a+b} = d_a + d_b, \quad d_{\alpha a} = \alpha^p d_a, \quad [d_a, d_b] = 0,$$
$$[r_a, d_b] = r_{\{a\, D_b\}}, \quad [l_a, d_b] = l_{\{a\, D_b\}},$$

and

$$(a+b)^{\cdot p} - a^{\cdot p} - b^{\cdot p} = \sum_{i=1}^{p-1} \Lambda_i(a,b),$$

where $i\Lambda_i(a,b)$ *is a coefficient at* t^{i-1} *of* $a\, ad^{p-1}(ta+b)$.

Corollary 1.2. *For any* $a, b \in A$, *take place the following relations*

$$r^p_{a+b} - r^p_a - r^p_b = r_{\{(a+b)^{\cdot p} - a^{\cdot p} - b^{\cdot p}\}} = r_{\left\{\sum_{i=1}^{p-1} \Lambda_i(a,b)\right\}}.$$

A Lie algebra L is restricted if and only if $ad\, x^p$ is interior derivation for any $x \in L$ [3]. By analogy of this statement we give the following definition.

Definition 1.3. *A right-symmetric algebra A over a field of characteristic* $p > 0$ *is called* restricted, *if* $R^p_a = R_{a^{\cdot p}}$, *for any* $a \in A$. *For restricted algebra A a map* $A \to A$, $a \mapsto a^{[p]}$, *such that* $R^p_a = R_{a^{[p]}}$, *is called as p-map.*

In particular, $a \mapsto a^{\cdot p}$ is a p-map.

Corollary 1.4. *If A is a restricted right-symmetric algebra, then elements* d_a *are in the center of the universal enveloping algebra* $U(A)$:

$$[d_a, r_b] = 0, \quad [d_a, l_b] = 0,$$

for any $a, b \in A$.

Definition 1.5. *An element e of right-symmetric algebra A is called left unit, if* $e \circ a = a$, *for any* $a \in A$. *An element e is called* unit, *if* $e \circ a = a \circ e = a$, *for any* $a \in A$. *A subspace* $Ann_r(A) = \{a \in A : R_a = 0\}$ *is called right annulator of A.*

If $a \mapsto a^{[p]_1}$ and $a \mapsto a^{[p]_2}$ are two different p-maps, then $a^{[p]_1} - a^{[p]_2} \in Ann_r(A)$, for any $a \in A$.

If A has left unit, then its right annulator is trivial:

$$R_a = 0 \Rightarrow a = e \circ a = 0.$$

Any algebra may have no more than 1 unit. If A has no unit, one can join it in external way: $A^* = A \oplus \langle e \rangle$, such that $e \circ a = a \circ e = a$, $\forall a \in A$, and $e \circ e = e$, is a right-symmetric algebra with unit e. In particular, any right-symmetric algebra can be completed to a right-symmetric algebra with left unit. For graded right-symmetric algebra A there is another imbedding to right-symmetric algebra with left unit. If $A = \oplus_{i\in Z} A_i$, $A_i \circ A_j \in A_{i+j}$, then an algebra $A^{*_1} = A \oplus \langle e_1 \rangle$, such that

$$e_1 \circ a = a, \quad a \circ e_1 = (\alpha|a| + 1)a,$$

for some fixed $\alpha \in k$, is an algebra with left unit. Here $|a| = i$ means that a is homogeneous: $a \in A_i$. In particular, A^{*_1} is graded and e_1 has degree 0.

Corollary 2.4. shows that a Lie algebra corresponding to restricted right-symmetric algebra is also restricted in Lie sense. A p-map in Lie sense can be given by right-symmetric p-map $a \mapsto a^{\cdot p}$.

Any associative algebra considered as a right-symmetric algebra is restricted. A p-map can be given by $a \mapsto a^p$.

