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N-commutators

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In memory of my wife Kuralay and son Arman

Abstract. The N-commutator

$$s_N(X_1,\dots,X_N)=\sum_{\sigma\in\mathfrak{S}_N}\operatorname{sign}\sigma\, X_{\sigma(1)}\cdots X_{\sigma(N)}$$

is conjecturally a well-defined nontrivial operation on $W(n)=Der\ \mathbb{K}[x]$ for $x=(x_1,\dots,x_n)$ if and only if $N=n^2+2n-2$. This is proved for $n=2$ and confirmed by computer experiments for $n<5$.

Under 2- and 5-commutators the algebra of divergence-free vector fields in two dimensions is an sh-Lie (strong homotopic Lie) algebra in the sense of Stasheff. Similarly, $W(2)$ is an sh-Lie algebra with respect to 2- and 6-commutators.

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1. Introduction

1.1. Notations

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. By $\mathbb{Z}_+$ we denote the set of nonnegative integers.

Let $U=\mathbb{K}[x]$ for $x=(x_1,\dots,x_n)$ and $W(n)=Der\ \mathbb{K}[x]$ be the Lie algebra of polynomial vector fields and $\mathrm{Diff}(n)=S_U(W(n))$ the associative algebra of differential operators with smooth or polynomial coefficients. When $\mathrm{Diff}(n)$ is considered with the commutator rather than juxtaposition as the product, we write $\mathfrak{Diff}(n)$; other products will also be used.

A vector space A is called a k-*algebra* with multiplication ω and denoted $A=(A,\omega)$, if ω is a polylinear map $A\otimes\cdots\otimes A\longrightarrow A$ with $k\geq 2$ arguments. Usually, *multiplication* is as a bilinear map and instead of $\omega(a,b)$ one writes $a\circ b$ or $a\cdot b$. In such cases we will call A just algebra and write $A=(A,\circ)$ or $A=(A,\cdot)$.

1.2. $W(n)$ with right-symmetric multiplication

On $W(n)$, let $\circ$ be the multiplication ($\partial_i = \partial/\partial x_i$)

$$u\partial_i \circ v\partial_j = v\partial_j(u)\partial_i.$$

Recall that the multiplication $\circ$ is *right-symmetric* if it satisfies the right-symmetric identity

$$(X_1, X_2, X_3) = (X_1, X_3, X_2),$$

where

$$(X_1, X_2, X_3) = X_1 \circ (X_2 \circ X_3) - (X_1 \circ X_2) \circ X_3$$

is the associator. Right-symmetric algebras are called also *pre-Lie, Vinberg,* or *Vinberg–Koszul* [1], [12], [17].

Main example. $(W(n), \circ)$ is right-symmetric.

Observe that usually the action of a vector field on a function is denoted by $X(u)$, but considering right-symmetric algebras $(W(n), \circ)$ and the associated Lie algebras we denote such action by $(u)X$. Therefore, the commutator given above for $W(n)$ and the commutator in the Lie algebra obtained from right-symmetric algebra $(W(n), \circ)$ differ by a sign.

1.3. Problem formulation

The subspace $W(n) \subset \mathrm{Diff}(n)$ is not a subalgebra with respect to composition. If $X = u_i\partial_i$, $Y = v_j\partial_j$ are differential operators of first order, then their composition

$$X \cdot Y = v_j\partial_j(u_i)\partial_i + u_i v_j \partial_i\partial_j$$

is a differential operator of second order. It has nontrivial quadratic differential part $u_i v_j \partial_i \partial_j$. But $W(n) \subset \mathfrak{Diff}(n)$ is a Lie subalgebra: it is closed under commutator since $\partial_i\partial_j = \partial_j\partial_i$. This well-known fact has the following interpretation in terms of skew-symmetric polynomials. Let $\mathfrak{S}_k$ be a permutation group. Let

$$s_k(t_1, \dots, t_k) = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}\sigma\, t_{\sigma(1)} \cdots t_{\sigma(k)}$$

be the standard skew-symmetric polynomial. Then instead of t_i we can substitute any differential operator from $\mathrm{Diff}(n)$.

Clearly, $s_2(X, Y) = [X, Y] \in W(n)$ for any $X, Y \in W(n)$. Does there exist $k > 2$, such that s_k is also a well defined operation on $W(n)$? Since $X_i = \sum_{j=1}^{n} u_{ij}\partial_j$, $i = 1, 2, \dots, k$, are first order differential operators, $X_{\sigma(1)} \cdots X_{\sigma(k)}$ is, in general, a k-th order differential operator and so is $s_k(X_1, \dots, X_k)$.

Surprisingly, for some special $k = k(n)$ it might happen that all higher degree differential parts of $s_k(X_1, \cdots, X_k)$, like quadratic differential part of s_2, can be cancelled for any $X_1, \dots, X_k \in W(n)$, but the first order part remains.

Let us consider $W(2)$. We prove that s_6 is a well defined non-trivial 6-linear map on $W(2)$:

$$s_6(X_1,\dots,X_6)\in W(2) \text{ for any } X_1,\dots,X_6\in W(2),$$

and

$$s_6(X_1,\dots,X_6)\neq 0 \text{ for some } \quad X_1,\dots,X_6\in W(2).$$

The number 6 here is unique:

$$s_k(X_1,\dots,X_k)=0, \quad \text{ for any } X_1,\dots,X_k\in W(2) \text{ and any } k>6$$

and $s_k(X_1,\dots,X_k)$ has a non-trivial quadratic differential part for some X_1, ..., $X_k\in W(2)$ and any $2<k<6$.

Consider $S(2)\subset W(2)$, the Lie subalgebra of divergence free vector fields. We will prove that on $S(2)$ the unique analog of the above is the 5-commutator:

$$s_5(X_1,\dots,X_5)\in S(2) \text{ for any } X_1,\dots,X_5\in S(2)$$

and

$$s_5(X_1,\dots,X_5)\neq 0 \text{ for some } X_1,\dots,X_5\in S(2).$$

Moreover,

$$k>5\Rightarrow s_k(X_1,\dots,X_k)=0 \text{ for any } X_1,\dots,X_k\in S(2).$$

If $2<k<5$, then $s_k(X_1,\dots,X_k)$ has a non-trivial quadratic differential part for some $X_1,\dots,X_k\in S(2)$.

So, the vector space $W(2)$ can be endowed with a Lie algebra structure with respect to s_2, usually denoted by [,], and the 6-commutator s_6. Similarly, $S(2)$ can be endowed by a structure of Lie algebra under 2-commutator s_2 and the 5-commutator s_5. These commutators have the following nice properties.

1.4. 5- and 6-commutators and right symmetric products

Let $A=W(2)$ or $S(2)$ and $X,Y,X_1,X_2,\ldots\in A$. It is well known that the right adjoint representation $ad\,X$, defined by $(Y)ad\,X=[Y,X]$, for any $X,Y\in A$ is a derivation. The commutator $[X,Y]$ can be represented in terms of right-symmetric multiplication: $X\cdot Y-Y\cdot X=X\circ Y-Y\circ X$. These facts have analogies for 5- and 6-commutators.

The following Leibniz rule holds: for any $X,X_1,\dots,X_5\in S(2)$ we have

$$[X,s_5(X_1,\dots,X_5)]=\sum_{i=1}^{5}s_5(X_1,\dots,X_{i-1},[X,X_i],X_{i+1},\dots,X_5).$$

To calculate the 5-commutator of $X_1,\dots,X_5$, one can use right-symmetric multiplication:

$$\sum_{\sigma\in\mathfrak{S}_5} \operatorname{sign}\sigma\, X_{\sigma(1)}\cdot X_{\sigma(2)}\cdot X_{\sigma(3)}\cdot X_{\sigma(4)}\cdot X_{\sigma(5)}$$
$$=\sum_{\sigma\in\mathfrak{S}_5} \operatorname{sign}\sigma\,(((X_{\sigma(1)}\circ X_{\sigma(2)})\circ X_{\sigma(3)})\circ X_{\sigma(4)})\circ X_{\sigma(5)}.$$

In other words,

$$s_5(X_1,\dots,X_5)=s_5^{rsym.r}(X_1,\dots,X_5)\quad \text{for any } X_1,\dots,X_5\in S(2).$$

Usage of right-symmetric multiplication simplifies calculation of 5-commutators. The 5-commutator satisfies the following 4-*left commutativity identity*:

$$\sum_{\sigma\in\mathfrak{S}_8} \operatorname{sign}\sigma\, s_5(X_{\sigma(1)},X_{\sigma(2)},X_{\sigma(3)},X_{\sigma(4)},s_5(X_{\sigma(5)},X_{\sigma(6)},X_{\sigma(7)},X_{\sigma(8)},X_0))=0$$

for any $X_0,X_1,\dots,X_8\in S(2)$.

Similar results are true for 6-commutator. One can calculate 6-commutator by right-symmetric multiplication:

$$\sum_{\sigma\in\mathfrak{S}_6} \operatorname{sign}\sigma\, X_{\sigma(1)}\cdot X_{\sigma(2)}\cdot X_{\sigma(3)}\cdot X_{\sigma(4)}\cdot X_{\sigma(5)}\cdot X_{\sigma(6)}$$
$$=\sum_{\sigma\in\mathfrak{S}_6} \operatorname{sign}\sigma\,((((X_{\sigma(1)}\circ X_{\sigma(2)})\circ X_{\sigma(3)})\circ X_{\sigma(4)})\circ X_{\sigma(5)})\circ X_{\sigma(6)},$$

for any $X_1,\dots,X_6\in W(2)$. The 6-commutator is 5-left commutative:

$$\sum_{\sigma\in\mathfrak{S}_{10}} \operatorname{sign}\sigma\, s_6\big(X_{\sigma(1)},X_{\sigma(2)},X_{\sigma(3)},X_{\sigma(4)},X_{\sigma(5)},$$
$$s_6(X_{\sigma(6)},X_{\sigma(7)},X_{\sigma(8)},X_{\sigma(9)},X_{\sigma(10)},X_0)\big)=0$$

for any $X_0,X_1,\dots,X_{10}\in W(2)$.

A property that the 5-commutator on $S(2)$ has, but the 6-commutator on $W(2)$ does not, is as follows. It is not true that the a composition of adjoint derivations is a derivation. Well known that ad is a Lie algebra homomorphism:

$$[ad\,X, ad\,Y]=ad\,[X,Y]\quad \text{for any } X,Y\in W(n).$$

In general, it is also false that $s_k(ad\,X_1,\dots,ad\,X_k)$ is a derivation. However, for $S(2)$ and $k=5$ it is:

$$s_5(ad\,X_1,\dots,ad\,X_5)=ad\,s_5(X_1,\dots,X_5)\ \text{ for any } X_1,\dots,X_5\in S(2).$$

A similar result for a 6-commutator is no longer true. For example,

$$F=s_6(ad\,\partial_1, ad\,\partial_2, ad\,x_1\partial_1, ad\,x_2\partial_1, ad\,x_1\partial_2, ad\,x_2\partial_2)\in \operatorname{End} W(2),$$

as a linear operator on $W(2)$ is defined by

$$(u_1\partial_1+u_2\partial_2)F=6(\partial_1\partial_2(u_1)+\partial_2^2(u_2))\partial_1-6(\partial_2^2(u_1)+\partial_1\partial_2(u_2))\partial_2.$$

We see that F has nontrivial quadratic differential part, so it is not even a derivation of the Lie algebra $(Vect(2), [\ ,\ ])$.

Note also the following relation between 5 and 6-commutators and divergences of vector fields:

$$s_6^{rsym.r}(X_1,\dots,X_6) = \sum_{i=1}^{6}(-1)^{i+1}(\operatorname{Div} X_i)\, s_5^{rsym.r}(X_1,\dots,\hat{X}_i,\dots,X_6),$$

for any $X_1,\dots,X_6 \in W(2)$. Here one can change on the right hand $s_5^{rsym.r}$ to s_5, despite of the fact that s_5 is not well defined on $W(2)$.

The quadratic differential part of the 5-commutator can be represented as a sum of three determinants (see lemma 7.4). All quadratic differential terms of s_5 are cancelled in taking alternative sum:

$$s_6^{rsym.r}(X_1,\dots,X_6) = \sum_{i=1}^{6}(-1)^{i+1}(\operatorname{Div} X_i)\, s_5(X_1,\dots,\hat{X}_i,\dots,X_6),$$

for any $X_1,\dots,X_6 \in W(2)$. Recall that $s_6 = s_6^{rsym.r}$ on $W(2)$.

Notice that here 6 and 5 can not be changed to smaller numbers. For example,

$$s_5(\partial_1,\partial_2,x_1\partial_1,x_2\partial_1,x_1\partial_2) - \operatorname{Div}(x_1\partial_1)s_4(\partial_1,\partial_2,x_2\partial_1,x_1\partial_2)$$
$$= -3\partial_1\partial_2 \neq 0.$$

Notice that these results, valid for 5 and 6-commutators, are not valid for lower degree commutators. Namely, s_3, s_4 for $S(2)$ and s_3, s_4, s_5 for $W(2)$ have no such properties. One can state some weaker versions of these statements.

Let $\mathfrak{gl}_2 \ni \mathbb{K}^2$ be the semi-direct sum of $\mathfrak{gl}_2$ and the identity module. For example, if $k = 3, 4, 5$, then

$$[X, s_k^{rsym.r}(X_1,\dots,X_k)] = \sum_{i=1}^{k} s_k^{rsym.r}(X_1,\dots,X_{i-1},[X,X_i],X_{i+1},\dots,X_k)$$

for any $X \in \mathfrak{gl}_2 \ni \mathbb{K}^2$, and $X_1,\dots,X_k \in W(2)$.

1.5. Strongly homotopic (sh-) algebras, n-Lie algebras, and $(n-1)$-left commutative algebras

For vector spaces M and N set $T^k(M,N) = \operatorname{Hom}(M^{\otimes k}, N)$ if $k > 0$, let $T^0(M,N) = N$ and $T^k(M,N) = 0$ if $k < 0$.

Let $T^*(M,N) = \oplus_k T^k(M,N)$. Let $\wedge^k M$ be the k-th exterior power of M. Set $C^k(M,N) = \operatorname{Hom}(\wedge^k M, N) \subseteq T^k(M,N)$, let $C^0(M,N) = N$, and $C^k(M,N) = 0$ if $k < 0$. Set $C^*(M,N) = \oplus_k C^k(M,N)$.

Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be a set of polylinear maps $\omega_i \in C^i(A,A)$. Let (A,Ω) be an algebra with vector space A and signature Ω [13] which means that A is endowed with the i-ary multiplication ω_i. Call A an Ω-*algebra* and in case where $\Omega = \{\omega_i\}$ set $A = (A, \omega_i)$.

An algebra (A, ω) is called an n-*Lie* [8], *Filipov* or *Nambu* algebra, if

$$\omega(a_1, \dots, a_{n-1}, \omega(a_n, \dots, a_{2n-1})) = \sum_{i=0}^{n-1} \omega(a_n, \dots, \omega(a_1, \dots, a_{n-1}, a_{n+i}), a_{i+1}, \dots, a_{2n-1});$$

(A, ω) is $(n-1)$-*left commutative* if

$$\sum_{\sigma \in \mathfrak{S}_{2n-2}} \operatorname{sign} \sigma\, \omega(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, \omega(a_{\sigma(n)}, \dots, a_{\sigma(2n-2)}, a_{2n-1})) = 0,$$

and n-*homotopy Lie* if

$$\sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sign} \sigma\, \omega(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, \omega(a_{\sigma(n)}, \dots, a_{\sigma(2n-2)}, a_{\sigma(2n-1)})) = 0,$$

for any $a_1, \dots, a_{2n-1} \in A$.

Finally, an algebra (A, Ω), where $\Omega = \{\omega_1, \omega_2, \dots\}$ is called a *strongly homotopy Lie* or *sh*-Lie [14], if

$$\sum_{i+j=k+1,\ i,j \geq 1} (-1)^{(j-1)i} \operatorname{sign} \sigma\, \omega_j(\omega_i(a_{\sigma(1)}, \dots, a_{\sigma(i)}), a_{\sigma(i+1)}, \dots, a_{\sigma(i+j-1)}) = 0,$$

for any $k = 1, 2, \dots$, and any $a_1, \dots, a_{i+j-1} \in A$.

In particular, an n-homotopy Lie algebra is an *sh*-algebra if Ω consists of only one non-zero multiplication, ω_n.

Suppose now that Ω consists of two elements ω_2 and ω_n. Then the condition that (A, Ω) is *sh*-Lie means that (A, ω_2) is a Lie algebra, (A, ω_n) is a n-homotopy Lie and ω_n is a n-cocycle of the adjoint module of the Lie algebra (A, ω_2).

In [5] it is established that, over the field of characteristic 0, any n-Lie algebra is $(n-1)$-left commutative and any $(n-1)$-left commutative algebra is n-homotopy Lie. Here we prove that $(S(2), s_5)$ and $(W(2), s_6)$ are 4- *and* 5-*left commutative*, respectively. Hence, $(S(2), s_5)$ *is* 5-*homotopy Lie* and $(W(2), s_6)$ *is* 6-*homotopy Lie.*

The algebra $S(2)$ endowed with multiplications $\omega_1, \omega_2, \omega_3, \dots$, such that $\omega_i = s_i$ for $i = 2, 5$ and $\omega_i = 0$ for $i \neq 2, 5$, is an *sh*-algebra. In particular, s_5 is a 5-cocycle of the adjoint module of the Lie algebra $S(2)$. Moreover, the following relations hold for any $X_1, \dots, X_6 \in S(2)$:

$$\sum_{1 \leq i < j \leq 6} (-1)^{i+j} s_5^{rsym.r}([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_6) = 0,$$

$$\sum_{1 \leq i \leq 6} (-1)^i [X_i, s_5^{rsym.r}(X_1, \dots, \hat{X}_i, \dots, X_6)] = 0.$$

1.6. Primitive commutators

For any two vector fields their commutator is once again a vector field. One can repeat this procedure $k-1$ times and construct from any k vector fields a new vector field. This can be done in many ways. One can get a linear combination of such commutators. So, in general there are many ways to construct invariant k-operation on $S(n)$ or $W(n)$. We call the invariant operations obtained in such ways *standard*. Call any k-linear invariant non-standard operation on $W(n)$ or its subalgebra a *primitive k-commutator*. We prove that the 5-commutator and the 6-commutator are primitive.

$S(2)$: Any divergence free vector field in two variables can be represented in terms of the generating function u as a Hamiltonian vector field

$$H_u = \partial_1(u)\partial_2 - \partial_2(u)\partial_1.$$

Let $u = (u_1, u_2, u_3, u_4, u_5)$, and $Du = (Du_1, Du_2, Du_3, Du_4, Du_5)$ be term-wise derivative along the field D; let $\partial_{12} = \partial_1\partial_2$. Set

$$[u] = \det \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \partial_1^2 u \\ \partial_{12} u \\ \partial_2^2 u \end{pmatrix}.$$

We find that the formula for the 5-commutator on $S(2)$ is rather simple:

$$s_5(H_{u_1}, H_{u_2}, H_{u_3}, H_{u_4}, H_{u_5}) = -3H_{[u]}.$$

$W(2)$: Let $X_i = u_{i,1}\partial_1 + u_{i,2}\partial_2 \in W(2)$ for $i = 1, \ldots, 6$. We will show that the 6-commutator $s_6(X_1, \ldots, X_6)$ can be presented as a linear combination with integral coefficients of fourteen 6×6 determinants of the form

$$\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ * & * & * & * & * & * \end{vmatrix} \partial_i$$

where $i = 1, 2$. The exact formula for 6-commutator is given in theorem 11.1.

I know similar formulas for $n = 3$ and 4. Perhaps, in better notations, they can be presented as understandable ones, but at the moment they look too lengthy and incomprehensible.

1.7. Related results

Left identities of $W(n)$ as a right-symmetric algebra was considered in [4]. There are many works about identities of $W(n)$ as a Lie algebra (see references in [15]). Identities of $W(2)$ as a Lie algebra was described in [11].

2. Main results

Theorem 2.1. *Let* $N = n^2 + 2n - 2$. *Then*

(i) $s_k = s_k^{rsym.r}$ *on* $W(n)$, *for any* $k \geq N$. *In particular,* s_k *is well defined on* $W(n)$, *if* $k \geq N$.

(ii) $s_k = 0$ *is an identity on* $W(n)$ *for any* $k > N + 1$.

(iii) $(W(n), s_N)$ *is* $(N-1)$*-left commutative. In particular,* $(W(n), s_N)$ *is* N*-homotopy Lie.*

(iv) $ad\,X \in Der\,(W(n), s_k)$ *for any* $X \in W(n)$ *and for any* $k \geq N$.

(v) $ad\,X \in Der\,(W(n), s_k)$ *for any* $X \in W(n)$ *such that* $\partial_i \partial_j (X) = 0, i, j = 1, \ldots, n$. *Here* k *is any integer* > 0.

Theorem 2.2. (i) $s_5 \neq 0$ *on* $S(2)$.

(ii) s_5 *is a* 5*-commutator on* $S(2)$.

(iii) $s_6 = 0$ *is an identity on* $S(2)$.

(iv) 5*-commutator* s_5 *on* $S(2)$ *is primitive.*

(v) $s_5 = s_5^{rsym.r}$ *on* $S(2)$.

(vi) $ad\,s_5(X_1, \ldots, X_5) = s_5(ad\,X_1, \ldots, ad\,X_5)$ *for any* $X_1, \ldots, X_5 \in S(2)$.

(vii) $(S(2), s_5)$ *is a* 4*-left commutative algebra.*

(viii) $(S(2), \{s_2, s_5\})$ *is an sh-Lie algebra.*

Theorem 2.3. (i) $s_6 \neq 0$ *on* $W(2)$.

(ii) s_6 *is a* 6*-commutator on* $W(2)$.

(iii) $s_7 = 0$ *is an identity on* $W(2)$.

(iv) 6*-commutator* s_6 *on* $W(2)$ *is primitive.*

(v) $s_6 = s_6^{rsym.r}$ *on* $W(2)$.

(vi) *For any* $X_1, \ldots, X_6 \in W(2)$,

$$s_6(X_1, \ldots, X_6) = \sum_{i=1}^{6} (-1)^{i+1} \mathrm{Div}\, X_i\, s_5(X_1, \ldots, \hat{X}_i, \ldots, X_6).$$

(vii) $(W(2), s_6)$ *is a* 5*-left commutative algebra.*

(viii) $(W(2), \{s_2, s_6\})$ *is an sh-Lie algebra.*

A natural question arises: Is it possible to construct nontrivial N- commutators on $W(n)$ for $n > 2$? If $N = n^2 + 2n - 2$, then s_N is a well defined N-commutator on $W(n), n > 1$ (theorem 2.1, (i)).

Conjecture. $s_N(X_1, \ldots, X_N) \neq 0$ *for some* $X_1, \ldots, X_N \in W(n)$.

We have checked this conjecture by a computer for $n = 2, 3, 4$.

3. k-commutators by right-symmetric multiplications

The aim of this section is to prove that for any n there exists $N = N(n)$ such that for any $k \geq N$, $s_k = s_k^{rsym.r}$ on $W(n)$.

3.1. The Lie algebra of polynomial vector fields

By setting $\deg x_i = 1$ for all i we endow U with the standard grading $U = \oplus_{s\geq 0} U_s$ and a filtration $\mathcal{U}_s = \oplus_{s\geq k} U_k$, so

$$U = \mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots .$$

These grading and filtration induce a grading and a filtration on $L = W(n)$:

$$L = \oplus_{s\geq -1} L_s, \quad L_s = \langle x^\alpha \partial_i \mid \alpha \in \Gamma_n^+, |\alpha| = s+1, i = 1, \dots, n\rangle,$$
$$\mathcal{L} = \mathcal{L}_{-1} \supseteq \mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \cdots, \quad \mathcal{L}_k = \oplus_{s\geq k} L_s.$$

These grading and filtration are compatible with the right-symmetric multiplication:

$$L_s \circ L_k \subseteq L_{s+k}, \quad \mathcal{L}_s \circ \mathcal{L}_k \subseteq \mathcal{L}_{k+s},$$

for any $k, s \geq -1$. In particular, they induce grading and filtration on $W(n)$ as a right-symmetric algebra and as a Lie algebra.

Since $W(n)^{rsym}$ is graded,

$$L_0 \circ L_s \subseteq L_s \text{ for any } s \geq -1.$$

In particular, L_0 is a right-symmetric subalgebra of $L = W(n)^{rsym}$. Hence, it is a Lie subalgebra of $W(n)$; clearly, $L_0 \subset W(n)^{rsym}$ is associative and isomorphic to the matrix algebra Mat_n, whereas $L_0 \subset W(n)$ as a Lie algebra isomorphic to $\mathfrak{gl}_n$.

3.2. $\mathcal{D}$-invariant polynomials

Let $f = f(t_1, \dots, t_k)$ be an associative noncommutative polynomial,

$$f = \sum_{i_1,\dots,i_l} \lambda_{i_1,\dots,i_l} t_{i_1} \dots t_{i_l},$$

i.e., f is a linear combination of monomials $t_{(i)} = t_{i_1} \dots t_{i_l}$, where $i_1, \dots, i_l$ run through elements $1, \dots, k$, possibly repeated. Later on we replace t_i with elements of some 2-algebras. Since our algebras may be non-associative, we assume that every monomial $t_{(i)}$ has a *left-normed* bracketing, i.e., $t_{(i)} = (\cdots((t_{i_1} t_{i_2}) t_{i_3}) \cdots) t_{i_l}$.

Let $A = (A, \circ)$ be an algebra with multiplication $\circ$. Suppose that B is a subspace of A. Since f is a linear combination of monomials of the form $t_{(i)}$, one can substitute instead of t_i elements of B and calculate f using the multiplication of the algebra A. So, we obtain a map $f_B : B \otimes \cdots \otimes B \to A$ defined by

$$f_B(b_1, \dots, b_k) = f(b_1, \dots, b_k).$$

Sometimes we endow A by several multiplications. In such cases, we will write f_B° instead of f_B when A is considered with multiplication $\circ$. Notice that B may be not closed under multiplication $\circ$. Whenever it is clear from the context, we reduce the notation f_B to f.

We endow the space of differential operators $\mathrm{Diff}(n)$ by three multiplications: $\cdot$, $\circ$ and $[\ ,\ ]$ stand for composition, right-symmetric multiplication and commutators. We will sometimes write $f^{\cdot}_{W(n)} = f$ and $f^{\circ}_{W(n)} = f^{rsym.r}$.

Define a multiplication $\bullet : \mathrm{Diff}(n) \otimes \mathrm{Diff}(n) \to \mathrm{Diff}(n)$ by setting

$$u\partial^{\alpha} \bullet v\partial^{\beta} = \sum_{\gamma \in \mathbb{Z}_+^n, \gamma \neq 0} \binom{\beta}{\gamma} v\partial^{\beta-\gamma}(u)\partial^{\alpha+\gamma}, \text{ for any } \alpha, \beta \in \mathbb{Z}_+^n.$$

Let us extend the right-symmetric multiplication $\circ$ from the space of first order differential operators to the space of all differential operators by setting

$$u\partial^{\alpha} \circ v\partial^{\beta} = v\partial^{\beta}(u)\partial^{\alpha}.$$

Let $\mathcal{D} = L_{-1}$. A polynomial $f = f(t_1, \ldots, t_k)$ or, more precisely, $f_{\mathrm{Diff}(n)}$, $\mathcal{D}$-*invariant* if

$$[f(X_1, \ldots, X_k), \partial_i] = \sum_{s=1}^{k} f(X_1, \ldots, X_{s-1}, [X_s, \partial_i], X_{s+1}, \ldots, X_k)$$

for any $X_1, \ldots, X_k \in \mathrm{Diff}(n)$ and $i = 1, \ldots, n$. Here we do not specify what multiplication in $\mathrm{Diff}(n)$ we use in the calculation of $f_{\mathrm{Diff}(n)}$ because

$$[X, \partial_i] = X \cdot \partial_i - \partial_i \cdot X = X \circ \partial_i - \partial_i \circ X, \text{ for any } X \in \mathrm{Diff}(n).$$

By this observation, $\mathcal{D}$-invariance of $f^{\cdot}_{\mathrm{Diff}(n)}$ and $f^{\circ}_{\mathrm{Diff}(n)}$ are equivalent notions.

