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S_n- and GL_n-module structures on free Novikov algebras

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ABSTRACT

An algebra with two identities $a(bc) = b(ac)$, $a(bc) - (ab)c = a(cb) - (ac)b$, is called Novikov. Module structures of a free Novikov algebra over permutation group and general linear group are studied.

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1. Introduction

Construction of free algebras is one of the important problems of modern algebra. They appear in studying the varieties of algebras, polynomial identities and in operads theory. Multilinear parts of free algebras contain essential information about free algebras. Multilinear parts are studied by combinatorial methods and by methods of representation theory. In our paper we focus our attention on module structures of multilinear parts of free Novikov algebras. By module structures we mean modules over permutation group and over general linear group. Recall that any irreducible S_n-module

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can be characterized by a partition of n or by Young diagrams. The so-called Specht modules S^α, where $\alpha \vdash n$ is a partition of n, give us a complete list of irreducible modules of permutation group S_n. There exist deep connections between irreducible S_n-modules and irreducible $GL(V)$-modules. Any irreducible $GL(V)$-module is isomorphic to a so-called Weyl module. Any Weyl module as Specht module can be characterized by Young diagrams. These facts are known and one can find details, for example, in [7,8].

The list of algebraic varieties with studied free objects is given in [12]. Let us recall some of such results.

If F_n^{ass} is a multilinear part of a free associative algebra with n generators, then F_n^{ass}, as a module over permutations group S_n, is isomorphic to a regular module. Any irreducible S_n-module is associative admissible. This means that any irreducible S_n-module appears in decomposition of F_n^{ass} to a direct sum of irreducible components with non-zero multiplicity. Moreover, for any irreducible S_n-module its multiplicity in such decomposition is equal to dimension of this module.

If F_n^{lie} is a multilinear part of a free Lie algebra with $n \geq 3$ generators, then by [6] any irreducible S_n-module S^α, except when $\alpha = (1^n), (n), (2^2), (2^3)$, is a Lie admissible module. Multiplicity of such a module in decomposition of F_n^{lie} can be calculated in terms of major indices of Young diagrams. It was done in [9].

Similar questions for multilinear parts of free bicommutative algebras are studied in [5]. For degrees $n \leq 7$ module structures over S_n for multilinear parts of free anti-commutative algebras were found in [1].

The aim of our paper is to study multilinear parts of free Novikov algebras. We introduce a notion of Novikov weights and we describe irreducible components of multilinear parts in terms of weights. We find a criterion for Novikov admissible irreducible modules. We prove that multiplicities of irreducible components are ruled by Kostka numbers.

2. Statement of main result

Let *rsym* (right-symmetric polynomial) and *lcom* (left-commutative polynomial) be non-commutative non-associative polynomials defined by

$$rsym = t_1(t_2t_3) - t_1(t_3t_2) - (t_1t_2)t_3 + (t_1t_3)t_2,$$
$$lcom = t_1(t_2t_3) - t_2(t_1t_3).$$

An algebra with identities $rsym = 0$ and $lcom = 0$ is called *right-Novikov*. In our paper we consider only right-Novikov algebras therefore the word "right" will be omitted. So, if $A = (A, \circ)$ is a Novikov algebra with multiplication $a \circ b$, then

$$(a, b, c) = (a, c, b),$$
$$a \circ (b \circ c) = b \circ (a \circ c),$$

for any $a, b, c \in A$. Here

$$(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$$

is an associator.

Example. Let $A = \mathbf{C}[x]$ and $a \circ b = \partial(a)b$, where $\partial = \frac{\partial}{\partial x}$ is partial derivation. Then $(A, \circ)$ is the Novikov algebra.

Free Novikov algebras were described in [3]. In [4] a base of free Novikov algebra in terms of Young diagrams is constructed. In our paper this base is used intensively.

A partition of a positive integer n is a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with positive integer components $\alpha_1 \geq \ldots \geq \alpha_k > 0$ such that $\sum_{i=1}^{k} \alpha_i = n$. If α is a partition of n, then we write $\alpha \vdash n$. Components α_i are called parts of the partition and we write $\alpha_i \in \alpha$. Let $i_l = |\{\alpha_s = l \mid s = 1, \ldots, k\}|$ be multiplicity of l, the number of parts of α equal to l. For a partition $\alpha \vdash n$ we will use also another notation $\alpha = (n^{i_n}, \ldots, 2^{i_2}, 1^{i_1})$. For example, $\alpha = (3, 2, 2, 1) = (3^1, 2^2, 1^1)$. For a sequence β let us denote by $sort(\beta)$ a partition whose parts are components of β sorted in non-increasing order. For example, if $\beta = (2, 3, 1, 5, 1)$, then $sort\,\beta = (5, 3, 2, 1, 1)$. Let $P(n)$ be a set of all partitions of n and $p(n)$ the number of partitions of n.

Definition. For $\alpha = (n^{i_n}, \ldots, 2^{i_2}, 1^{i_1}) \in P(n)$ the partition $w(\alpha) \in P(n+1)$ defined by

$$w(\alpha) = sort\left(n + 1 - \sum_{j=1}^{n} i_j, i_1, i_2, \ldots, i_n\right)$$

is called the *weight* of α.