Let us check now right-symmetric Witt algebra $W_n(\mathbf{m})$ for restrictness. Long calculations show that

$$R^p_{u\partial_i} - R_{u\partial_i{}^{\cdot p}} = u^p \partial_i^p.$$

Therefore, Witt algebra $W_n(\mathbf{m})$ is restricted, if and only if $m_i = 1$, for all $i = 1, \ldots, n$. In other words, $W_n(\mathbf{m})$ is restricted as a right-symmetric algebra if and only if $W_n(\mathbf{m})$ is restricted as a Lie algebra. Since right-symmetric Witt algebra $W_n(\mathbf{m})$ has left unit $e = \sum_{i=1}^n x_i \partial_i$, the algebra $W_n(\mathbf{1})$ has unique p-structure. Some examples for right-symmetric p powers:

$$\begin{aligned}
p = 2, \quad (u\partial_i)^{\cdot p} &= uu'\partial_i,\\
p = 3, \quad (u\partial_i)^{\cdot p} &= (u(u')^2 + u^2 u'')\partial_i,\\
p = 5, \quad (u\partial_i)^{\cdot p} &= (u(u')^4\\
&\quad + u^2(u')^2 u'' - u^3(u'')^2 + 2u^3 u' u''' + u^4 u'''')\partial_i,\\
p = 7, \quad (u\partial_i)^{\cdot p} &= (u(u')^6 + u^2(u')^4 u'' + 5u^3(u')^2(u'')^2\\
&\quad - u^4(u'')^3 + 3u^3(u')^3 u''' + 3u^4 u' u'' u''' + u^5(u''')^2\\
&\quad + 6u^4(u')^2 u'''' + 5u^5 u'' u'''' + 2u^5 u' u''''' + u^6 u'''''')\partial_i,
\end{aligned}$$

where $u' = \partial_i(u), u'' = \partial_i^2(u)$, etc.

2. SOME USEFUL FORMULAS

If A is associative (even alternative), then $[r_a, l_a] = 0$. Therefore, by Newton binomial formula

$$(r_a - l_a)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i l_a^i r_a^{k-i},$$

for any $k \in \mathbf{Z}_+$. For right-symmetric algebras analog of this formula is the following.

Lemma 2.1. *For any element a of a right-symmetric algebra A and for any positive integer k,*

$$(r_a - l_a)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i l_{a^{\cdot i}} r_a^{k-i}$$

Here we set $l_{a^{\cdot 0}} = 1$.

Proof. We argue by induction on $k = 1, 2, \ldots$. The statement is true for $k = 1$. Suppose that it is true for k. Then

$$\begin{aligned}(r_a - l_a)^{k+1} &= (r_a - l_a)(r_a - l_a)^k \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i r_a l_{a^{\cdot i}} r_a^{k-i} - \binom{k}{i} (-1)^i l_a l_{a^{\cdot i}} r_a^{k-i} \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i (l_{a^{\cdot i}} r_a) r_a^{k-i} + \binom{k}{i} (-1)^i [r_a l_{a^{\cdot i}}] r_a^{k-i} \\ &\quad - \binom{k}{i} (-1)^i l_a l_{a^{\cdot i}} r_a^{k-i}.\end{aligned}$$

By (2),

$$[r_a, l_{a^{\cdot i}}] = l_a l_{a^{\cdot i}} - l_{a^{\cdot \{i+1\}}}.$$

Therefore,

$$\begin{aligned}(r_a - l_a)^{k+1} &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i l_{a^{\cdot i}} r_a^{k-i+1} + \binom{k}{i} (-1)^i l_a l_{a^{\cdot i}} r_a^{k-i} \\ &\quad - \binom{k}{i} (-1)^i l_{a^{\cdot \{i+1\}}} r_a^{k-i} - \binom{k}{i} (-1)^i l_a l_{a^{\cdot i}} r_a^{k-i} \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i l_{a^{\cdot i}} r_a^{k-i+1} - \binom{k}{i} (-1)^i l_{a^{\cdot \{i+1\}}} r_a^{k-i} \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i l_{a^{\cdot i}} r_a^{k-i+1}.\end{aligned}$$

Lemma 2.1 is proved completely. □

Corollary 2.2. $(r_a - l_a)^p = r_a^p - l_{a^{\cdot p}}$.