For $X = \sum_{\alpha \in \mathbb{Z}^n - +} u_{\alpha}\partial^{\alpha} \in \mathrm{Diff}(n)$, where $u_{\alpha} \in U$, set

$$|X| = \text{the highest degree with respect to } \partial$$

and

$$||X|| = \text{ the lowest degree with respect to } \partial.$$

Obviously,

$$|X + Y| \geq \min\{|X|, |Y|\}, \quad ||X + Y|| \leq \max\{||X||, ||Y||\} \text{ for any } X, Y \in \mathrm{Diff}(n).$$

For any $X, Y \in \mathrm{Diff}(n)$ we have

$$X \cdot Y = X \circ Y + X \bullet Y, \quad |X \bullet Y| > |X| \text{ if } X \bullet Y \neq 0. \tag{1}$$

Lemma 3.1. *Let $X_1, \ldots, X_k \in \mathrm{Diff}(n)$ be such that $|X_i| \geq s$ for any $i = 1, 2, \ldots, k$ and $||s_k(X_1, \ldots, X_k)|| \leq s$. Then*

$$s_k(X_1, \ldots, X_k) = pr_s(s_k^{rsym.r}(X_1, \ldots, X_k)),$$

where pr_s is the projection onto the space $\langle u\partial^{\alpha} \mid \alpha \in \mathbb{Z}_+^n, |\alpha| = s\rangle$.

Proof. By formula (1) we can express the composition $X_{\sigma(1)} \cdot \ldots \cdot X_{\sigma(k)}$ as a linear combination of elements of the form $X'_\sigma = (\ldots((X_{\sigma(1)} \circ X_{\sigma(2)})\ldots) \circ X_{\sigma(k)}$, without any $\bullet$, and elements of the form $X''_\sigma = (\ldots((X_{\sigma(1)} \star X_{\sigma(2)})\ldots) \star X_{\sigma(k)}$, where $\star = \circ$ or $\bullet$ and the number of the $\bullet$'s is at least one. Notice that $|X \bullet Y| > |X|$ if $|X \bullet Y| \neq 0$. Therefore,

$$|X''_\sigma| > s, \quad \text{if } X''_\sigma \neq 0.$$

Since

$$s_k(X_1, \ldots, X_k) = \sum_{\sigma \in \mathfrak{S}_k} X'_\sigma + X''_\sigma,$$

$$||s_k(X_1, \ldots, X_k)|| \le s, \quad \Big|\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}\sigma\, X'_\sigma\Big| \ge s, \quad \Big|\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sign}\sigma\, X''_\sigma\Big| > s,$$

it follows that

$$||s_k(X_1, \ldots, X_k)|| = |s_k(X_1, \ldots, X_k)| = s$$

and

$$s_k(X_1, \ldots, X_k) = pr_s(s_k^{rsym.r}(X_1, \ldots, X_k)).$$

Corollary 3.2. *Suppose that $s_k(X_1, \ldots, X_k) \in W(n)$ for any $X_1, \ldots, X_k \in W(n)$. Then*

$$s_k(X_1, \ldots, X_k) = s_k^{rsym.r}(X_1, \ldots, X_k).$$

Lemma 3.3. *Let $s_k = s_k^{rsym.r}$ on $W(n)$ and D a derivation of $(\mathrm{Diff}(n), \cdot)$ that preserves $W(n)$. Then D is a derivation of the k-algebra $(W(n), s_k^{rsym.r})$, i.e.,*

$$D(s_k^{rsym.r}(X_1, \ldots, X_k)) = \sum_{i=1}^{k} s_k^{rsym.r}(X_1, \ldots, X_{i-1}, D(X_i), X_{i+1}, \ldots, X_k)$$

for any $X_1, \ldots, X_k \in W(n)$.

Proof. We have

$$D(s_k^{rsym.r}(X_1, \ldots, X_k)) =$$

(corollary 3.2)

$$= D(s_k(X_1, \ldots, X_k)) =$$

(since $D \in Der\, \mathrm{Diff}(n)$)

$$= \sum_{i=1}^{k} s_k(X_1, \ldots, X_{i-1}, D(X_i), X_{i+1}, \ldots, X_k).$$

Since $D(W(n)) \subseteq W(n)$ by hypothesis,

$$s_k(X_1, \ldots, X_{i-1}, D(X_i), X_{i+1}, \ldots, X_k) \in W(n)$$

for any $X_1, \dots, X_k \in W(n)$. Thus by corollary 3.2,

$$\begin{aligned} &s_k(X_1, \dots, X_{i-1}, D(X_i), X_{i+1}, \dots, X_k) \\ &= s_k^{rsym.r}(X_1, \dots, X_{i-1}, D(X_i), X_{i+1}, \dots, X_k). \end{aligned}$$

Hence, for any $X_1, \dots, X_k \in W(n)$ we have

$$D(s_k^{rsym.r}(X_1, \dots, X_k)) = \sum_{i=1}^{k} s_k^{rsym.r}(X_1, \dots, X_{i-1}, D(X_i), X_{i+1}, \dots, X_k).$$

Corollary 3.4. *Let $s_k = s_k^{rsym.r}$ on $W(n)$. Then for any $X \in W(n)$ the derivation $\mathrm{ad}\, X$ generates also a derivation of $(W(n), s_k^{rsym.r})$. In particular, $\mathrm{ad}\, X$ is a derivation of the algebra $(W(n), s_{n^2+2n-2})$. Similarly, $\mathrm{ad}\, X$ is a derivation of $(S(2), s_5)$ for any $X \in S(2)$.*

4. How to calculate L_{-1}-invariants

4.1. (L, U)-modules

Let $L = W(n)$ and $U = \mathbb{C}[x_1, \dots, x_n]$ with the standard grading and M a graded L-module. The subspace of L_{-1}-invariants, $M_0 = M^{L_{-1}} = \langle m \in M \mid (m)\partial_i = 0 \rangle$, has a natural structure of an L_0-module. Make M_0 into an $\mathcal{L}_0$-module by setting $\mathcal{L}_1 M_0 = 0$. Call the $\mathcal{L}_0$-module M_0 the *base* of the L-module M.

Let M an (L, U)*-module*, if M is a right U-module such that

$$(mu)X = m[u, X] + (mX)u, \text{ for any } m \in M, u \in U, X \in L.$$

Let M be an (L, U)-module. Call M an (L, U)*-module with base* $M_0 = M^{L_{-1}}$ if M, as a U-module, is a free module with base M_0.

The main construction of (L, U)-modules ([16]) is the following. Let M_0 be an $\mathcal{L}_0$-module such that $M_0 \mathcal{L}_1 = 0$. Set

$$M = \mathrm{Hom}_{U(\mathcal{L}_0)}^{poly}(U(L), M_0) \simeq U \otimes M_0.$$

Clearly, $M = T^{poly}(M_0)$ is the space of tensor fields with polynomials coefficients with fiber M_0.

Examples. $U = T^{poly}(\mathbb{K})$, where $\mathbb{K}$ is the trivial L_0-module, is the space of functions; (L, U)-module L itself has base L_{-1}, the dual to the identity representation of L_0.

If $L = S_{n-1}$, then the adjoint module has no structure of an (L, U)-module.

We will use realization of (L, U)-modules given in [2]. Notice that our construction is general than Rudakov's construction. For example, in case of two-sided Witt algebras (L, U)-modules in our sense can not be obtained as a co-induced modules. The algorithm how to calculate L_{-1}-invariants [2] is given below.

4.2. Escorts and supports

Let $\epsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$ (all coordinates except i-th are 0). Let $\mathcal{E}$ be the root system on $W(n)$ with respect to Cartan subalgebra spanned by $\{x_i\partial_i | i = 1, \dots, n\}$ and $\alpha^1, \dots, \alpha^k \in \mathcal{E}$. Then

$$[x_i\partial_i, x^\beta\partial_j] = (\beta_i - \delta_{i,j})x^\beta\partial_j.$$

So, we can identify any root with n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} \alpha_i\epsilon_i$, where $-1 \le \alpha_i$, for all $i = 1, \dots, n$. For $\alpha \in \mathcal{E}$ denote by

$$L_\alpha = \langle x^{\alpha+\epsilon_i}\partial_i | i = 1, \dots, n\rangle$$

its root space.

Assign to any $\psi \in T^k(L, L)$ a polylinear map $\mathrm{esc}\,(\psi) \in T^k(L, L_{-1})$, called the *escort* of ψ, by the rule

$$\mathrm{esc}\,(\psi)(X_1, \dots, X_k) = \psi(X_1, \dots, X_k),$$

if $X_i \in L_{\alpha^i}$, and $\alpha^1 + \cdots + \alpha^k = -\epsilon_s$ for some $s = 1, \dots, n$. Here $\alpha^1, \dots, \alpha^k$ are some roots from $\mathcal{E}$. If $\alpha^1 + \cdots + \alpha^k \neq -\epsilon_s$ for any $s \in \{1, 2, \dots, n\}$, then set

$$\mathrm{esc}\,(\psi)(X_1, \dots, X_k) = 0.$$

So, having defined $\mathrm{esc}\,(\psi)(X_1, \dots, X_k)$ for root elements $X_1, \dots, X_k$ we extend $\mathrm{esc}\,(\psi)(X_1, \dots, X_k)$ by polylinearity to any $X_1, \dots, X_k$. We see that $\mathrm{esc}\,(\psi) \in T(L, L_{-1})$ if and only if ψ is of degree 0.

Observe that $-\epsilon_s$ can only be represented as a sum of k roots in finitely many ways. Therefore, the space

$$\mathrm{supp}_s = \oplus_{\alpha^1,\dots,\alpha^k\in\mathcal{E},\alpha^1+\dots+\alpha^k=-\epsilon_s} L_{\alpha^1} \otimes \cdots \otimes L_{\alpha^k},$$

called the *s-support of ψ* or just s-support, is finite dimensional for any $s \in \{1, \dots, n\}$. Call

$$\mathrm{supp} = \oplus_{s=1}^n \mathrm{supp}_s$$

the *support*. So, the escort of any $\mathcal{D}$-invariant $\mathcal{E}$-graded map $\psi \in T^k(L, L)$ of degree 0 is uniquely defined by the restriction to its support.

Let $V(L)$ be the monomial basis of $L = W(n)$ consisting of the vectors of the form $x^\alpha\partial_i$, where $\alpha \in \mathbb{Z}^n_+$ and $i \in \{1, \dots, n\}$. Denote by V the basis of supp obtained by tensoring the basis vectors of $V(L)$. We will write $(a_1, \dots, a_k)$ instead of $a_1 \otimes \cdots \otimes a_k$. So,

$$V = \cup_{s=1}^n V_s,$$

where

$$V_s = \big\{(a_1, \dots, a_k) \mid a_l \in V(L_{\alpha^l}),\ \alpha^l \in \mathcal{E},\ \text{where } \alpha^1 + \dots + \alpha^k = -\epsilon_s, l = 1, \dots, k\big\}.$$

As was shown in [2], any $\mathcal{E}$-graded $\mathcal{D}$-invariant map can be uniquely recovered by its escort. Namely,

$$\psi(X_1, \dots, X_k) = \sum_{(a_1,\dots,a_k)\in V} E_{a_1}(X_1)\cdots E_{a_k}(X_k)\mathrm{esc}\,(\psi)(a_1,\dots,a_k), \qquad (2)$$

where

$$E_{x^\alpha\partial_i}(v\partial_j) = \delta_{i,j}\frac{\partial^\alpha(v)}{\alpha!}.$$

4.3. Cup-products

Given an algebra A with multiplication $\star$, define the cup-product on the space $T^*(A, A)$ by setting

$$\psi \smile \phi(a_1,\dots,a_{k+l}) = \sum_{\sigma\in\mathfrak{S}_{k,l}} \operatorname{sign}\sigma\, \psi(a_{\sigma(1)},\dots,a_{\sigma(k)}) \star \phi(a_{\sigma(k+1)},\dots,a_{\sigma(k+l)})$$

for $\psi \in T^k(A, A)$ and $\phi \in T^l(A, A)$ then define $\psi \smile \phi \in T^{k+l}(A, A)$, where

$$\mathfrak{S}_{k,l} = \{\sigma \in \mathfrak{S}_{k+l} \mid \sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l)\}.$$

Suppose that A has an associative multiplication $\cdot$ and a right-symmetric multiplication $\circ$. Denote by $\overset{\cdot}{\smile}$ and $\overset{\circ}{\smile}$ cup-products induced by multiplications $\cdot$ and $\circ$ correspondingly.

If B is a subspace of A then one can consider cup-products

$$\smile : C^*(B, A) \times C^*(B, A) \to C^*(B, A).$$

We use the cup-products for $A = \mathrm{Diff}(n)$ and $B = W(n)$ or $S(n)$. Sometimes the cup-product of $\psi \in C^k(A, B)$ and $\phi \in C^l(A, B)$ lies in $C^{k+l}(B, B)$. Such fortunate situations occur in calculating of s_{k+l} for sufficiently large $k + l$.

5. Sufficient condition for a $\mathcal{D}$-invariant form with skew-symmetric arguments to be zero

Lemma 5.1. *Consider the following problem of linear programming*

$$\begin{cases} \sum\limits_{i=-1}^{m} x_i = r, \quad 0 < x_i \le l_i, \quad i = -1, 0, \dots, m-1, \\ f(x_{-1}, x_0, \dots, x_m) := \sum\limits_{i=-1}^{m} i x_i \to \min. \end{cases}$$

Then

$$\min f(x_{-1}, x_0, \dots, x_m) = mr - \sum_{i=-1}^{m-1} (m-i)x_i$$

and this value is attained for $x_{-1} = l_{-1}, x_0 = l_0, \ldots, x_{m-1} = l_{m-1}$ *and* $x_m = r - \sum_{i=-1}^{m-1} l_i$.

Proof. Since $x_m = r - \sum_{i=-1}^{m-1} x_i$, it follows that

$$f(x_{-1}, x_0, \ldots, x_m) = m\,r - \sum_{i=-1}^{m-1} (m-i)x_i.$$

Thus,

$$f(x_{-1}, x_0, \ldots, x_m) \leq m\,r - \sum_{i=-1}^{m-1} (m-i)l_i$$

and the inequality can be converted to equality if $x_i = l_i, i = -1, 0, \ldots, m-1$, and $x_m = r - \sum_{i=-1}^{m-1} l_i$.

Theorem 5.2. *Let* $A = \oplus_{i\geq -1} A_i$ *be graded algebra,* $\mathcal{D} = A_{-1}$, *and* $M = \oplus_{i\geq q} M_i$ *be* $\mathcal{D}$*-graded module,*

$$A_{-1}M_i \subseteq M_{i-1}, \quad i \geq q,$$

such that $A^{\mathcal{D}} = A_{-1}$ *and* $M^{\mathcal{D}} = M_q$. *Suppose that* $\psi \in T^k(A, M)$ *is a* 0*-graded polylinear map and skew-symmetric in* r *arguments. Let* i_0 *be number such that*

$$\sum_{-1\leq i\leq i_0} \dim\, A_i \leq r < \sum_{-1\leq i\leq i_0+1} \dim\, A_i.$$

If

$$k + q < r(i_0+2) - \sum_{-1\leq i\leq i_0} (i_0+1-i)\dim\, A_i$$

and ψ *is* $\mathcal{D}$*-invariant then* $\psi = 0$.