Example. Let us construct the weight function $w : P(5) \to P(6)$. We have

$$P(5) = \{(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3), (1^5)\}.$$
$$w((5)) = sort(5, 1) = (5, 1),$$
$$w((4, 1)) = w((3, 2)) = sort(4, 1, 1) = (4, 1, 1) = (4, 1^2),$$
$$w((3, 1^2)) = w((2^2, 1)) = sort(3, 1, 2) = (3, 2, 1),$$
$$w((2, 1^3)) = sort(2, 3, 1) = (3, 2, 1), \qquad w((1^5)) = sort(1, 5) = (5, 1).$$

Recall that for $\alpha, \beta \vdash n$, the Kostka number $K_{\alpha\beta}$ is equal to the number of semi-standard Young tableaux of shape α and content β. Recall also that for a partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vdash n$ its Young subgroup in S_n is defined as

$$S_\alpha = S_{\{1,2,\ldots,\alpha_1\}} \times S_{\{\alpha_1+1,\alpha_1+2,\ldots,\alpha_1+\alpha_2\}} \times \cdots \times S_{\{n-\alpha_l+1,n-\alpha_l+2,\ldots,n\}},$$

where each component in this product is a symmetric group constructed by permutations on its corresponding sets. A partition $\beta \vdash n+1$ is called *Novikov admissible* if the Specht module S^β is an S_{n+1}-module of a multilinear part of a free Novikov algebras with nontrivial coefficients. Let $N(V)$ be a Novikov algebra freely generated by basis elements of V.

The aim of our paper is to describe S_n- and $GL(V)$-module structures on free Novikov algebras. The main field is supposed to be a field of characteristic 0. The structure of a free Novikov algebra as an S_n-module is given below.

Theorem 2.1. *Let F_{n+1} be a multilinear part of free Novikov algebra of degree $n+1$.*

a. *Then F_{n+1} as an S_{n+1}-module is isomorphic to a direct sum of modules induced by trivial module of Young subgroups $S_{w(\alpha)}$.*

$$F_{n+1} \cong \bigoplus_{\alpha \vdash n} Ind_{S_{w(\alpha)}}^{S_{n+1}}(\mathbf{1}).$$

b. *Further, induced modules are direct sums of irreducible ones, and*

$$F_{n+1} \cong \bigoplus_{\beta \vdash n+1} \Big(\sum_{\alpha \vdash n} K_{\beta w(\alpha)} \Big) S^\beta,$$

where S^β is a Specht module corresponding to a partition β.

c. *$\beta = (\beta_1, \ldots, \beta_k) \vdash n+1$ is Novikov admissible if and only if:*

$$\beta_1 - 1 \geq \beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k.$$

The structure of a free Novikov algebra as $GL(V)$-module is given below.

Corollary 2.2. *Let N_{n+1} be the homogenous part of $N(V)$ of degree $n+1$. Then N_{n+1} as $GL(V)$-module is isomorphic to*

$$N_{n+1} \cong \bigoplus_{\alpha \vdash n} \bigotimes_{w_i \in w(\alpha)} Sym^{w_i} V$$

where $Sym^{w_i} V$ is the symmetric power of V, and

$$N_{n+1} \cong \bigoplus_{\beta \vdash n+1} \Big(\sum_{\alpha \vdash n} K_{\beta w(\alpha)} \Big) S_\beta(V)$$

where $S_\beta(V)$ is a Weyl module corresponding to a partition β whose number of parts is not more than $\dim V$.

Remark 2.3. We see that multiplicities of Novikov admissible modules can be characterized by sums of Kostka numbers. Let us recall some results about sums of Kostka numbers. Let $\lambda \vdash n$. In general, by definition of Schur functions

$$\sum_{\mu} K_{\lambda\mu} = s_{\lambda}\bigl(1^{n}\bigr)$$

where $\mu = (\mu_1, \dots, \mu_n)$ such that $\mu_1 + \dots + \mu_n = n$ and $\mu_i \geq 0$ for any i, and $s_\lambda(x_1, \dots, x_n)$ is a Schur function corresponding to the partition λ. By Corollary 7.21.4 in [11],

$$\sum_{\mu} K_{\lambda\mu} = \prod_{u \in \lambda} \frac{n + j - i}{h(u)}$$

where $u = (i, j)$ is a square in λ, and $h(u)$ is the hook length of λ at u.

For partitions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vdash n$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l) \vdash n$ we say that α *dominates* β, and write $\alpha \trianglerighteq \beta$, if

$$\alpha_1 + \alpha_2 + \dots + \alpha_i \geq \beta_1 + \beta_2 + \dots + \beta_i$$

for all $i \geq 1$. If $i > k$ (respectively, $i > l$), then we take α_i (respectively, β_i) to be zero.