Proof. Use Lemma 2.1 for $k = p$ and the following arithmetic result:

$$\binom{p}{i} \equiv 0 (mod\, p),$$

if $0 < i < p$. □

Corollary 2.3. $(r_a - l_a)^{p-1} = \sum_{i=0}^{p-1} l_{a^{\cdot i}} r_a^{\{p-1-i\}}$.

Proof. Use Lemma 2.1 for $k = p - 1$ and the following number-theoretic result:

$$\binom{p-1}{i} \equiv (-1)^i (mod\, p),$$

for any $0 \leq i < p$. □

Corollary 2.4. *If A is restricted right-symmetric algebra, then* A^{lie} *is a restricted Lie algebra. Corresponding p-map on* A^{lie} *can be given by* $a \mapsto a^{\cdot p}$.

Proof. By corollary 2.2, $ad^p a = r_a^p - l_a^p$. If $r_a^p = r_{a^{\cdot p}}$, then

$$ad^p a = r_{a^{\cdot p}} - l_{a^{\cdot p}} = ad\, a^{\cdot p}$$

Therefore, A^{lie} is restricted.

Denote by Γ_n a set of vectors $\alpha = (\alpha_1, \ldots, \alpha_n)$ with integral coordinates $\alpha_i \in \mathbf{Z}$. Let $\Gamma_n^+ = \{\alpha \in \Gamma : \alpha_i > 0, i = 1, \ldots, n\}$. Let $\Gamma = \cup_n \Gamma_n$ and $\Gamma^+ = \cup_n \Gamma_n^+$. For $\alpha \in \Gamma$, set

$$|\alpha| = \sum_i \alpha_i.$$

For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \Gamma$, such that $|\alpha| = \alpha_1 + \cdots + \alpha_l \geq 0$, let

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1! \cdots \alpha_l!}$$

be multinomial coefficient. Set

$$\binom{|\alpha|}{\alpha} = 0,$$

if $\alpha_s < 0$, for some s. Recall that ϵ_i is the vector with ith coordinate 1 (other coordinates are 0, number of coordinates will be clear from context). For $\alpha \in \Gamma$, denote by α_0 its last coordinate and let $\bar{\alpha} \in \Gamma$ be α without α_0. So, $\alpha = (\bar{\alpha}, \alpha_0)$ and

$$\binom{|\alpha|}{\alpha} = \binom{|\alpha|}{\bar{\alpha}\, \alpha_0} = \binom{|\bar{\alpha}|}{\bar{\alpha}} \binom{|\alpha|}{\alpha_0}.$$

The following relation for multinomial coefficients is well known:

$$\binom{|\alpha|}{\alpha} = \sum_i \binom{|\alpha-\epsilon_i|}{\alpha-\epsilon_i} \tag{3}$$

Let

$$\Omega^+(q,n) = \{\alpha \in \Gamma_n^+ : |\alpha| = q\},$$
$$\Omega(q,n) = \{\alpha = (\bar\alpha, \alpha_0) \in \Gamma_n : |\alpha| = q, \quad \bar\alpha \in \Omega^+(q-\alpha_0, n-1),\ \alpha_0 \geq 0\}.$$

Notice that

$$|\Omega(q,n)| = \binom{q}{n-1}, \quad |\Omega(q)| = 2^q.$$

Let

$$\Omega(q) = \bigcup\nolimits_{n\geq 1} \Omega(q,n).$$

For $\alpha \in \Omega^+(q)$, set $h(\alpha) = n$, if $\alpha \in \Omega^+(q,n)$. For $\alpha \in \Omega(q)$, set $l(\alpha) = h(\bar\alpha)$.