Proof. We prove that $\mathrm{esc}\,(\psi) = 0$. Suppose that it is not true and $\psi \neq 0$. Then there exist homogeneous $a_1, \ldots, a_k \in A$ such that $\psi(a_1, \ldots, a_k) \neq 0$ and $|a_1| + \cdots + |a_k|$ is minimal. We have

$$\psi \in T^k(A, M)^{\mathcal{D}} \Rightarrow$$

$$\partial_i \psi(a_1, \ldots, a_k) = \sum_{j=1}^{k} \psi(a_1, \ldots, a_{j-1}, \partial_i(a_j), a_{j+1}, \ldots, a_k).$$

As $|a_1| + \ldots + |a_k|$ is minimal with property $\psi(a_1, \ldots, a_k) \neq 0$ and

$$|a_1| + \cdots + |a_k| > |a_1| + \cdots + |a_{j-1}| + |\partial_i(a_j)| + |a_{j+1}| + \cdots + |a_k|,$$

we obtain that

$$0 \neq \psi(a_1, \dots, a_k) \in M^{\mathcal{D}} = M_q.$$

Since ψ is graded with degree 0, this means that we can choose homogeneous elements $a_1, \dots, a_k \in A$ such that

$$\psi(a_1, \dots, a_k) \neq 0, \quad |a_1| + \cdots + |a_k| = |\psi(a_1, \dots, a_k)| = q.$$

As ψ is skew-symmetric in r arguments, the set $\{a_1, \dots, a_k\}$ should have at least r linear independent elements. Denote them by $a_{i_1}, \dots, a_{i_r}$.

Suppose that among $a_{i_1}, \dots, a_{i_r}$ there are l_i elements of A_i. Then

$$r = \sum_{i \geq -1} l_i \tag{3}$$

and

$$l_i \leq \dim A_i. \tag{4}$$

Since $r \leq \sum_{i=-1}^{i_0+1} l_i$, from (3) it follows that

$$l_i = 0, \quad i > i_0 + 1$$

and

$$r = \sum_{i=-1}^{i_0+1} l_i. \tag{5}$$

So, among elements $a_{i_1}, \dots, a_{i_r}$ there are l_{-1} elements of degree -1, l_0 elements of degree 0, etc, l_{i_0} elements of degree i_0 and finally $r - \sum_{i=-1}^{i_0} l_i \geq r - \sum_{i=-1}^{i_0} \dim A_i$ elements of degree $i_0 + 1$. Since $|a_i| \geq -1$ for any $i \in \{1, \dots, k\}$, we obtain that

$$|\psi(a_1, \dots, a_k)| = \sum_{i=1}^{k} |a_i| \geq (-1)(k - r) + \sum_{s=-1}^{r} |a_{i_s}| \geq f(l_{-1}, l_0, \dots, l_{i_0+1}),$$

where

$$f(l_{-1}, l_0, \dots, l_{i_0+1}) = r - k + \sum_{i=-1}^{i_0+1} i\, l_i.$$

According to lemma 5.1 and our condition,

$$\min f(l_{-1}, l_0, \dots, l_{i_0+1}) = r - k + (i_0 + 1)r - \sum_{i=-1}^{i_0+1} (i_0 + 1 - i) \dim A_i > q.$$

Therefore,

$$|\psi(a_1, \dots, a_k)| > q.$$

In particular,

$$\psi(a_1, \dots, a_k) \notin M^{\mathcal{D}},$$

which is a contradiction.

Corollary 5.3. *Let $\psi \in T^9(S(2), S(2))$ be a $\mathcal{D}$-invariant 0-graded form with 8 skew-symmetric arguments. Then $\psi = 0$.*

Proof. Recall that $S(2)$ denotes the subalgebra of $W(2)$ consisting of derivations with divergence 0. Let $A = S(2)$. Then $\dim A_{-1} = 2, \dim A_0 = 3, \dim A_1 = 4$, since $A_{-1} = \langle \partial_1, \partial_2 \rangle$, $A_0 = \langle x_1\partial_2, x_1\partial_1 - x_2\partial_2, x_2\partial_1 \rangle$, $A_1 = \langle x_2^2\partial_1, x_1^2\partial_1 - 2x_1x_2\partial_2, x_2^2\partial_2 - 2x_1x_2\partial_1, x_1^2\partial_2 \rangle$. Thus,

$$\dim A_{-1} + \dim A_0 = 2 + 3 \leq r = 8 < \dim A_{-1} + \dim A_0 + \dim A_1 = 2 + 3 + 4.$$

In other words, $i_0 = 0$ for $r = 8$. Furthermore, for $k = 9, q = -1, r = 8, i_0 = 0$. We see that

$$k + q = 8 < 9 = r(i_0 + 2) - \sum_{-1 \leq i \leq i_0} (i_0 + 1 - i) \dim A_i = 8(0 + 2) - 2 \cdot 2 - 1 \cdot 3.$$

Therefore, all conditions of theorem 5.2 are fulfilled and $\psi = 0$ for $A = S(2)$.

Corollary 5.4. *Let $\psi \in T^{11}(W(2), W(2))$ be a $\mathcal{D}$-invariant form with 10 skew-symmetric arguments. Then $\psi = 0$.*

Proof. Take $A = W(2)$. Then $\dim A_{-1} = 2, \dim A_0 = 4, \dim A_1 = 6$, since $A_{-1} = \langle \partial_1, \partial_2 \rangle$, $A_0 = \langle x_i\partial_j : i, j = 1, 2 \rangle$, $A_1 = \langle x_ix_j\partial_s : i, j, s = 1, 2, i \leq j \rangle$. For $r = 10, k = 11, q = -1$ we see that

$$\dim A_{-1} + \dim A_0 = 2 + 4 \leq r = 10 < \dim A_{-1} + \dim A_0 + \dim A_1 = 2 + 4 + 6.$$

Hence $i_0 = 0$, and

$$k + q = 10 < 12 = r(i_0 + 2) - \sum_{-1 \leq i \leq i_0} (i_0 + 1 - i) \dim A_i = 10(0 + 2) - 2 \cdot 2 - 1 \cdot 4.$$

Therefore, by theorem 5.2, $\psi = 0$ on $A = W(2)$.

Corollary 5.5. *$s_k = 0$ is an identity on $W(n)$, if $k \geq n^2 + 2n$.*

Proof. Let $A = W(n)$.

We have $\dim A_{-1} = n, \dim A_0 = n^2, \dim A_1 = n^2(n + 1)/2$. We see that for $r = k \geq n^2 + 2n$,

$$\dim A_{-1} + \dim A_0 = n + n^2 \leq r.$$

Therefore $i_0 \geq 0$. Hence, if $i_0 = 0$ then

$$k + q = r - 1 < r + r - 2n - n^2 = r(0 + 2) - 2\dim A_{-1} - \dim A_0.$$

If $i_0 > 0$, $n > 1$, then $2\dim A_{-1} \leq \dim A_1$ and

$$r(i_0 + 2) - \sum_{-1 \leq i \leq i_0} (i_0 + 1 - i) \dim A_i$$

$$= r - 2\dim A_{-1} + \sum_{1\le i\le i_0} i\dim A_i + (i_0+1)(r - \sum_{-1\le i\le i_0} \dim\, A_i)$$

$$\ge r - 2\dim A_{-1} + \sum_{1\le i\le i_0} i\dim A_i > r-1.$$

Notice that s_k are graded $\mathcal{D}$-invariant of degree 0. Let ψ be a composition of s_k with the projection onto $M =< u\partial^\alpha, |\alpha| = q >$. We see that ψ and M obey the hypotheses of theorem 5.2, if $n > 1$.

If $n = 1$, it is easy to check that $s_3 = 0$, and $s_k = 0$, for any $k > 3$.

So, we have proved $\psi = 0$ for $A = W(n)$.

Corollary 5.6. *Let $\psi \in T^{2n^2+4n-5}(W(n), W(n)), n > 1$, be skew-symmetric in $r \ge (3n^2 + 6n - 5)/2$ arguments. Then $\psi = 0$. In particular, $(W(n), s_{n^2+2n-2})$ is $(n^2 + 2n - 3)$-left commutative.*

Proof. Let $A = W(n)$. For $q = -1, k = 2n^2 + 4n - 5, r \ge (3n^2 + 6n - 5)/2$, it is easy to see that $i_0 \ge 0$.

Check that the case $i_0 > 0$ is impossible. If $n = 2$ then we obtain a contradiction with the conditions

$$r \le k = 11$$

and

$$\dim A_{-1} + \dim A_0 + \dim A_1 = 12 \le r.$$

Let $n > 2$. Then we will have

$$\dim A_{-1} + \dim A_0 + \dim A_1 = n + n^2 + n^2(n+1)/2 \le r < k = 2n^2 + 4n - 5,$$

and

$$n^3 - n^2 - 6n + 5 \le 0.$$

For $n \ge 3$,

$$n^3 - n^2 - 6n + 5 \ge 2n^2 - 6n + 5 > 0,$$

and again obtain a contradiction.

So, $i_0 = 0$. Then

$$k + q = 2n^2 + 4n - 6 < 2n^2 + 4n - 5 \le 2r - 2\dim A_{-1} - \dim A_0.$$

Hence, the condition of theorem 5.2 is satisfied. Thus, $\psi = 0$ for $A = W(n)$.

Notice that $2(n^2 + 2n - 3) \ge (3n^2 + 6n - 5)/2$ if $n > 1$. Therefore, the $(n^2 + 2n - 3)$-left commutativity condition, as a condition for a $\mathcal{D}$-invariant form with $2(n^2 + 2n - 3)$ skew-symmetric arguments, is an identity on $W(n)$.

6. Invariant N-operation on vector fields

Let $\pi_1, \dots, \pi_{n-1}$ are fundamental weights of sl_n. Let $R(\gamma)$ be irreducible sl_n-module with highest weight γ.

Lemma 6.1. *The sl_n-module $\wedge^{n-2}(R(2\pi_1)\otimes R(\pi_{n-1}))$ does not contain $R(2\pi_{n-1})$ as a submodule.*

Proof. I am grateful to R.Howe for the following elegant proof of this lemma.

One can argue that the full $(n-2)$ tensor power of $R(2\pi_1)\otimes R(\pi_{n-1})$ does not contain $R(2\pi_{n-1})$. Indeed, the $(n-2)$ tensor power of this tensor product is equal to the tensor product of the $(n-2)$ tensor powers of each factor.

The representation $R(2\pi_1)$ corresponds to the diagram with one row of length two. The representation $R(\pi_{n-1})$ corresponds to the diagram with one column of length $n-1$. So, the question then becomes, does the Young diagram with $n-3$ columns of length n, and two columns of length $n-1$, appear in the indicated tensor product?

Since the diagram of $R(\pi_{n-1})$ has only one column, all of the components of its $(n-2)$ tensor power will have at most $(n-2)$ columns. Since the diagram of $R(2\pi_1)$ has only one row, all the components of its $(n-2)$ tensor power will contain at most $n-2$ rows. Now taking the tensor product of these two representations, we can say that all components of the tensor product will have diagrams which fit in an Γ-shaped region with $(n-2)$ columns and $(n-2)$ rows. But the diagram of the representation we are asking about does not fit in to this region, so it cannot be a component.

Corollary 6.2. *s_{n^2+2n-2} has no quadratic differential part on $W(n)$.*

Proof. Since, as sl_n-modules,

$$L_1 \cong R(2\pi_1+\pi_{n-1})\oplus R(\pi_1)\cong R(2\pi_1)\otimes R(\pi_{n-1}),$$

we obtain an isomorphism of sl_n-modules

$$\wedge^k L_1 \cong \wedge^k(R(2\pi_1)\otimes R(\pi_{n-1})).$$

Consider the homomorphism of sl_n-modules

$$\rho_{k,s}:\wedge^{k-n^2-n}L_1\to R(s\pi_{n-1}),$$

induced by

$$\rho(X_1\wedge\ldots\wedge X_{k-n^2-n})$$

$$= pr_s(s_{k,\mathrm{Diff}(n)}(\partial_1,\ldots,\partial_n,x_1\partial_1,\ldots,x_n\partial_1,\ldots,x_1\partial_n,\ldots,x_n\partial_n,X_1,\ldots,X_{k-n^2-n})),$$

where $pr_s:\mathrm{Diff}(n)\to\langle\partial^\alpha:|\alpha|=s\rangle\cong R(s\pi_{n-1})$ is the projection map.

Since $\wedge^n L_{-1}\otimes\wedge^{n^2}L_0\cong\mathbb{C}$, it is clear that $\rho_{n^2+2n-2,2}$ should give a homomorphism of $\wedge^{n-2}(R(\pi_1)\otimes R(\pi_1+\pi_{n-1}))$ to $R(2\pi_{n-1})$. By lemma 6.1 this homomorphism is trivial. Thus,

$$s_{n^2+2n-2}(X_1,\ldots,X_{n^2+2n-2})\in W(n) \text{ for any } X_1,\ldots,X_{n^2+2n-2}\in W(n).$$

Lemma 6.3. *If $k = n^2+2n-2$ and $k = n^2+2n-1$, then $s_k(X_1,\dots,X_k) \in W(n)$ for any $X_1,\dots,X_k \in W(n)$.*

Proof. For $k = n^2+2n-2$ this follows from corollary 6.2. For $k = n^2+2n-1$ we see that $\mathrm{esc}\,(s_k)$ has support $\wedge^n L_{-1} \otimes \wedge^{n^2-1} L_0 \otimes \wedge^{n-1} L_1$ and $s_k(\partial_1, \dots, \partial_n, a_1, \dots, a_{n^2-1}, X_1, \dots, X_{n-1})$, where $a_1,\dots,a_{n^2-1} \in L_0$, $X_1,\dots,X_{n-1} \in L_1$, never gives quadratic terms, as

$$|s_k(\partial_1,\dots,\partial_n,a_1,\dots,a_{n-1},X_1,\dots,X_{n-1})| = -1.$$

7. The quadratic differential parts for k-commutators in two variables

Let $\mathrm{Diff}_s(n)$ be the subspace of differential operators of order s, and $pr_s : \mathrm{Diff}(n) \to \mathrm{Diff}_s(n)$ the projection.