Example. Let us describe S_6-module of F_6 as a direct sum of irreducible S_6-modules. Novikov admissible partitions are

$$\beta = \bigl\{(6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1)\bigr\}.$$

Recall that if $\alpha, \beta \vdash n$, then $K_{\alpha\beta} \neq 0 \Leftrightarrow \alpha \trianglerighteq \beta$. Therefore

$$K_{(5,1)(5,1)} = K_{(6)(5,1)} = K_{(4,1^2)(4,1^2)} = K_{(4,2)(4,1^2)} = K_{(6)(4,1^2)} = 1,$$
$$K_{(5,1)(4,1^2)} = K_{(4,2)(3,2,1)} = K_{(5,1)(3,2,1)} = 2,$$
$$K_{(3,2,1)(3,2,1)} = K_{(4,1^2)(3,2,1)} = K_{(3^2)(3,2,1)} = K_{(6)(3,2,1)} = 1.$$

Thus,

$$F_6 \cong 7S^{(6)} \oplus 12S^{(5,1)} \oplus 8S^{(4,2)} \oplus 5S^{(4,1^2)} \oplus 3S^{(3^2)} \oplus 3S^{(3,2,1)}.$$

Example. If $\mathit{dim}\, V > 1$ then as $GL(V)$-module

$$N_3 \cong 2S_{(3)}(V) \oplus 2S_{(2,1)}(V).$$

Let $p(n,k)$ be the number of partitions of n into k different parts. Then by [2]

$$p(n,k) = \sum_{\alpha \vdash n} \binom{k}{k} i_1 i_2 \dots i_k - \binom{k+1}{k} i_1 i_2 \dots i_{k+1} + \binom{k+2}{k} i_1 i_2 \dots i_{k+2} - \dots,$$

where $\alpha = (n^{i_n}, \dots, 2^{i_2}, 1^{i_1})$. For example, $p(5,2) = 5$, since the list of partitions of 5 into 2 different parts is as follows: $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$.

Corollary 2.4. *For F_{n+1}, the following statements are true:*

- **a.** *Multiplicity of trivial module is $p(n)$.*
- **b.** *Multiplicity of the irreducible module $S^{(n+1-l,1^l)}$ is $\sum_{k\geq l} p(n,k)\binom{k}{l}$.*
- **c.** *A partition with two parts is Novikov admissible.*
- **d.** *If $k > 2$, then the partition (l^k) is not Novikov admissible.*

Let

$$p_\lambda(x_1,\ldots,x_m) = \prod_{i=1}^{l}\left(x_1^{\lambda_i} + \ldots + x_m^{\lambda_i}\right)$$

be power sum symmetric polynomial and

$$C(\lambda,\mu) = \left[x_1^{\mu_1}\ldots x_m^{\mu_m}\right]p_\lambda(x_1,\ldots,x_m)$$

the coefficient at $x_1^{\mu_1}\ldots x_m^{\mu_m}$ in $p_\lambda(x_1,\ldots,x_m)$.

Corollary 2.5. *Let $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_l) \vdash n+1$. The character of the representation F_{n+1} evaluated at an element of S_{n+1} with cycle type λ is equal to*

$$\chi(\lambda) = \sum_{\alpha\vdash n} C\big(\lambda, w(\alpha)\big).$$

Example. Let $n = 3$. Then $P(3) = \{(3),(2,1),(1,1,1)\}$ and

$$w\big((3)\big) = (3,1), \qquad w\big((2,1)\big) = (2,1,1), \qquad w\big((1,1,1)\big) = (3,1).$$

Let us calculate the character value of $(12)(3)(4) \in S_4$. We see that the cycle type of $(12)(3)(4)$ is $\lambda = (2,1,1)$.

$$\begin{aligned}
\chi\big((2,1,1)\big) &= 2C\big((2,1,1),(3,1)\big) + C\big((2,1,1),(2,1,1)\big)\\
&= 2\left[x_1^3x_2^1\right]p_{(2,1,1)}(x_1,x_2) + \left[x_1^2x_2^1x_3^1\right]p_{(2,1,1)}(x_1,x_2,x_3)\\
&= 2\left[x_1^3x_2^1\right]\left(x_1^2+x_2^2\right)(x_1+x_2)(x_1+x_2)\\
&\quad + \left[x_1^2x_2^1x_3^1\right]\left(x_1^2+x_2^2+x_3^2\right)(x_1+x_2+x_3)(x_1+x_2+x_3)\\
&= 2\cdot 2 + 2 = 6.
\end{aligned}$$

3. Young diagrams, Novikov diagrams and blocks

In [3], a base of free Novikov algebras in terms of rooted trees and so-called r-elements is constructed. In [4] this base is given in terms of partitions and Young diagrams. For

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n$ denote by Y_α its Young diagram, i.e. a diagram with α_i boxes in the i-th row.

For a partition $\alpha \vdash n$ denote by $\widetilde{\alpha} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_k)$ a partition of $n+1$ such that $\widetilde{\alpha}_1 = \alpha_1 + 1$, $\widetilde{\alpha}_i = \alpha_i$, $i > 1$. Note that $Y_{\widetilde{\alpha}}$ is a Young diagram. Call it *a Novikov* diagram.