Notice that

$$\Omega(q+1, n+1) = \bigcup\nolimits_{l=1}^{q+1} \Omega(l,n).$$

For $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_0) \in \Omega(q,n)$, set

$$l_{b^{\cdot\bar\alpha}} = l_{b^{\cdot\alpha_1}} \cdots l_{b^{\cdot\alpha_{n-1}}},$$
$$l_{b^{\cdot\alpha}} = l_{b^{\cdot\bar\alpha}} l_{a R_b^{\alpha_0}}.$$

□

Lemma 2.5. *For any $q \in \mathbf{Z}_+$,*

$$l_a\, ad^q r_b = \sum_{\alpha\in\Omega(q)} (-1)^{l(\alpha)} \binom{|\alpha|}{\alpha} l_{b^{\cdot\bar\alpha}} l_{a R_b^{\alpha_0}}.$$

Example. Let $q = 3$. Then

$$l_a\, ad^3 r_b = -l_{(b\circ b)\circ b} l_a + 3 l_{b\circ b} l_b l_a + 3 l_b l_{b\circ b} l_a - 6 l_b^3 l_a$$
$$- 3 l_{b\circ b} l_{a\circ b} + 6 l_b^2 l_{a\circ b} - 3 l_b l_{(a\circ b)\circ b} + l_{((a\circ b)\circ b)\circ b}.$$

and

$$\Omega^+(3,1) = \{(3)\},\ \Omega^+(3,2) = \{(1,2),(2,1)\},\ \Omega^+(3,3) = \{(1,1,1)\},$$
$$\Omega^+(3,i) = \emptyset, \quad \text{if } i > 3 \text{ or } i \leq 0,$$
$$\Omega(3,1) = \{(3)\},\ \Omega(3,2) = \{(1,2),(2,1),(3,0)\}$$
$$\Omega(3,3) = \{(1,1,1),(1,2,0),(2,1,0)\},\ \Omega(3,4) = \{(1,1,1,0)\},$$
$$\Omega^+(3,i) = \emptyset, \quad \text{if } i > 4 \text{ or } i \leq 0.$$

Proof of Lemma 2.5. For some statement $\mathcal{X}$ set $\delta(\mathcal{X}) = 1$, if $\mathcal{X}$ is true and $= 0$, if $\mathcal{X}$ is false. Let

$$l_a \, ad^q r_b = \sum_{s\geq 0} \sum_{\beta\in\Omega^+(q-s)} \lambda_{\beta,s} l_{b\cdot\beta} l_{a\,R_b^s},$$

for some $\lambda_{\beta,s} \in k$. We should prove that

$$\lambda_{\beta,s} = (-1)^{h(\beta)} \binom{|\beta| + \delta(s>0)}{\beta \quad s}.$$

We use induction on q. For $q = 1$, the statement follows from (2). Suppose that it is true for $q - 1$. Then

$$\begin{aligned}
l_a \, ad^{q+1} r_b &= -[r_b, l_a \, ad^q r_b] \\
&= \sum_{s\geq 0} \sum_{\alpha\in\Omega^+(q-s)} (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} [r_b, l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_{aR_b^s}] \\
&= \sum_{s\geq 0} \sum_{\alpha\in\Omega^+(q-s)} \sum_{r=1}^{h(\alpha)} (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{r-1}} l_b l_{b\cdot\alpha_r} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_{aR_b^s} \\
&\quad + (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_b l_{aR_b^s} \\
&\quad - (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_{aR_b^{s+1}} \\
&= \sum_{\alpha\in\Omega^+(q)} \sum_{r=1}^{h(\alpha)} (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{r-1}} l_b l_{b\cdot\alpha_r} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_a \\
&\quad + (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_b l_a \\
&\quad + \sum_{s\geq 0} \sum_{\alpha\in\Omega^+(q-s)} \sum_{r=1}^{h(\alpha)} (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{r-1}} l_b l_{b\cdot\alpha_r} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_{aR_b^{s+1}} \\
&\quad + (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_b l_{aR_b^{s+1}} \\
&\quad - (-1)^{h(\alpha)} \binom{|\alpha|}{\alpha} l_{b\cdot\alpha_1} \cdots l_{b\cdot\alpha_{h(\alpha)}} l_{aR_b^{s+1}}.
\end{aligned}$$