Lemma 7.1. *For any $X_1,\dots,X_k \in W(2)$,*

$$pr_l(s_k(X_1,\dots,X_k)) = 0,$$

if $l > 2$.

Proof. If $k > 6$ then by corollary 5.5 and lemma 12.1, $s_k = 0$ is an identity. If $k = 6$, then $s_{k,W(2)}$ has only a linear part. If $k \le 5$ then s_k can be decomposed into a cup-product of s_2 and s_{k-2}. We know that s_2 can only give differential operators of first order. So, $s_3 = s_2 \smile s_1$ and $s_4 = s_2 \smile s_2$ can give differential operators at most second order. As far as $s_5 = s_3 \smile s_2$, the following reasoning shows that the differential operators of third order can not be represented as $s_5(X_1,\dots,X_5)$ for any $X_1,\dots,X_5 \in W(2)$. For $L = W(2)$, support for an escort map of $pr_l s_5$ with a maximal l should contain $\{\partial_1, \partial_2, x_2\partial_1, a, b, c\}$, where $a,b,c \in L_0$. Easy calculations then show that $l \le 2$ if $k = 5$.

Remark. One can prove that if $l > n$ then $pr_l(s_k(X_1,\dots,X_k)) = 0$ for any k and $X_1,\dots,X_k \in W(n)$.

Lemma 7.2.

$$\begin{aligned} &pr_2(s_3(X_1,X_2,X_3)) \\ &= -\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 \\ \partial_2((x_1)X_1) & \partial_2((x_1)X_2) & \partial_2((x_1)X_3) \end{vmatrix} \partial_1^2 \\ &+ \begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 \\ \partial_1((x_1)X_1) & \partial_1((x_1)X_2) & \partial_1((x_1)X_3) \end{vmatrix} \partial_1\partial_2 \end{aligned}$$

$$
-\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 \\ \partial_2((x_2)X_1) & \partial_2((x_2)X_2) & \partial_2((x_2)X_3) \end{vmatrix} \partial_1\partial_2
$$

$$
+\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 \\ \partial_1((x_2)X_1) & \partial_1((x_2)X_2) & \partial_1((x_2)X_3) \end{vmatrix} \partial_2^2,
$$

for any $X_1, X_2, X_3 \in W(2)$.

Lemma 7.3.

$$
pr_2(s_4(X_1,\dots,X_4))
$$

$$
= -2\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 \\ \partial_1((x_1)X_1) & \partial_1((x_1)X_2) & \partial_1((x_1)X_3) & \partial_1((x_1)X_4) \\ \partial_2((x_1)X_1) & \partial_2((x_1)X_2) & \partial_2((x_1)X_3) & \partial_2((x_1)X_4) \end{vmatrix} \partial_1^2
$$

$$
-2\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 \\ \partial_1((x_1)X_1) & \partial_1((x_1)X_2) & \partial_1((x_1)X_3) & \partial_1((x_1)X_4) \\ \partial_2((x_2)X_1) & \partial_2((x_2)X_2) & \partial_2((x_2)X_3) & \partial_2((x_2)X_4) \end{vmatrix} \partial_1\partial_2
$$

$$
-2\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 \\ \partial_1((x_2)X_1) & \partial_1((x_2)X_2) & \partial_1((x_2)X_3) & \partial_1((x_2)X_4) \\ \partial_2((x_1)X_1) & \partial_2((x_1)X_2) & \partial_2((x_1)X_3) & \partial_2((x_1)X_4) \end{vmatrix} \partial_1\partial_2
$$

$$
-2\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 \\ \partial_1((x_2)X_1) & \partial_1((x_2)X_2) & \partial_1((x_2)X_3) & \partial_1((x_2)X_4) \\ \partial_2((x_2)X_1) & \partial_2((x_2)X_2) & \partial_2((x_2)X_3) & \partial_2((x_2)X_4) \end{vmatrix} \partial_2^2,
$$

for any $X_1, \dots, X_4 \in W(2)$.

Lemma 7.4.

$$
pr_2(s_5(X_1,\dots,X_5))
$$

$$
= -\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 & (x_1)X_5 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 & (x_2)X_5 \\ \partial_1((x_1)X_1) & \partial_1((x_1)X_2) & \partial_1((x_1)X_3) & \partial_1((x_1)X_4) & \partial_1((x_1)X_5) \\ \partial_2((x_2)X_1) & \partial_2((x_2)X_2) & \partial_2((x_2)X_3) & \partial_2((x_2)X_4) & \partial_2((x_2)X_5) \\ \partial_2((x_1)X_1) & \partial_2((x_1)X_2) & \partial_2((x_1)X_3) & \partial_2((x_1)X_4) & \partial_2((x_1)X_5) \end{vmatrix} \partial_1^2
$$

$$
-\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 & (x_1)X_5 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 & (x_2)X_5 \\ \operatorname{Div} X_1 & \operatorname{Div} X_2 & \operatorname{Div} X_3 & \operatorname{Div} X_4 & \operatorname{Div} X_5 \\ \partial_2((x_1)X_1) & \partial_2((x_1)X_2) & \partial_2((x_1)X_3) & \partial_2((x_1)X_4) & \partial_2((x_1)X_5) \\ \partial_1((x_2)X_1) & \partial_1((x_2)X_2) & \partial_1((x_2)X_3) & \partial_1((x_2)X_4) & \partial_1((x_2)X_5) \end{vmatrix} \partial_1\partial_2
$$

$$-\begin{vmatrix} (x_1)X_1 & (x_1)X_2 & (x_1)X_3 & (x_1)X_4 & (x_1)X_5 \\ (x_2)X_1 & (x_2)X_2 & (x_2)X_3 & (x_2)X_4 & (x_2)X_5 \\ \partial_1((x_1)X_1) & \partial_1((x_1)X_2) & \partial_1((x_1)X_3) & \partial_1((x_1)X_4) & \partial_1((x_1)X_5) \\ \partial_2((x_2)X_1) & \partial_2((x_2)X_2) & \partial_2((x_2)X_3) & \partial_2((x_2)X_4) & \partial_2((x_2)X_5) \\ \partial_1((x_2)X_1) & \partial_1((x_2)X_2) & \partial_1((x_2)X_3) & \partial_1((x_2)X_4) & \partial_1((x_2)X_5) \end{vmatrix} \partial_2^2,$$

for any $X_1, \ldots, X_5 \in W(2)$.

To prove these statements one needs to calculate their escorts. A sufficient number of examples of similar calculations will be given below.

8. Exact formula for 5-commutator

Theorem 8.1. *Let* U *be an associative commutative algebra with two commuting derivations* ∂_1 *and* ∂_2. *Then*

$$s_5(D_{12}(u_1), D_{12}(u_2), D_{12}(u_3), D_{12}(u_4), D_{12}(u_5)) = -3D_{12}([u_1, u_2, u_3, u_4, u_5]),$$

for any $u_1, \ldots, u_5 \in U$, *where*

$$[u_1, u_2, u_3, u_4, u_5] = \begin{vmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 & \partial_1 u_4 & \partial_1 u_5 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 & \partial_2 u_4 & \partial_2 u_5 \\ \partial_1^2 u_1 & \partial_1^2 u_2 & \partial_1^2 u_3 & \partial_1^2 u_4 & \partial_1^2 u_5 \\ \partial_1\partial_2 u_1 & \partial_1\partial_2 u_2 & \partial_1\partial_2 u_3 & \partial_1\partial_2 u_4 & \partial_1\partial_2 u_5 \\ \partial_2^2 u_1 & \partial_2^2 u_2 & \partial_2^2 u_3 & \partial_2^2 u_4 & \partial_2^2 u_5 \end{vmatrix}$$

and $D_{12}(u) = \partial_1(u)\partial_2 - \partial_2(u)\partial_1$.

Proof. By polynomial principle [6] we can assume that $U = \mathbb{Z}[x_1, x_2]$ with $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial}{\partial x_2}$.

Let L_i be graded components for $S(2) = \langle X \in W_2 : \operatorname{Div} X = 0\rangle$ and $a, b, c \in L_0, X \in L_1$. Notice that

$$s_3^{rsym.r}(\partial_i, a, X) = [\partial_i, a] \circ X + [a, X] \circ \partial_i + [X, \partial_i] \circ a$$
$$= -a \circ \partial_i X + X \circ \partial_i a \in L_0.$$

Therefore,

$$s_5(\partial_1, \partial_2, a, b, X)$$
$$= -s_3^{rsym.r}(\partial_1, b, X) \circ \partial_2(a) + s_3^{rsym.r}(\partial_2, b, X) \circ \partial_1(a)$$
$$+s_3^{rsym.r}(\partial_1, a, X) \circ \partial_2(b) - s_3^{rsym.r}(\partial_2, a, X) \circ \partial_1(b)$$
$$= +(b \circ \partial_1 X - X \circ \partial_1 b) \circ \partial_2 a - (b \circ \partial_2 X - X \circ \partial_2 b) \circ \partial_1 a$$
$$+(X \circ \partial_1 a - a \circ \partial_1 X) \circ \partial_2 b - (X \circ \partial_2 a - a \circ \partial_2 X) \circ \partial_1 b$$
$$= -(a \circ \partial_1 X - X \circ \partial_1 a) \circ \partial_2 b + (a \circ \partial_2 X - X \circ \partial_2 a) \circ \partial_1 b$$

$$+(b\circ\partial_1 X - X\circ\partial_1 b)\circ\partial_2 a - (b\circ\partial_2 X - X\circ\partial_2 b)\circ\partial_1 a$$
$$= -a\circ[\partial_1 X,\partial_2 b] + [X,\partial_2 b]\circ\partial_1 a + a\circ[\partial_2 X,\partial_1 b] - [X,\partial_1 b]\circ\partial_2 a$$
$$+b\circ[\partial_1 X,\partial_2 a] - [X,\partial_2 a]\circ\partial_1 b - b\circ[\partial_2 X,\partial_1 a] + [X,\partial_1 a]\circ\partial_2 b.$$

We see that non-zero components for $\mathrm{esc}\,(s_5)$ are

$$s_5(\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,x_1^2\partial_2)=6\partial_2,$$
$$s_5(\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,x_1^2\partial_1-2x_1x_2\partial_2)=6\partial_1,$$
$$s_5(\partial_1,\partial_2,x_2\partial_1,x_1\partial_2,x_1^2\partial_1-2x_1x_2\partial_2)=-6\partial_2,$$
$$s_5(\partial_1,\partial_2,x_2\partial_1,x_1\partial_2,x_2^2\partial_2-2x_1x_2\partial_1)=-6\partial_1,$$
$$s_5(\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,x_2^2\partial_2-2x_1x_2\partial_1)=-6\partial_2,$$
$$s_5(\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,x_2^2\partial_1)=-6\partial_1.$$

It is easy to check, that

$$\mathrm{esc}\,(s_5^{rsym.r})(D_{12}(u_1),\dots,D_{12}(u_5)) = -3pr_{-1}\ D_{12}([u_1,\dots,u_5]),$$

for any $u_1,\dots,u_5\in\mathbb{C}[x_1,x_2]$, such that $|u_1|+\cdots+|u_5|=11$.

It remains to use (2) for $\mathcal{D}$-invariant form $s_5^{rsym.r}$ and use lemma 7.4.

9. 5-commutator of adjoint derivations

Lemma 9.1. *Let U be $\{\partial_1,\partial_2\}$-differential algebra, i.e., an associative commutative algebra with two commuting derivations ∂_1,∂_2, and $S(2)$ be the subspace of vector fields without divergence of $W(2)$. Then*

$$ad\,s_5(X_1,\dots,X_5) = s_5(ad\,X_1,\dots,ad\,X_5),$$

for any $X_1,\dots,X_5\in S(2)$.

Proof. Consider a multilinear polynomial f with 6 variables defined by

$$f(t_0,t_1,\dots,t_5) = (t_0)s_5^{ad}(t_1,\dots,t_5) = \sum_{\sigma\in\mathfrak{S}_5}\mathrm{sign}\,\sigma\,[\cdots[[t_0,t_{\sigma(1)}],t_{\sigma(2)}],\cdots].$$

We see that f is polylinear and skew-symmetric in all variables except the first one. Important properties for us are: $f_{S(2)}$ is $\mathcal{D}$-invariant and $\mathcal{E}$-graded. Therefore, f can be uniquely restored from its escort. We see that

$$\begin{aligned}\mathrm{supp}=\mathrm{supp}(f) &= L_{-1}\otimes\wedge^2L_{-1}\otimes\wedge^2L_0\otimes L_2\\ &\oplus\quad L_{-1}\otimes\wedge^2L_{-1}\otimes L_0\otimes\wedge^2L_1\\ &\oplus\quad L_{-1}\otimes L_{-1}\otimes\wedge^3L_0\otimes L_1\\ &\oplus\quad L_0\otimes\wedge^2L_{-1}\otimes\wedge^2L_0\otimes L_1\end{aligned}$$

$$\oplus \quad L_1 \otimes \wedge^2 L_{-1} \otimes \wedge^3 L_0.$$

Here by L_i we denote the graded components for $S(2) = \langle X \in W(2) : \operatorname{Div} X = 0\rangle$.

Let $(a, b, c) = (x_2\partial_1, x_1\partial_1 - x_2\partial_2, x_1\partial_2)$. Then

$$F := s_5^{ad}(\partial_1, \partial_2, a, b, c) \in \operatorname{End} W(n),$$

is defined by

$$(u_1\partial_1 + u_2\partial_2)F = 6(\partial_1\partial_2(u_1) + 6\partial_2^2(u_2))\partial_1 - 6(\partial_1^2 u_1 + \partial_1\partial_2 u_2)\partial_2. \tag{6}$$

In other words,

$$(u_1\partial_1 + u_2\partial_2)F = -6D_{12}(\partial_1(u_1) + \partial_2(u_2)).$$

Set $s_k^{ad}(X_1, \ldots, X_k) = s_k(ad\, X_1, \ldots, ad\, X_k))$, where $ad : L \to \operatorname{End} L$.