For a Novikov diagram we call its first column *nape*, its complement *face* and added box in the first row *nose*. So, nose is a part of face and

$$Y_{\widetilde{\alpha}} = nape(Y_{\widetilde{\alpha}}) \cup face(Y_{\widetilde{\alpha}}).$$

For example, if $\alpha = (5, 4, 4, 3, 3, 3)$, then

$$Y_{\widetilde{\alpha}} = \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & & \end{array}$$

and the box corresponding to white circle is nose. To show nape and face, we denote nape boxes by x and face boxes by y

$$Y_{\widetilde{\alpha}} = \begin{array}{cccccc} x & y & y & y & y & y \\ x & y & y & y & & \\ x & y & y & y & & \\ x & y & y & & & \\ x & y & y & & & \\ x & y & y & & & \end{array}$$

Now we introduce so-called *blocks* of Novikov diagram. Suppose that a Novikov diagram $Y_{\widetilde{\alpha}}$ corresponds to a partition α such that

$$\alpha_1 = \ldots = \alpha_{l_1} > \alpha_{l_1+1} = \ldots = \alpha_{l_1+l_2} > \alpha_{l_1+l_2+1}$$
$$= \ldots > \alpha_{l_1+\ldots+l_{s-1}+1} = \ldots = \alpha_{l_1+\ldots+l_s} > 0,$$
$$k = l_1 + \ldots + l_s.$$

In other words, the partition $\alpha \vdash n$ in terms of multiplicities can be presented in the following way

$$\alpha = \left(\alpha_1^{l_1}, \alpha_{l_1+1}^{l_2}, \ldots, \alpha_{l_1+\ldots+l_{s-1}+1}^{l_s}\right).$$

Then Novikov diagram $Y_{\widetilde{\alpha}}$ has $s+1$ blocks, denoted by $B_0, B_1, \ldots, B_s$. By definition, the block B_0 is the face of the Novikov diagram. Other blocks are parts of the nape, B_1 corresponds to rows $1, 2, \ldots, l_1$, B_2 corresponds to rows $l_1+1, \ldots, l_2$, etc., B_s corresponds to rows $l_1 + \ldots + l_{s-1} + 1, \ldots, l_1 + \ldots + l_{s-1} + l_s$. The number of boxes in a block is called the *length* of block. So,

$$nape(Y_{\widetilde{\alpha}}) = B_1 \cup \ldots \cup B_s, \qquad face(Y_{\widetilde{\alpha}}) = B_0,$$
$$|B_i| = l_i, \quad 1 \le i \le s, \qquad |B_0| = n + 1 - k = l_0.$$

For example, $\alpha = (5, 4, 4, 3, 3, 3)$ has 4 blocks since $\alpha = (5^1, 4^2, 3^3)$. Below we denote boxes of block B_i of Novikov diagram $Y_{\widetilde{\alpha}}$ by x_i,

$$Y_{\widetilde{\alpha}} = \begin{array}{llllll} x_1 & x_0 & x_0 & x_0 & x_0 & x_0 \\ x_2 & x_0 & x_0 & x_0 & & \\ x_2 & x_0 & x_0 & x_0 & & \\ x_3 & x_0 & x_0 & & & \\ x_3 & x_0 & x_0 & & & \\ x_3 & x_0 & x_0 & & & \end{array}$$

Example. The list of Novikov diagrams with blocks for $n = 4$.

$$Y_{\widetilde{(4)}} = \star \quad \bullet \quad \bullet \quad \bullet \quad \bullet, \qquad |B_0| = 4, \qquad |B_1| = 1,$$

$$Y_{\widetilde{(3,1)}} = \begin{array}{llll} \star & \bullet & \bullet & \bullet \\ \diamond & & & \end{array}, \qquad |B_0| = 3, \qquad |B_1| = 1, \qquad |B_2| = 1,$$

$$Y_{\widetilde{(2,2)}} = \begin{array}{lll} \star & \bullet & \bullet \\ \star & \bullet & \end{array}, \qquad |B_0| = 3, \qquad |B_1| = 2,$$

$$Y_{\widetilde{(2,1^2)}} = \begin{array}{lll} \star & \bullet & \bullet \\ \diamond & & \\ \diamond & & \end{array}, \qquad |B_0| = 2, \qquad |B_1| = 1, \qquad |B_2| = 2,$$

$$Y_{\widetilde{(1^4)}} = \begin{array}{ll} \star & \bullet \\ \star & \\ \star & \\ \star & \end{array}, \qquad |B_0| = 1, \qquad |B_1| = 4.$$

Now we are ready to introduce Novikov tableau. It is a Novikov diagram with filling. Below we give an exact definition of Novikov tableau.

Let Ω be an ordered set. For a Novikov tableau $Y_{\widetilde{\alpha}}$ call a function $f : Y_{\widetilde{\alpha}} \to \Omega$ *a filling* of $Y_{\widetilde{\alpha}}$. This means that any box of $Y_{\widetilde{\alpha}}$ is labelled by elements of Ω. Denote by $Y_{\widetilde{\alpha},f}$ a diagram $Y_{\widetilde{\alpha}}$ with filling f. Let $B_0, B_1, \ldots, B_s$ be blocks of Novikov diagram $Y_{\widetilde{\alpha}}$. For block B_l, $0 \le l \le s$, denote by $B_{l,f}$ a sequence of its elements under filling f. Call it *a block sequence.* Here we assume, to be definite, that boxes of blocks are numerated from bottom to top and in each row (in case of face) boxes are numerated from left to right.