So,

$$\lambda_{\beta,s} = \sum_{t\geq 1,\ \beta_t>0} (-1)^{h(\beta-\epsilon_t)-h(\beta)} \lambda_{\beta-\epsilon_t,s} + \lambda_{\beta,s-1}.$$

For $s = 0$ by inductive proposal

$$\lambda_{\beta,0} = \sum_{t\geq 1,\, \beta_t>0} (-1)^{h(\beta-\epsilon_t)-h(\beta)}\lambda_{\beta-\epsilon_t,0}$$
$$= \lambda_{\beta,0} = \sum_{t\geq 1,\, \beta_t>0} (-1)^{h(\beta)}\binom{|\beta-\epsilon_t|}{\beta-\epsilon_t\ \ 0}.$$

Therefore, by (3),

$$\lambda_{\beta,0} = \sum_{t\geq 1,\, \beta_t>0} (-1)^{h(\beta)}\binom{|\beta-\epsilon_t|}{\beta-\epsilon_t} = (-1)^{h(\beta)}\binom{|\beta|}{\beta}.$$

By similar reasons, for $s > 0$, we have

$$\lambda_{\beta,s} = \sum_{t\geq 1,\, \beta_t>0} (-1)^{h(\beta)}\binom{|\beta-\epsilon_t|+s}{\beta-\epsilon_t\ \ s} + (-1)^{h(\beta)}\binom{|\beta|+s-1}{\beta\ \ s-1}$$
$$= (-1)^{h(\beta)}\binom{|\beta\ \ s|}{\beta\ \ s}.$$

Lemma is proved completely. □

Lemma 2.6. $[l_a, d_b] = l_{\{a\,D_b\}}$.

Proof. By Lemma 2.5 for $q = p$, we have

$$[l_a, r_b^p] = l_a\, ad^p r_a = l_{aR_b^p} - l_{b^{\cdot p}} l_a.$$

It remains to notice that

$$[l_a, r_{b^{\cdot p}}] = -l_{b^{\cdot p}} l_a + l_{aR_{b^{\cdot p}}}.$$ □

Corollary 2.7. $[L_a, D_b] = L_{\{a\,D_b\}}$.

Lemma 2.8. $[r_a, d_b] = r_{\{a\,D_b\}}$.

Proof. We have

$$[r_a, d_b] = [r_a, r_b^p - r_{b^{\cdot p}}] = r_a\, ad^p r_b - r_{\{[a,b^{\cdot p}]\}} = r_{\{a\, ad^p b - a\, ad\, b^{\cdot p}\}}.$$

By Corollary 2.2, $a\, ad^p b = a\, r_b^p - b^{\cdot p}\circ a$. Thus

$$[r_a, d_b] = r_{\{aR_b^p - b^{\cdot p}\circ a - a\circ b^{\cdot p} + b^{\cdot p}\circ a\}} = r_{\{aR_b^p - a\, R_{b^{\cdot p}}\}} = r_{\{a\,D_b\}}.$$ □

Corollary 2.9. $[R_a, D_b] = R_{\{a\,D_b\}}$.