Let us prove (6). We have

$$F = F_1 + F_2 + F_3,$$

where

$$F_1 = s_3^{ad}(\partial_1, a, b)\cdot ad[\partial_2, c] + s_3^{ad}(\partial_1, b, c)\cdot ad[\partial_2, a] + s_3^{ad}(\partial_1, c, a)\cdot ad[\partial_2, b],$$

$$F_2 = -s_3^{ad}(\partial_2, a, b) \cdot ad[\partial_1, c] - s_3^{ad}(\partial_2, b, c) \cdot ad[\partial_1, a] - s_3^{ad}(\partial_2, c, a) \cdot ad[\partial_1, b],$$

$$F_3 = s_3^{ad}(\partial_1, \partial_2, a) \cdot ad\,[b, c] + s_3^{ad}(\partial_1, \partial_2, b) \cdot ad[c, a] + s_3^{ad}(\partial_1, \partial_2, c) \cdot ad[a, b].$$

Further,

$$F_1 = -s_3^{ad}(\partial_1, b, c) \cdot \partial_1 + s_3^{ad}(\partial_1, c, a) \cdot \partial_2,$$

$$F_2 = s_3^{ad}(\partial_2, a, b) \cdot \partial_2 + s_3^{ad}(\partial_2, c, a) \cdot \partial_1,$$

$$F_3 = -2\, s_3^{ad}(\partial_1, \partial_2, a) \cdot ad\, c - s_3^{ad}(\partial_1, \partial_2, b) \cdot ad\, b - 2\, s_3^{ad}(\partial_1, \partial_2, c) \cdot ad\, a.$$

It is easy to see that

$$\begin{gathered} F_3 \\ = 2(\partial_1 \cdot \partial_2 a) \cdot ad\, c - 2(\partial_2 \cdot \partial_1 a) \cdot ad\, c \\ +(\partial_1 \cdot \partial_2 b) \cdot ad\, b - (\partial_2 \cdot \partial_1 b) \cdot ad\, b \\ +2(\partial_1 \cdot \partial_2 c) \cdot ad\, a - 2(\partial_2 \cdot \partial_1 c) \cdot ad\, a \\ = 2(\partial_1 \cdot \partial_1) \cdot ad\, c - (\partial_1 \cdot \partial_2) \cdot ad\, b - (\partial_2 \cdot \partial_1) \cdot ad\, b - 2(\partial_2 \cdot \partial_2) \cdot ad\, a \\ = 2\partial_1^2 \cdot ad\, c - 2(\partial_1\partial_2) \cdot ad\, b - 2\partial_2^2 \cdot ad\, a. \end{gathered}$$

Note that

$$\begin{gathered} ((u_1\partial_1 + u_2\partial_2)\partial_1^2) ad\, c \\ = [\partial_1^2(u_1)\partial_1 + \partial_1^2(u_2)\partial_2, x_1\partial_2] = -\partial_1^2(u_1)\partial_2, \\ (u_1\partial_1 + u_2\partial_2)(\partial_1\partial_2 \cdot ad\, b) \\ = [\partial_1\partial_2(u_1)\partial_1 + \partial_1\partial_2(u_2)\partial_2, x_1\partial_1 - x_2\partial_2] = 0, \end{gathered}$$

$$((u_1\partial_1+u_2\partial_2)\partial_2^2)ad\,a$$
$$=[\partial_2^2(u_1)\partial_1+\partial_2^2(u_2)\partial_2,x_2\partial_1]=-\partial_2^2(u_2)\partial_1.$$

Therefore,

$$(u_1\partial_1+u_2\partial_2)F_3$$
$$=-2\partial_1^2(u_1)\partial_2+2\partial_2^2(u_2)\partial_1.$$

Similarly,

$$(u_1\partial_1+u_2\partial_2)(F_1+F_2)$$
$$=4\partial_1\partial_2(u_1)\partial_1-4\partial_1^2(u_1)\partial_2+4\partial_2^2(u_2)\partial_1-4\partial_1\partial_2(u_2)\partial_2.$$

From these expressions of F_1, F_2 and F_3 we obtain (6).

So, by (6), $F=0$ on $S(2)$. In particular

$$\operatorname{esc}(f)(X,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,x_1\partial_2)=0,$$

for any $X\in L_1$.

Similar calculations show that non-zero components for $\operatorname{esc}(f)$ are

$$f(x_2\partial_1,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,D_{12}(x_1^3))=18\partial_1,$$
$$f(x_2\partial_1,\partial_1,\partial_2,x_2\partial_1,x_1\partial_2,D_{12}(x_1^2x_2))=6\partial_1,$$
$$f(x_2\partial_1,\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_1x_2^2))=-6\partial_1,$$
$$f(x_1\partial_1-x_2\partial_2,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,D_{12}(x_1^3))=-18\partial_2,$$
$$f(x_1\partial_1-x_2\partial_2,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,D_{12}(x_1^2x_2))=-6\partial_1,$$
$$f(x_1\partial_1-x_2\partial_2,\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_1x_2^2))=6\partial_2,$$
$$f(x_1\partial_1-x_2\partial_2,\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_2^3))=18\partial_1,$$
$$f(x_1\partial_2,\partial_1,\partial_2,x_2\partial_1,x_1\partial_2,D_{12}(x_1x_2^2))=-6\partial_2,$$
$$f(x_1\partial_2,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,D_{12}(x_1^2x_2))=-6\partial_2,$$
$$f(x_1\partial_2,\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_2^3))=18\partial_2,$$

and

$$f(\partial_1,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,D_{12}(x_1^4))=-72\partial_2,$$
$$f(\partial_1,\partial_1,\partial_2,x_2\partial_1,x_1\partial_1-x_2\partial_2,D_{12}(x_1^3x_2))=18\partial_1,$$
$$f(\partial_1,\partial_1,\partial_2,x_2\partial_1,x_1\partial_2,D_{12}(x_1^3x_2)=-18\partial_2,$$
$$f(\partial_1,\partial_1,\partial_2,x_2\partial_1,x_1\partial_2,D_{12}(x_1^2x_2^2)=12\partial_1,$$
$$f(\partial_1,\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_1^2x_2^2))=12\partial_2,$$
$$f(\partial_1,\partial_1,\partial_2,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_1x_2^3))=-18\partial_1,$$
$$f(\partial_1,\partial_1,x_2\partial_1,x_1\partial_1-x_2\partial_2,x_1\partial_2,D_{12}(x_1^3))=-18\partial_2,$$

$$f(\partial_1, \partial_1, x_2\partial_1, x_1\partial_1 - x_2\partial_2, x_1\partial_2, D_{12}(x_1^2 x_2)) = 6\partial_1,$$

$$f(\partial_1, \partial_2, x_2\partial_1, x_1\partial_1 - x_2\partial_2, x_1\partial_2, D_{12}(x_1^2 x_2)) = -6\partial_2,$$

$$f(\partial_1, \partial_2, x_2\partial_1, x_1\partial_1 - x_2\partial_2, x_1\partial_2, D_{12}(x_1 x_2^2)) = 6\partial_1,$$

$$f(\partial_1, \partial_1, \partial_2, x_2\partial_1, D_{12}(x_1^3), D_{12}(x_1^2 x_2)) = -72\partial_2,$$

$$f(\partial_1, \partial_1, \partial_2, x_2\partial_1, D_{12}(x_1^3), D_{12}(x_1 x_2^2)) = 36\partial_1,$$

$$f(\partial_1, \partial_1, \partial_2, x_1\partial_1 - x_2\partial_2, D_{12}(x_1^3), D_{12}(x_1 x_2^2)) = 72\partial_2,$$

$$f(\partial_1, \partial_1, \partial_2, x_1\partial_1 - x_2\partial_2, D_{12}(x_1^3), D_{12}(x_2^3)) = -108\partial_1,$$

$$f(\partial_1, \partial_1, \partial_2, x_1\partial_2, D_{12}(x_1^2 x_2), D_{12}(x_2^3)) = -36\partial_1,$$

$$f(\partial_1, \partial_1, \partial_2, x_1\partial_2, D_{12}(x_1^2 x_2), D_{12}(x_1 x_2^2)) = 24\partial_2.$$

Other components of the form $f(\partial_1, \partial_1, \partial_2, a, b, X)$, $f(\partial_1, \partial_i, a, b, c, Y)$, and $f(\partial_1, \partial_1, \partial_2, a, Y, Z)$ are equal to 0, where $a, b, c \in L_0$, $X \in L_2, Y, Z \in L_1$ and $i = 1, 2$. To find $f(\partial_2, \partial_1, \partial_2, a, b, X)$, $f(\partial_1, \partial_i, a, b, c, Y)$ and $f(\partial_2, \partial_1, \partial_2, a, Y, Z)$ one should use the involutive automorphism of $W(n)$ induced by changing of variables $(x_1, x_2) \mapsto (x_2, x_1)$.

Using lemma 9.1 we see that

$$\operatorname{esc}(f) = \operatorname{esc}(g),$$

where $\mathcal{D}$-invariant map $g : S(2) \otimes \wedge^5 S(2) \to S(2)$ is given by

$$g(X_1, X_1, \ldots, X_5) = [X_0, s_5^{rsym.r}(X_1, \ldots, X_5)].$$

It remains to use (2), theorem 8.1 for $\mathcal{D}$-invariant forms f and g to obtain that $s_5^{ad} = ad\, s_5$ for $S(2)$.

10. Exact formulas for the 6-commutator

In this section we prove that 6-commutator s_6 on $W(2)$ can be given as a sum of fourteen 6×6 determinants.

Theorem 10.1. *Let U be associative commutative algebra with two commuting derivations $\{\partial_1, \partial_2\}$, $X_i = u_{i,1}\partial_1 + u_{i,2}\partial_2 \in W(2)$, $u_{i,1}, u_{i,2} \in U$, for $i = 1, \ldots, 6$ and*

$$s_6(X_1, \ldots, X_6) = F_1(X_1, \ldots, X_6)\partial_1 + F_2(X_1, \ldots, X_6)\partial_2,$$

where $F_s(X_1, \ldots, X_6) \in U, s = 1, 2$. Then the polynomial F_1 is a sum of seven 6×6 determinants:

$$F_1(X_1, \ldots, X_6)$$

$$= -\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_2^2 u_{1,2} & \partial_2^2 u_{2,2} & \partial_2^2 u_{3,2} & \partial_2^2 u_{4,2} & \partial_2^2 u_{5,2} & \partial_2^2 u_{6,2} \end{vmatrix}$$

$$-\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_1^2 u_{1,1} & \partial_1^2 u_{2,1} & \partial_1^2 u_{3,1} & \partial_1^2 u_{4,1} & \partial_1^2 u_{5,1} & \partial_1^2 u_{6,1} \end{vmatrix}$$

$$-\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2^2 u_{1,2} & \partial_2^2 u_{2,2} & \partial_2^2 u_{3,2} & \partial_2^2 u_{4,2} & \partial_2^2 u_{5,2} & \partial_2^2 u_{6,2} \end{vmatrix}$$

$$2\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_{12} u_{1,1} & \partial_{12} u_{2,1} & \partial_{12} u_{3,1} & \partial_{12} u_{4,1} & \partial_{12} u_{5,1} & \partial_{12} u_{6,1} \end{vmatrix}$$

$$2\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_{12} u_{1,1} & \partial_{12} u_{2,1} & \partial_{12} u_{3,1} & \partial_{12} u_{4,1} & \partial_{12} u_{5,1} & \partial_{12} u_{6,1} \end{vmatrix}$$

$$2\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_2 u_{1,1} & \partial_2 u_{2,1} & \partial_2 u_{3,1} & \partial_2 u_{4,1} & \partial_2 u_{5,1} & \partial_2 u_{6,1} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_{12} u_{1,2} & \partial_{12} u_{2,2} & \partial_{12} u_{3,2} & \partial_{12} u_{4,2} & \partial_{12} u_{5,2} & \partial_{12} u_{6,2} \end{vmatrix}$$

$$-3\begin{vmatrix} u_{1,1} & u_{2,1} & u_{3,1} & u_{4,1} & u_{5,1} & u_{6,1} \\ u_{1,2} & u_{2,2} & u_{3,2} & u_{4,2} & u_{5,2} & u_{6,2} \\ \partial_1 u_{1,1} & \partial_1 u_{2,1} & \partial_1 u_{3,1} & \partial_1 u_{4,1} & \partial_1 u_{5,1} & \partial_1 u_{6,1} \\ \partial_1 u_{1,2} & \partial_1 u_{2,2} & \partial_1 u_{3,2} & \partial_1 u_{4,2} & \partial_1 u_{5,2} & \partial_1 u_{6,2} \\ \partial_2 u_{1,2} & \partial_2 u_{2,2} & \partial_2 u_{3,2} & \partial_2 u_{4,2} & \partial_2 u_{5,2} & \partial_2 u_{6,2} \\ \partial_2^2 u_{1,1} & \partial_2^2 u_{2,1} & \partial_2^2 u_{3,1} & \partial_2^2 u_{4,1} & \partial_2^2 u_{5,1} & \partial_2^2 u_{6,1} \end{vmatrix}$$

Here $\partial_{12} = \partial_1\partial_2$. Other seven matrices for the ∂_2-part $F_2(X_1, \ldots, X_6)$ are obtained from these matrices by interchanging all the indices 1 with 2.

Proof. By polynomial principle [6] we can assume that $U = \mathbb{Z}[x_1, x_2]$ with $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial}{\partial x_2}$.

Let $X_1 = \partial_1, X_2 = \partial_2$, $a_1 = x_1\partial_1, a_2 = x_2\partial_1, a_3 = x_1\partial_2, a_4 = x_2\partial_2$. Let V be the set of 6-tuples of the form $(X_1, X_2, a_1, \ldots \hat{a_i}, \ldots, a_4, X_6)$, where $i = 1, 2, 3, 4$ and X_6 runs over the basic elements of W_2 with order $|X_6| = 1$.

We see that supp is generated by elements $X_1 \otimes X_2 \otimes a_1 \otimes \cdots \hat{a_i} \otimes \cdots \otimes a_4 \otimes X_6$, where $(X_1, X_2, a_1, \ldots, \hat{a_i}, \ldots, a_4, X_6) \in V$.