Say that a filling $f : Y_{\widetilde{\alpha}} \to \Omega$ is *Novikov*, if

- *block sequence $B_{l,f}$, is non-decreasing for any $0 \le l \le s$.*

For a filling $g : Y_{\widetilde{\alpha}} \to \Omega$ denote by $sort(g) : Y_{\widetilde{\alpha}} \to \Omega$ a filling, such that each sequence $B_{l,sort(g)}$ is a sequence $sort(B_{l,g})$, $0 \le l \le s$, that is sequence $B_{l,f}$ sorted in non-decreasing order. Note that $Y_{\widetilde{\alpha},sort(g)}$ is a Novikov tableau.

Let $Y_{\widetilde{\alpha},f}$ be a Novikov tableau

$$\begin{array}{lllllll} f_{1,1} & f_{1,2} & \cdots & \cdots & \cdots & f_{1,\alpha_1} & f_{1,\alpha_1+1} \\ f_{2,1} & f_{2,2} & \cdots & \cdots & f_{2,\alpha_2} & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ f_{k,1} & f_{k,2} & \cdots & f_{k,\alpha_k} & & & \end{array}$$

Then the base element of free Novikov algebra constructed from $Y_{\widetilde{\alpha},f}$ is

$$e_{\alpha,f} = X_{k,f} \circ \big(\cdots (X_{2,f} \circ X_{1,f}) \cdots\big),$$

where

$$X_{i,f} = \big(\big(\cdots (f_{i,1} \circ f_{i,2}) \cdots\big) \circ f_{i,\alpha_i-1}\big) \circ f_{i,\alpha_i}, \quad k \ge i > 1,$$
$$X_{1,f} = \big(\big(\cdots (f_{1,1} \circ f_{1,2}) \cdots\big) \circ f_{1,\alpha_1}\big) \circ f_{1,\alpha_1+1}.$$

In [4] it was established that elements $e_{\alpha,f}$ corresponding to Novikov tableaux $Y_{\widetilde{\alpha},f}$, where $\alpha \in P(n)$, form base of free Novikov algebra.

Denote by $Base_{n+1}$ the set of Novikov base elements $e_{\alpha,f}$, where $\alpha \vdash n$ and f is a Novikov filling. For a base element $v \in Base_{n+1}$, say that v has *degree* α and denote $deg\, v = \alpha$, if $v = e_{\alpha,f}$.

Example. Suppose that $a_1 < a_2 < \ldots < a_8$. Then

$$Y_{\widetilde{(3,3,1)},f} = \begin{array}{llll} a_2 & a_5 & a_6 & a_7 \\ a_1 & a_3 & a_4 & \\ a_8 & & & \end{array}$$

is a Novikov tableau and

$$Y_{\widetilde{(3,3,1)},g} = \begin{array}{llll} a_1 & a_5 & a_6 & a_3 \\ a_2 & a_7 & a_4 & \\ a_8 & & & \end{array}$$

is not a Novikov tableau. Note that $sort(g) = f$ and $Y_{\widetilde{(3,3,1)},sort(g)} = Y_{\widetilde{(3,3,1)},f}$. The base element v corresponding to Novikov tableau $Y_{\widetilde{(3,3,1)},f}$ is the element

$$v = e_{(3,3,1),f} = a_8 \circ \big(\big((a_1 \circ a_3) \circ a_4\big) \circ \big(\big((a_2 \circ a_5) \circ a_6\big) \circ a_7\big)\big)$$

with degree $deg\, v = (3,3,1)$ and block sequences

$$B_{0,f} = a_3a_4a_5a_6a_7, \qquad B_{1,f} = a_1a_2, \qquad B_{2,f} = a_8.$$

4. Filtration and grading on F_{n+1}

Recall that our alphabet Ω has a linear order. In this section we give prolongation of linear order in alphabet Ω to a linear order on base $Base_{n+1}$.

Let $e_{\alpha,f}, e_{\beta,g} \in Base_{n+1}$ be two Novikov base elements with $\alpha, \beta \vdash n$ and fillings f and g. Suppose that $B_{\alpha,f} = (B_{0,f}, B_{1,f}, \dots, B_{l,f})$ and $C_{\alpha,g} = (C_{0,g}, C_{1,g}, \dots, C_{m,g})$ are block structures of $e_{\alpha,f}$ and $e_{\beta,g}$ respectively.

We say that $e_{\alpha,f}$ is *no less than* $e_{\beta,f}$, and denote it $e_{\alpha,f} \geq e_{\beta,g}$, if one of the following cases holds

- $\alpha > \beta$.
- $\alpha = \beta$, $B_{0,f} = C_{0,g}$, $\dots$, $B_{i-1,f} = C_{i-1,g}$ and $B_{i,f} > C_{i,g}$ for some $i \geq 0$.

It is clear that this order is a linear order.