Lemma 2.10. *For any $a \in A$, $D_a \in Der\, A$.*

Proof. This statement and Corollary 2.7 and Corollary 2.9 are equivalent. □

Corollary 2.11. $(a\,D_b)\,ad^{p-1}a - a^{\cdot p}\,D_b = 0$

Proof. By Lemma 2.10,

$$\begin{aligned} a^{\cdot p} D_b &= (a^{\cdot p-1} \circ a) D_b = (a^{\cdot p-1}\, D_b) \circ a + a^{\cdot p-1} \circ (a\, D_b) \\ &= (a^{\cdot p-2} D_b)\, R_a^2 + (a^{\cdot p-2} \circ (a\, D_b))\, R_a + a^{\cdot p-1} \circ (a\, D_b) \\ &= \cdots = (a\, D_b) R_a^{p-1} + \cdots + (a^{\cdot p-2} \circ (a\, D_b))\, R_a + a^{\cdot p-1} \circ (a\, D_b). \end{aligned}$$

On the other hand, by Corollary 2.3,

$$(a\, D_b)\, ad^{p-1} a = \sum_{i=0}^{p-1} (a^{\cdot i} \circ (a\, D_b)) R_a^{p-1-i}.$$

This proves our corollary. □

Lemma 2.12. $[d_a, d_b] = 0, \quad a, b \in A.$

Proof. By Lemma 2.8,

$$\begin{aligned} [d_a, d_b] &= [r_a^p, d_b] - [r_{a^{\cdot p}}, d_b] = -d_b\, ad^p r_a - r_{\{a^{\cdot p}\, D_b\}} \\ &= [r_a, d_b]\, ad^{p-1} r_a - r_{\{a^{\cdot p}\, D_b\}} \\ &= r_{\{a\, D_b\}}\, ad^{p-1} r_a - r_{\{a^{\cdot p}\, D_b\}} \\ &= r_{\{(a\, D_b)\, ad^{p-1} a - a^{\cdot p}\, D_b\}}. \end{aligned}$$

It remains to use Corollary 2.11. □

Lemma 2.13. *For left-normed p-powers of right-symmetric algebra takes place the following formula*

$$(a+b)^{\cdot p} - a^{\cdot p} - b^{\cdot p} = \sum_{i=1}^{p-1} \Lambda_i(a, b),$$

where $i\Lambda_i(a,b)$ *is a coefficient at* t^{i-1} *of* $a\, ad^{p-1}(ta+b)$.

Proof. We repeat arguments of the proof of Jacobson formula for the pth power of a sum of two elements in restricted Lie algebras. Present element $X = (ta+b)^{\cdot p}$ as a sum of polynomials on t:

$$X = t^p a^{\cdot p} + \sum_{i=1}^{p-1} t^i \Lambda_i(a, b) + b^{\cdot p}, \tag{4}$$

for some $\Lambda_i(a,b) \in A$. Prove that

$$a\,ad^{p-1}(ta+b) = \sum_{i=1}^{p-1} i\Lambda_i(a,b)t^{i-1}. \tag{5}$$

We have

$$\frac{\partial X}{\partial t} = \sum_{i=1}^{p-1} it^{i-1}\Lambda_i(a,b), \tag{6}$$

By Leibniz rule,

$$\frac{\partial X}{\partial t} = \sum_{i=1}^{p}((ta+b)^{\cdot\{i-1\}} \circ a)r_{\{ta+b\}}^{\{p-i\}} = \sum_{i=1}^{p} a\,l_{(ta+b)^{\cdot\{i-1\}}}r_{\{ta+b\}}^{\{p-i\}}.$$

By Lemma 2.2,

$$a\,ad^{p-1}(ta+b) = \sum_{i=0}^{p-1} a\,l_{(ta+b)^{\cdot i}}r_{(ta+b)}^{p-1-i}.$$

So, (6) is proved. Lemma 2.13 follows from relation (4) for $t = 1$. □

3. PROOF OF THEOREM 1.1

Lemma 2.8, 2.6, 2.10, 2.12, 2.13.

ACKNOWLEDGMENTS

I am grateful to Swedish Royal Society for support.

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Received January 2001