One calculates that

$$s_6(\partial_1, \partial_2, x_i\partial_i, x_2\partial_1, x_1\partial_2, x_1^2\partial_1) = -2\partial_2,$$
$$s_6(\partial_1, \partial_2, x_i\partial_i, x_2\partial_1, x_1\partial_2, x_1x_2\partial_1) = 2\partial_1,$$
$$s_6(\partial_1, \partial_2, x_i\partial_i, x_2\partial_1, x_1\partial_2, x_1x_2\partial_2) = 2\partial_2,$$
$$s_6(\partial_1, \partial_2, x_i\partial_i, x_2\partial_1, x_1\partial_2, x_2^2\partial_2) = -2\partial_1,$$

for $i = 1, 2$ and

$$s_6(\partial_1, \partial_2, x_1\partial_1, x_2\partial_1, x_2\partial_2, x_1^2\partial_1) = -2\partial_1,$$
$$s_6(\partial_1, \partial_2, x_1\partial_1, x_2\partial_1, x_2\partial_2, x_1x_2\partial_2) = 2\partial_1,$$
$$s_6(\partial_1, \partial_2, x_1\partial_1, x_2\partial_1, x_2\partial_2, x_1^2\partial_2) = -6\partial_2,$$
$$s_6(\partial_1, \partial_2, x_1\partial_1, x_1\partial_2, x_2\partial_2, x_1x_2\partial_1) = 2\partial_2,$$
$$s_6(\partial_1, \partial_2, x_1\partial_1, x_1\partial_2, x_2\partial_2, x_2^2\partial_1) = -6\partial_1,$$
$$s_6(\partial_1, \partial_2, x_1\partial_1, x_1\partial_2, x_2\partial_2, x_2^2\partial_2) = -2\partial_2,$$

For other $(X_1, \ldots, X_6) \in V$,

$$s_6(X_1, \ldots, X_6) = 0.$$

Calculations here are not difficult, but tedious. We perform them in one example. Let us calculate $s_6(X_1, \ldots, X_6)$ for

$$X_1 = \partial_1, X_2 = \partial_2, X_3 = x_2\partial_1, X_4 = x_1\partial_1 - x_2\partial_2, X_5 = x_1\partial_2, X_6 = x_1^2\partial_1.$$

Since $|s_6(X_1, \ldots, X_6)| = -1$,

$$s_6(X_1, \ldots, X_6) = \sum_{\sigma \in \mathfrak{S}_{3,3}} \operatorname{sign}\sigma\, s_3^{rsym}(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) \circ s_3(X_{\sigma(4)}, X_{\sigma(5)}, X_{\sigma(6)}).$$

Recall that $\mathfrak{S}_{3,3}$ is the set of shuffle permutations, $\sigma(1) < \sigma(2) < \sigma(3)$, $\sigma(4) < \sigma(5) < \sigma(6)$. So,

$$s_6(X_1,\dots,X_6) = s_6'(X_1,\dots,X_6) + s_6''(X_1,\dots,X_6),$$

where

$$s_6' = s_3^{rsym} \smile^{\circ} s_3^{rsym},$$

$$s_6'' = s_3^{rsym} \smile^{\circ} (\text{quadratic differential part of } s_3).$$

Then

$$s_6''(X_1,\dots,X_5,x_1^2\partial_1)$$
$$= 2(3x_1^2\partial_1 + 2x_1x_2\partial_2)\circ\partial_1\partial_2 + 2x_1x_2\partial_1\circ\partial_2^2 = 4\partial_2.$$

We see that

$$s_3^{rsym}(X_1,X_3,X_6) = s_3(\partial_1, x_2\partial_1, x_1^2\partial_1) = 2x_2\partial_1\circ x_1\partial_1 = 0,$$

and

$$s_3^{rsym}(X_1,X_3,X_6)\circ s_3^{rsym}(X_2,X_4,X_5) = 0.$$

Furthermore,

$$s_3^{rsym}(X_1,X_4,X_6) = s_3^{rsym}(\partial_1, x_1\partial_1 - x_2\partial_2, x_1^2\partial_1)$$
$$= 2(x_1\partial_1 - x_2\partial_2)\circ x_1\partial_1 - x_1^2\partial_1\circ\partial_1 = 2x_1\partial_1 - 2x_1\partial_1 = 0,$$

and

$$s_3^{rsym}(X_1,X_4,X_6)\circ s_3^{rsym}(X_2,X_3,X_5) = 0.$$

At last,

$$s_3^{rsym}(X_1,X_5,X_6) = s_3^{rsym}(\partial_1, x_1\partial_2, x_1^2\partial_1)$$
$$= 2x_1\partial_2\circ x_1\partial_1 - x_1^2\partial_1\circ\partial_2 = 2x_1\partial_2,$$
$$s_3^{rsym}(X_2,X_3,X_4) = s_3^{rsym}(\partial_2, x_2\partial_1, x_1\partial_1 - x_2\partial_2)$$
$$= -x_2\partial_1\circ\partial_2 - (x_1\partial_1 - x_2\partial_2)\circ\partial_1 = -2\partial_1,$$

and

$$s_3^{rsym}(X_1,X_5,X_6)\circ s_3^{rsym}(X_2,X_3,X_4) = -4\partial_2.$$

Similarly,

$$s_3^{rsym}(X_2,X_3,X_6) = s_3^{rsym}(\partial_2, x_2\partial_1, x_1^2\partial_1) = -x_1^2\partial_1\circ\partial_1 = -2x_1\partial_1,$$
$$s_3^{rsym}(X_1,X_4,X_5) = s_3^{rsym}(\partial_1, x_1\partial_1 - x_2\partial_2, x_1\partial_2)$$
$$= (x_1\partial_1 - x_2\partial_2)\circ\partial_2 - x_1\partial_2\circ\partial_1 = -2\partial_2,$$

and

$$s_3^{rsym}(X_2,X_3,X_6)\circ s_3^{rsym}(X_1,X_4,X_5) = 0.$$

We have:

$$s_3^{rsym}(X_2,X_4,X_6) = s_3^{rsym}(\partial_2, x_1\partial_1 - x_2\partial_2, x_1^2\partial_1)$$

$$= x_1^2\partial_1 \circ \partial_2 = 0,$$

and

$$s_3^{rsym}(X_2, X_4, X_6) \circ s_3^{rsym}(X_1, X_3, X_5) = 0.$$

Finally,

$$s_3^{rsym}(X_2, X_5, X_6) = s_3^{rsym}(\partial_2, x_1\partial_2, x_1^2\partial_1) = 0,$$

and

$$s_3^{rsym}(X_2, X_5, X_6) \circ s_3^{rsym}(X_1, X_3, X_4) = 0.$$

Thus,

$$s_6'(X_1, \ldots, X_5, x_1^2\partial_1) = s_3^{rsym}(X_1, X_5, X_6) \circ s_3^{rsym}(X_2, X_3, X_4) = -4\partial_2.$$

Hence

$$\begin{aligned} s_6(X_1, \ldots, X_5, x_1^2\partial_1) &= s_6'(X_1, \ldots, X_5, x_1^2\partial_1) + s_6''(X_1, \ldots, X_5, x_1^2\partial_1) \\ &= -4\partial_2 + 4\partial_2 = 0. \end{aligned}$$

So, we have constructed esc $(s_{6,W_2}^{rsym.r})$. A reconstruction of $s_{6,W_2}^{rsym.r}$ by its escort (see (2)) gives us the formula for s_6. By lemma 6.3, $s_6 = s_6^{rsym.r}$ on $W(2)$.

11. $s_6 = 0$ is an identity on $S(2)$

Lemma 11.1. *$s_6 = 0$ is an identity on $S(2)$.*

Proof. Set

$$X_1 = \partial_1, X_2 = \partial_2, X_3 = x_2\partial_1, X_4 = x_1\partial_1 - x_2\partial_2, X_5 = x_1\partial_2,$$
$$V = \{(X_1, X_2, \ldots, X_6) : |X_6| = 1, X_6 \in S(2)\}.$$

Since supp $=$ supp(s_6) is generated by elements $X_1 \otimes \cdots \otimes X_6$, where $(X_1, \ldots, X_6) \in V$, we need to check that $s_6(X_1, \ldots, X_6) = 0$, for all $(X_1, \ldots, X_6) \in V$. By lemma 6.3,

$$s_6^{rsym.r} = s_6.$$

We have,

$$s_6 = s_3 \smile s_3.$$

Let $(X_1, \ldots, X_6) \in V$ and $F = s_6(X_1, \ldots, X_6)$.

We see that, $s_6(X_1, \ldots, X_6)$ is the alternating sum of elements of the form $s_3(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) \cdot s_3(X_{\sigma(4)}, X_{\sigma(5)}, X_{\sigma(6)})$, where $\sigma \in \mathfrak{S}_{3,3}$ are shuffle permutations, i.e., $\sigma(1) < \sigma(2) < \sigma(3)$, $\sigma(4) < \sigma(5) < \sigma(6)$. Moreover,

$$s_6(X_1, \ldots, X_6)$$
$$= \sum_{\sigma \in \mathfrak{S}_{3,3}, \sigma(1) < \sigma(4)} \operatorname{sign} \sigma \, [s_3(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}), s_3(X_{\sigma(4)}, X_{\sigma(5)}, X_{\sigma(6)})].$$

Since $|s_6(X_1,\dots,X_6)| = -1$,

$$s_6(X_1,\dots,X_6) \in \langle \partial_i \rangle,$$

for some $i = 1,2$. Therefore, in calculating $F = s_6(X_1,\dots,X_6)$ we can make summation only in $\sigma \in \mathfrak{S}_{3,3}$ such that

$$s_3(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) \in \langle u\partial_i : |u| = 1,2 \rangle,$$

$$s_3(X_{\sigma(4)}, X_{\sigma(5)}, X_{\sigma(6)}) \in \langle \partial^\alpha : |\alpha| = 1,2 \rangle.$$

Since

$$s_3(X_1, X_2, X) = s_3(\partial_1, \partial_2, X) = [\partial_2, X] \cdot \partial_1 + [X, \partial_1] \cdot \partial_2,$$

there are two possibilities:

- if $1,2 \in \{\sigma(1),\sigma(2),\sigma(3)\}$ or $1,2 \in \{\sigma(4),\sigma(5),\sigma(6)\}$ then $(\sigma(4),\sigma(5),\sigma(6)) = (1,2,s)$, and $(\sigma(1),\sigma(2),\sigma(3)) = (q,r,6)$, where $\{q,r,s\} = \{3,4,5\}$, and $q < r$.
- if each of the following subsets $\{\sigma(1),\sigma(2),\sigma(3)\}$ and $\{\sigma(4),\sigma(5),\sigma(6)\}$ contains exactly one element $s \in \{1,2\}$.

Therefore,

$$s_6(X_1,\dots,X_6) = s_6'(X_1,\dots,X_6) + s_6''(X_1,\dots,X_6),$$

where

$$\begin{aligned} &s_6'(X_1,\dots,X_6)\\ &= s_3^{rsym}(X_1,X_3,X_6) \circ s_3^{rsym}(X_2,X_4,X_5) - s_3^{rsym}(X_1,X_4,X_6) \circ s_3^{rsym}(X_2,X_3,X_5)\\ &+ s_3^{rsym}(X_1,X_5,X_6) \circ s_3^{rsym}(X_2,X_3,X_4) - s_3^{rsym}(X_2,X_3,X_6) \circ s_3^{rsym}(X_1,X_4,X_5)\\ &+ s_3^{rsym}(X_2,X_4,X_6) \circ s_3^{rsym}(X_1,X_3,X_5) - s_3^{rsym}(X_2,X_5,X_6) \circ s_3^{rsym}(X_1,X_3,X_4),\\ &s_6''(X_1,\dots,X_6)\\ &= -s_3^{rsym}(X_4,X_5,X_6) \circ s_3(X_1,X_2,X_3) + s_3^{rsym}(X_3,X_5,X_6) \circ s_3(X_1,X_2,X_4)\\ &\quad - s_3^{rsym}(X_3,X_4,X_6) \circ s_3(X_1,X_2,X_5). \end{aligned}$$

Here we use notation s_3^{rsym} instead of $s^{rsym.r}$ or $s_3^{rsym.l}$, because $s^{rsym.r} = s_3^{rsym.l}$ for any right-symmetric algebra.

Notice that

$$s_3(\partial_1, \partial_2, x_1\partial_1) = \partial_1\partial_2,$$

$$s_3(\partial_1, \partial_2, x_2\partial_1) = -\partial_1^2,$$

$$s_3(\partial_1, \partial_2, x_1\partial_2) = \partial_2^2,$$

$$s_3(\partial_1, \partial_2, x_2\partial_2) = -\partial_1\partial_2.$$

Therefore,

$$\begin{aligned} &s_6''(X_1,\dots,X_6)\\ &= s_3^{rsym}(x_1\partial_1 - x_2\partial_2, x_1\partial_2, X_6) \circ \partial_1^2 + 2s_3^{rsym}(x_2\partial_1, x_1\partial_2, X_6) \circ \partial_1\partial_2 \end{aligned} \tag{7}$$

$$-s_3^{rsym}(x_2\partial_1, x_1\partial_1 - x_2\partial_2, X_6) \circ \partial_2^2.$$

Now calculate $s_6''(X_1, \dots, X_6)$ for $X_6 = x_1^2\partial_2$. By (7) we have

$$s_6''(X_1, \dots, X_5, x_1^2\partial_2)$$
$$= 2x_1^2\partial_2 \circ \partial_1\partial_2 + (4x_1^2\partial_1 + 6x_1x_2\partial_2) \circ \partial_2^2 = 0.$$

Check that $s_6'(X_1, \dots, X_6) = 0$ for $X_6 = x_1^2\partial_2$.