Example. Elements of $Base_3$ sorted in decreasing order:

$$(a \circ b) \circ c \geq (b \circ a) \circ c \geq (c \circ a) \circ b \geq a \circ (b \circ c) \geq a \circ (c \circ b) \geq b \circ (c \circ a).$$

Let $X = \sum_{v \in Base_{n+1}} \lambda_v v$ be an element of F_{n+1}, where $\lambda_v \in K$. We say that X has *degree* α and write $deg(X) = \alpha$, if $\lambda_u = 0$ as soon as $deg\, u > \alpha$ and $\lambda_v \neq 0$, for some $v \in Base_{n+1}$ with $deg(v) = \alpha$. To define order on any X and Y, first write the linear combinations of base elements for X and Y, and then compare corresponding base elements of X and Y. Note that

$$deg\, X < deg\, Y \quad \Rightarrow \quad X < Y.$$

By using the definition of the order one can easily show that

Proposition 4.1. *For any $X, Y \in F_{n+1}$,*

$$deg(X + Y) \leq max(deg\, X, deg\, Y),$$
$$deg(\lambda X) \leq deg\, X, \quad \lambda \in K.$$

Now we are able to construct filtration on F_{n+1}. By Proposition 4.1 for any $\alpha \vdash n$ the set of elements of degree no more than α forms a linear subspace. Denote it by $\mathcal{F}_\alpha$. So, $\mathcal{F}_\alpha$ is a linear space generated by base elements $e_{\beta,g}$, where $\beta \leq \alpha$. For instance,

$$\mathcal{F}_{(2)} = \big\langle (a \circ b) \circ c, (b \circ a) \circ c, (c \circ a) \circ b, a \circ (b \circ c), a \circ (c \circ b), b \circ (c \circ a) \big\rangle,$$
$$\mathcal{F}_{(1^2)} = \big\langle a \circ (b \circ c), a \circ (c \circ b), b \circ (c \circ a) \big\rangle.$$

Hence,

$$F_3 = \mathcal{F}_{(2)} \supseteq \mathcal{F}_{(1^2)}.$$

Definition. Let $\alpha, \beta \vdash n$. We say that β is predecessor of α and write $\alpha \succ \beta$, if $\alpha > \beta$ and there is no $\gamma \vdash n$ such that $\alpha > \gamma > \beta$.

Since the relation $\leq$ is a lexicographic order on a set of partitions of n, it is a linear order. Therefore we are able to sort all elements of $Par(n)$ in decreasing order,

$$Par(n) = \{\alpha_1, \dots, \alpha_{p(n)}\}$$

It is clear that the maximal element of $Par(n)$ will be $\alpha_1 = (n)$ and the minimal element will be $\alpha_{p(n)} = (1^n)$, so

$$(n) = \alpha_1 \succ \alpha_2 \succ \cdots \succ \alpha_{p(n)} = \left(1^n\right).$$

The filtration

$$F_{n+1} = \mathcal{F}_{\alpha_1} \supseteq \mathcal{F}_{\alpha_2} \supseteq \cdots \supseteq \mathcal{F}_{\alpha_{p(n)}}$$

induces a grading on F_{n+1},

$$F_{\alpha_i} = \mathcal{F}_{\alpha_i} / \mathcal{F}_{\alpha_{i+1}}.$$

It is clear that F_α is generated by classes of elements $e_{\alpha,f}$, where f runs Novikov fillings of α. In particular,

$$F_{(1^n)} = \mathcal{F}_{(1^n)}.$$

The aim of the following section is to prove that these filtrations and gradings are compatible with the action of symmetric group.

5. S_{n+1}-submodules of a multilinear part of a free Novikov algebra

We consider F_{n+1} as S_{n+1}-modules with a natural action

$$\sigma X(a_1, \dots, a_{n+1}) = X(a_{\sigma(1)}, \dots, a_{\sigma(n+1)}).$$

In this section we prove that $\mathcal{F}_\alpha$ is an S_{n+1}-submodule of F_{n+1} for any $\alpha \vdash n$. Then, F_α as a factor-module will have S_{n+1}-module structure as well. Therefore, F_{n+1} will be a direct sum of submodules F_α. We will see that F_α is isomorphic to a permutation module $M^{w(\alpha)}$.

Lemma 5.1. *For any* $a_1, \dots, a_n \in \Omega$,

$$deg\big(\big(\cdots\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ (a_k \circ a_{k+1})\big)\cdots\big) \circ a_n\big) \\ < deg\big(\big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_k\big) \circ a_{k+1}\big)\cdots\big) \circ a_n\big).$$

In particular,

$$\big(\cdots\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ (a_k \circ a_{k+1})\big)\cdots\big) \circ a_n \\ \le \big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_k\big) \circ a_{k+1}\big)\cdots\big) \circ a_n.$$

Proof. We use an induction on n. If $n = 3$, then it is trivial. Assume that our statement is true for $n-1$ elements. We prove our lemma for n by induction on $n-k = 1, \dots, n-1$.

Base of induction. Let $n - k = 1$. By left-commutative rule

$$\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{n-2}\big) \circ (a_{n-1} \circ a_n) = a_{n-1} \circ \big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{n-2}\big) \circ a_n\big).$$

On the other hand

$$deg\big(a_{n-1} \circ \big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{n-2}\big) \circ a_n\big)\big) \\ = (n-2, 1) < (n-1) = deg\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_n\big).$$

Therefore,

$$deg\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{n-2}\big) \circ (a_{n-1} \circ a_n)\big) \\ < deg\big(\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{n-2}\big) \circ a_{n-1}\big) \circ a_n\big).$$

So, base of induction is established.

Assume that the proposition is true for $n - k - 1$, i.e.

$$deg\big(\big(\cdots\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ (a_{k+1} \circ a_{k+2})\big)\cdots\big) \circ a_n\big) \\ < deg\big(\big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{k+1}\big) \circ a_{k+2}\big)\cdots\big) \circ a_n\big).$$

We want to show the following inequality

$$deg\big(\big(\cdots\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ (a_k \circ a_{k+1})\big)\cdots\big) \circ a_n\big) \\ < deg\big(\big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_k\big) \circ a_{k+1}\big)\cdots\big) \circ a_n\big).$$

Let

$$X = \big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{k-1}, \qquad Y = a_k \circ a_{k+1}, \qquad Z = a_{k+2}.$$

By right-symmetric rule

$$(X \circ Y) \circ Z = X \circ (Y \circ Z) - X \circ (Z \circ Y) + (X \circ Z) \circ Y,$$

and

$$\begin{aligned}&\bigl(\cdots\bigl(\bigl((X \circ Y) \circ Z\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n \\ &\quad= \bigl(\cdots\bigl(\bigl(X \circ (Y \circ Z)\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n - \bigl(\cdots\bigl(\bigl(X \circ (Z \circ Y)\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n \\ &\qquad+ \bigl(\cdots\bigl(\bigl(\bigl((X \circ Z) \circ Y\bigr)\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n.\end{aligned}$$

So,

$$\bigl(\cdots\bigl(\bigl((X \circ Y) \circ Z\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n = A - B + C, \tag{1}$$

where

$$\begin{aligned}A &= \bigl(\cdots\bigl(\bigl(X \circ (Y \circ Z)\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n, \\ B &= \bigl(\cdots\bigl(\bigl(X \circ (Z \circ Y)\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n\end{aligned}$$

and

$$C = \bigl(\cdots\bigl(\bigl(\bigl((X \circ Z) \circ Y\bigr)\bigr) \circ a_{k+3}\bigr)\cdots\bigr) \circ a_n.$$

Let us prove that

$$deg(C) < deg\bigl(\bigl(\cdots\bigl(\bigl(\bigl(\cdots(a_1 \circ a_2)\cdots\bigr) \circ a_k\bigr) \circ a_{k+1}\bigr)\cdots\bigr) \circ a_n\bigr) \tag{2}$$

By inductive suggestion on $n - k$

$$\begin{aligned}&deg\bigl(\bigl(\cdots\bigl(\bigl(\bigl(\cdots(a_1 \circ a_2)\cdots\bigr) \circ a_{k+1}\bigr) \circ a_{k+2}\bigr)\cdots\bigr) \circ a_n\bigr) \\ &\quad> deg\bigl(\bigl(\cdots\bigl(\bigl(\bigl(\bigl(\cdots(a_1 \circ a_2)\cdots\bigr) \circ a_{k-1}\bigr) \circ a_k\bigr) \circ (a_{k+1} \circ a_{k+2})\bigr)\cdots\bigr) \circ a_n\bigr).\end{aligned}$$

Therefore,

$$\begin{aligned}&deg\bigl(\bigl(\cdots\bigl(\bigl(\bigl(\bigl(\bigl(\cdots(a_1 \circ a_2)\cdots\bigr) \circ a_{k-1}\bigr) \circ a_{k+2}\bigr) \circ a_k\bigr) \circ a_{k+1}\bigr)\cdots\bigr) \circ a_n\bigr) \\ &\quad> deg\bigl(\bigl(\cdots\bigl(\bigl(\bigl(\bigl(\cdots(a_1 \circ a_2)\cdots\bigr) \circ a_{k-1}\bigr) \circ a_{k+2}\bigr) \circ (a_k \circ a_{k+1})\bigr)\cdots\bigr) \circ a_n\bigr).\end{aligned}$$

Since

$$deg\bigl(\bigl(\cdots\bigl(\bigl(\bigl(\cdots(a_1 \circ a_2)\cdots\bigr) \circ a_{k+1}\bigr) \circ a_{k+2}\bigr)\cdots\bigr) \circ a_n\bigr) = (n-1)$$

is the highest degree,

$$deg\big(\big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{k+1}\big) \circ a_{k+2}\big)\cdots\big) \circ a_n\big)$$
$$\geq deg\big(\big(\cdots\big(\big(\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{k-1}\big) \circ a_{k+2}\big) \circ a_k\big) \circ a_{k+1}\big)\cdots\big) \circ a_n\big)$$

Thus

$$deg\big(\big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{k+1}\big) \circ a_{k+2}\big)\cdots\big) \circ a_n\big)$$
$$> deg\big(\big(\cdots\big(\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_{k-1}\big) \circ a_{k+2}\big) \circ (a_k \circ a_{k+1})\big)\cdots\big) \circ a_n\big) = deg(C).$$

So, (2) is proved.