Let $a, b, c \in \langle x_2\partial_1, x_1\partial_1 - x_2\partial_2, x_1\partial_2\rangle$. Notice that

$$s_3^{rsym}(\partial_i, a, X_6) = a \circ [X_6, \partial_i] + X_6 \circ [\partial_i, \circ a]$$
$$= a \circ \partial_i(X_6) - X_6 \circ \partial_i(a),$$
$$s_3^{rsym}(\partial_j, b, c) = b \circ [c, \partial_j] + c \circ [\partial_j, b] = b \circ \partial_j(c) - c \circ \partial_j(b).$$

By these formulas, it is easy to calculate that

$$s_3^{rsym}(X_1, X_3, X_6) = s_3(\partial_1, x_2\partial_1, x_1^2\partial_2) = 2x_2\partial_1 \circ x_1\partial_2 = 2x_1\partial_1,$$
$$s_3^{rsym}(X_2, X_4, X_5) = s_3(\partial_2, x_1\partial_1 - x_2\partial_2, x_1\partial_2) = x_1\partial_2 \circ \partial_2 = 0,$$

and

$$s_3^{rsym}(X_1, X_3, X_6) \circ s_3(X_2, X_4, X_5) = 0.$$

Furthermore,

$$s_3^{rsym}(X_1, X_4, X_6) = s_3^{rsym}(\partial_1, x_1\partial_1 - x_2\partial_2, x_1^2\partial_2)$$
$$= 2(x_1\partial_1 - x_2\partial_2) \circ x_1\partial_2 - x_1^2\partial_2 \circ \partial_1$$
$$= -2x_1\partial_2 - 2x_1\partial_2 = -4x_1\partial_2,$$
$$s_3^{rsym}(X_2, X_3, X_5) = s_3^{rsym}(\partial_2, x_2\partial_1, x_1\partial_2) = -x_1\partial_2 \circ \partial_1 = -\partial_2,$$

and

$$s_3^{rsym}(X_1, X_4, X_6) \circ s_3(X_2, X_3, X_5) = 0.$$

Finally,

$$s_3^{rsym}(X_1, X_5, X_6) = s_3(\partial_1, x_1\partial_2, x_1^2\partial_2)$$
$$= 2x_1\partial_2 \circ x_1\partial_2 - x_1^2\partial_2 \circ \partial_2 = 0,$$

and

$$s_3^{rsym}(X_1, X_5, X_6) \circ s_3^{rsym}(X_2, X_3, X_4) = 0.$$

Similarly,

$$s_3^{rsym}(X_2, X_3, X_6) \circ s_3^{rsym}(X_1, X_4, X_5) = 0,$$
$$s_3^{rsym}(X_2, X_4, X_6) \circ s_3^{rsym}(X_1, X_3, X_5) = 0,$$
$$s_3^{rsym}(X_2, X_5, X_6) \circ s_3^{rsym}(X_1, X_3, X_4) = 0.$$

So, we have established that $s_6'(X_1, \dots, X_6) = 0$ for $X_6 = x_1^2\partial_2$. Thus,

$$s_6(X_1, \dots, X_5, x_1^2\partial_2) = s_6'(X_1, \dots, X_5, x_1^2\partial_2) + s_6''(X_1, \dots, X_5, x_1^2\partial_2) = 0.$$

Similarly, one can calculate that

$$s_6(X_1,\dots,X_5,X_6)=0,$$

for any $X_6 = x_1^2\partial_1 - 2x_1x_2\partial_2, x_2^2\partial_2 - 2x_1x_2\partial_1, x_2^2\partial_1$. In other words,

$$\operatorname{esc}(s_6)(X_1,\dots,X_6)=0, \text{ for any } (X_1,\dots,X_6)\in V.$$

Therefore, by (2) $s_6=0$ is an identity on $S(2)$.

12. $s_7 = 0$ is an identity on $W(2)$

Lemma 12.1. *$s_7 = 0$ is an identity on $W(2)$.*

Proof. We see that $\operatorname{esc}(s_7)$ is uniquely defined by the homomorphism of sl_2-modules $f : L_1 \to L_{-1}$ given by

$$f(X) = s_7(\partial_1,\partial_2,x_1\partial_1,x_2\partial_2,x_1\partial_1,x_2\partial_2,X).$$

Notice that

$$L_1 \cong R(\pi_1)\oplus R(2\pi_1+\pi_2).$$

This isomorphism of sl_2-modules can be given by divergence map,

$$\operatorname{Div} : L_1 \to U,$$

$$\bar{L}_1 = \{X : \operatorname{Div} X = 0\} \cong R(2\pi_1+\pi_2),\quad \tilde{L}_1 = \{\operatorname{Div} X : X\in L_1\}\cong R(\pi_1).$$

Thus $f(X) = \lambda \operatorname{Div}(X)$ for some $\lambda\in\mathbb{C}$. Using the decomposition $s_7^{rsym.r} = s_4^{rsym.r} \overset{\circ}{\smile} s_3$, one can calculate that

$$s_7(\partial_1,\partial_2,x_1\partial_1,x_2\partial_2,x_1\partial_1,x_2\partial_2,x_1^2\partial_1)=0.$$

Therefore, $\lambda=0$ and $s_7=0$ is an identity on $W(2)$.

13. 5- and 6-commutators are primitive

Assume that $g = g(t_1,\dots,t_k)$ is a skew-symmetric multilinear polynomial. We call g a *k-commutator* on a class of vector fields, if for any k vector fields $X_1,\dots,X_k$ of this class $g(X_1,\dots,X_k)$ is again a vector field of this class.

Suppose that f is a Lie polynomial with left-normed brackets. Let $(A,[\ ,\ ])$ be a Lie algebra. As we have explained above, $f_A^{[\,,\,]} : A\otimes\cdots\otimes A\to A$ is a map obtained from f by substituting elements of A as arguments t_j and using the commutator $[\ ,\ ]$ for the product.

Suppose now that $(A,[\ ,\ ])$ is a Lie algebra of vector fields. Then $f_A^{[\,,\,]}$ is the standard k-commutator for any vector field algebra A. We call the k-commutator g *primitive* on A if $g_A^{\cdot}$ can not be represented as $f_A^{[\,,\,]}$ for any left-normed polynomial f.

Lemma 13.1. *There does not exist a Lie polynomial $f = f(t_1, \ldots, t_5)$ such that $s_5(X_1, \ldots, X_5) = f(X_1, \ldots, X_5)$, for any $X_1, \ldots, X_5 \in S(2)$. Similarly, one can not represent a 6-commutator on $W(2)$ in the form $s_6(X_1, \ldots, X_6) = g(X_1, ..., X_6)$, for any $X_1, ..., X_6 \in W(2)$, where g a is Lie polynomial in 6 variables.*

Proof. Let L be a Lie algebra, $U(L)$ its universal enveloping algebra and

$$\Delta : U(L) \to U(L) \otimes U(L), \quad \Delta(X) = X \otimes 1 + 1 \otimes X, \ \forall X \in L,$$

a comultiplication. For any $X_1, \ldots, X_k \in L$,

$$\Delta(X_1 \cdot \ldots \cdot X_k) = \sum_{l=0}^{k} \sum_{\sigma \in \mathfrak{S}_{l,k-l}} X_{\sigma(1)} \cdot \cdots \cdot X_{\sigma(l)} \otimes X_{\sigma(l+1)} \cdot \ldots \cdot X_{\sigma(k)}.$$

Thus, for any $X_1, \ldots, X_k \in L$,

$$\Delta(s_k(X_1, \ldots, X_k)) = \sum_{l=0}^{k} \sum_{\sigma \in \mathfrak{S}_{l,k-l}} s_l(X_1, \ldots, X_l) \otimes s_{k-l}(X_{l+1}, \ldots, X_k).$$

Therefore, if s_k is the standard k-commutator, i.e., if s_k is obtained from Lie polynomial, then [10]

$$G_k = \sum_{l=1}^{k-1} s_l(X_1, \ldots, X_l) \otimes s_{k-l}(X_{l+1}, \ldots, X_k)$$

should be identically 0 for any $X_1, \ldots, X_k \in L$. Here $L = W(2)$ if $k = 6$, and $L = S(2)$ if $k = 5$.

In a calculation of G_k below we use formulas for quadratic parts of k commutators (lemmas 7.2, 7.3, 7.4).

Consider the case of 5-commutators. Take

$$(X_1, X_2, X_3, X_4, X_5) = (\partial_1, \partial_2, x_1\partial_1 - x_2\partial_2, x_2\partial_1, x_1\partial_2).$$

One can calculate that

$$\begin{gathered} G_5 \\ = -4\partial_1 \otimes \partial_2 - 4\partial_2 \otimes \partial_1 - 2\partial_2 \otimes x_1\partial_1^2 - 4\partial_2 \otimes x_2\partial_1\partial_2 + 4\partial_1\partial_2 \otimes x_1\partial_1 \\ -4\partial_1\partial_2 \otimes x_2\partial_2 + 4x_1\partial_1 \otimes \partial_1\partial_2 - 2x_1\partial_1^2 \otimes \partial_2 - 4x_2\partial_2 \otimes \partial_1\partial_2 - 4x_2\partial_1\partial_2 \otimes \partial_2 \\ \neq 0. \end{gathered}$$

So, s_5 on $S(2)$ can not be obtained from any Lie polynomial.

Consider now the case of 6-commutator. Take

$$(X_1, X_2, X_3, X_4, X_5, X_6) = (\partial_1, \partial_2, x_1\partial_1, x_2\partial_1, x_1\partial_2, x_1^2\partial_1).$$

We see that

$$s_3(X_1, X_2, X_3) \otimes s_3(X_4, X_5, X_6) = \partial_1\partial_2 \otimes (3x_1^2\partial_1 + 2x_1x_2\partial_2 + x_1^3\partial_1^2 + 2x_1^2x_2\partial_1\partial_2),$$

Therefore, G_6 has the term of the form $\partial_1\partial_2 \otimes x_1^3\partial_1^2$. Collect all terms of G_6 of the form $\lambda\ \partial_1\partial_2 \otimes x_1^3\partial_1^2$. Then their sum, denoted by R, should be 0 if s_6 is standard 5-commutator. As a differential operator of second order, $x_1^3\partial_1^2$ can not appear in $s_2(X_i, X_j)$. Direct calculations then show that the elements of the form $s_l(X_{j_1}, \ldots, X_{j_l}), j_1 < \cdots < j_l, l = 3, 4, 5$, may have the part $\mu\, x_1^3\partial_1^2, \mu \neq 0$ only in one case: $l = 3, (j_1, j_2, j_3) = (4, 5, 6)$. So, $R = \partial_1\partial_2 \otimes x_1^3\partial_1^2 \neq 0$. This contradiction shows that 6-commutator on $W(2)$ is primitive.

14. s_5 and s_6 are cocycles

Let $d : C^k(L, L) \to C^{k+1}(L, L)$ be the coboundary operator. Then

$$d\psi = d'\psi + d''\psi,$$

where

$$d'\psi(X_1, \ldots, X_{k+1}) = \sum_{i<j}(-1)^{i+j}\psi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

$$d''\psi(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}[X_i, \psi(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})].$$

Lemma 14.1. *$d's_k^{rsym.r} = 0$, if k is even and $d's_k^{rsym.r} = -s_{k+1}^{rsym.r}$, if k is odd.*

Proof. This follows from induction in n and the following relation

$$s_{k+1}^{rsym.r} = \sum_{i=1}^{k+1}(-1)^{i+k+1}(s_k^{rsym.r}(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))X_i.$$

Lemma 14.2. $(2d' + d'')s_k = 0$, *for any $k \geq n^2 + 2n - 2$.*

Proof. By corollary 3.4, $ad\, X \in Der(W(n), s_k)$, if $k \geq n^2 + 2n - 2$. Therefore,

$$\begin{aligned}&[X_i, s_k(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})]\\ &= \sum_{j=1}^{i-1}(-1)^{1+j}s_k([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})\\ &+ \sum_{j=i+1}^{k+1}(-1)^j s_k([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),\end{aligned}$$

and

$$d''s_k(X_1, \ldots, X_{k+1})$$

$$= -2\sum_{i<j}(-1)^{i+j}s_k([X_i,X_j],\dots\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}).$$

In other words, $d''s_k = -2d's_k$, if $k \geq n^2+2n-2$.

Corollary 14.3. *$ds_5 = 0$ on $S(2)$.*

Proof. By corollary 3.2 and theorems 8.1 and 11.1 $s_5 = s_5^{rsym.r}$ and $s_6 = s_6^{rsym.r} = 0$ are identities on $S(2)$. Therefore, $d_{s_5} = d's_5 + d''s_5 = -d's_5 = s_6 = 0$ is an identity on $S(2)$.

Corollary 14.4. *$ds_6 = 0$ on $W(2)$.*

Proof. By corollary 3.2, lemma 6.3 and lemma 12.1 $s_6 = s_6^{rsym.r}$ and $s_7 = s_7^{rsym.r}$ is an identity on $W(2)$. Therefore, by lemma 14.1 and 14.2 $d_{s_6} = d's_6 + d''s_6 = -d's_6 = 0$ is an identity on $W(2)$.

Remark. One can prove that $(L,\{s_2,s_l\})$ is also *sh*-Lie, for $l = n^2+2n-2$, if $L = W(n)$ and $l = n^2+2n-3$, if $L = S(n)$.

Our results can be formulated in terms of generalized cohomology operators. There are two ways to do it. In the first way one saves the index of nilpotency $d^2 = 0$, but changes the grading degree. In the second way one saves grading degree, but changes the index of nilpotency from $d^2 = 0$ to $d^N = 0$. A cohomology theory for $d^N = 0$ was developed in [7].

Let us show how to do it for left multiplication operators. Let $L = W(n)$ be the right-symmetric algebra of vector fields, r_a right multiplication operator and l_a left multiplication operator, $(b)r_a = b\circ a$, $(b)l_a = a\circ b$. Define a linear operator $d : \wedge^*(L,L) \to \wedge^*(L,L)$ by

$$d : C^k(L,L) \to C^{k+n}(L,L),$$

$$d\psi(a_1,\dots,a_{k+n}) = \sum_{\sigma\in\mathfrak{S}_{n,k}} \operatorname{sign}\sigma\,(\cdots((\psi(a_{\sigma(n+1)},\dots,a_{\sigma(k+n)}))l_{a_{\sigma(1)}})\cdots)l_{a_{\sigma(n)}}.$$

Then the condition $d^2 = 0$ follows from theorem 3.3 of [4].

In the second case we need to consider a coboundary operator with grading degree $+1$,

$$d_l : \wedge^*(L,L) \to \wedge^*(L,L),$$

$$d_l : \wedge^k(L,L) \to \wedge^{k+1}(L,L),$$

$$d_l\psi(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1}(-1)^i(\psi(a_1,\dots,\hat{a}_i,\dots,a_{k+1}))l_{a_i}.$$

Then $d_l^{2n} = 0$.

One can construct similar coboundary operators corresponding to right-multiplication operators. For example,

$$d_r : \wedge^*(L,L) \to \wedge^*(L,L),$$
$$d_r : \wedge^k(L,L) \to \wedge^{k+1}(L,L),$$
$$d_r\psi(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1}(-1)^i(\psi(a_1,\dots,\hat{a_i},\dots,a_{k+1}))r_{a_i}.$$

has the property $d_r^{n^2+2n-1} = 0$.

These constructions have some other modifications that include the case of more general right-symmetric algebras and their modules.

15. Proofs of main results

Proof of theorem 2.1. This follows from lemmas 6.3, 5.6 and corollary 5.5.

Proof of theorem 2.2. This follows from lemmas 5.3, 9.1, 11.1, 13.1 and corollary 14.3.

Proof of theorem 2.3. This follows from lemmas 12.1, 13.1, 5.4 and corollary 14.4. We have proved that $W(3)$ has nontrivial 10-commutator. Its restriction to $S(3)$ is also nontrivial. So, $W(3)$ has two well-defined nontrivial N-commutators: 13-commutator s_{13} and 10-commutator s_{10}. Divergenceless vector fields subalgebra $S(3)$ has only one nontrivial N-commutator: 10-commutator s_{10}.

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