Let us prove that

$$deg(B) < deg(C), \tag{3}$$
$$deg(A) < (n-1). \tag{4}$$

We consider $a_k \circ a_{k+1}$ as one element $Y = a_k \circ a_{k+1}$. Then we can consider A and B as products of $n-1$ elements and we can use inductive suggestion on n. So,

$$deg(B) = deg\big(\big(\cdots\big(\big(X \circ (Z \circ Y)\big) \circ a_{k+3}\big)\cdots\big) \circ a_n\big)$$
$$< deg\big(\big(\cdots\big(\big(\big((X \circ Z) \circ Y\big)\big) \circ a_{k+3}\big)\cdots\big) \circ a_n\big) = deg(C)$$

This is relation (3).

$$deg(A) = deg\big(\big(\cdots\big(\big(X \circ (Y \circ Z)\big) \circ a_{k+3}\big)\cdots\big) \circ a_n\big)$$
$$< deg\big(\big(\cdots\big(\big((X \circ Y) \circ Z\big) \circ a_{k+3}\big)\cdots\big) \circ a_n\big) \leq (n-1).$$

This is relation (4). By (2), (3) and (4)

$$deg\, C \leq (n-2,1), \qquad deg\, B < (n-2,1), \qquad deg\, A \leq (n-2,1).$$

By Proposition 4.1,

$$deg(A - B + C) \leq (n-2,1) < (n-1)$$
$$= deg\big(\big(\cdots\big(\big(\big(\cdots(a_1 \circ a_2)\cdots\big) \circ a_k\big) \circ a_{k+1}\big)\cdots\big) \circ a_n\big).$$

Our lemma is proved completely. □

Lemma 5.2. *For elements $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_n$, of the following forms*

$$A'_i = \big(\cdots\big(\big(\cdots(a_{i,1} \circ a_{i,2})\cdots\big) \circ (a_{i,k} \circ a_{i,k+1})\big)\cdots\big) \circ a_{i,n_i}$$

and

$$A_i = \big(\cdots\big(\big(\big(\cdots(a_{i,1} \circ a_{i,2})\cdots\big) \circ a_{i,k}\big) \circ a_{i,k+1}\big)\cdots\big) \circ a_{i,n_i},$$

where $a_{i,j} \in \Omega$, the following inequality holds

$$deg\big(A_n \circ \big(\cdots \big(A'_i \circ \big(\cdots (A_2 \circ A_1) \cdots\big)\big) \cdots\big)\big)$$
$$< deg\big(A_n \circ \big(\cdots \big(A_i \circ \big(\cdots (A_2 \circ A_1) \cdots\big)\big) \cdots\big)\big).$$

Proof. By Lemma 5.1 $deg(A'_i \circ a_{1,n_1}) < deg(A_i \circ a_{1,n_1}) = (n_i)$. Then

$$deg\big(A'_i \circ a_{1,n_1}\big) = \alpha = (\alpha_1, \ldots, \alpha_m) \vdash n_i,$$

such that $\alpha_1 < n_i$.

Suppose that

$$A'_i \circ a_{1,n_1} = \sum_f \lambda_f e_{\alpha,f} + \ldots + \sum_g \lambda_g e_{\beta,g} \tag{5}$$

where $\alpha > \cdots > \beta$, and f, g run through Novikov fillings. Let

$$e_{\alpha,f} = C_{\alpha_m} \circ \big(\cdots (C_{\alpha_2} \circ C_{\alpha_1}) \cdots\big) \tag{6}$$

where $C_{\alpha_j} = (\cdots (c_{j,1} \circ c_{j,2}) \cdots) \circ c_{j,\alpha_j}$ and $\{c_{1,1}, \ldots, c_{m,\alpha_m}\} = \{a_{i,1}, \ldots, a_{i,n_i}, a_{1,1}\}$. Let $B_1 = (\cdots (a_{1,1} \circ a_{1,2}) \cdots) \circ a_{1,n_1-1}$, then $A_1 = B_1 \circ a_{1,n_1}$. So,

$$deg\big(A_n \circ \big(\cdots \big(A_2 \circ (B_1 \circ e_{\alpha,f})\big) \cdots\big)\big)$$

(by (6))

$$= deg\big(A_n \circ \big(\cdots \big(A_2 \circ \big(B_1 \circ \big(C_{\alpha_m} \circ \big(\cdots (C_{\alpha_2} \circ C_{\alpha_1}) \cdots\big)\big)\big)\big) \cdots\big)\big)$$

(by definition of degree)

$$< deg\big(A_n \circ \big(\cdots \big(A_2 \circ \big(B_1 \circ (A_i \circ a_{1,n_1})\big)\big) \cdots\big)\big)$$

(by left-commutativity rule)

$$= deg\big(A_n \circ \big(\cdots \big(A_i \circ \big(\cdots (A_2 \circ A_1) \cdots\big)\big) \cdots\big)\big). \tag{7}$$

Hence,

$$deg\big(A_n \circ \big(\cdots \big(A'_i \circ \big(\cdots \big(A_2 \circ (B_1 \circ a_{1,n_1})\big) \cdots\big)\big) \cdots\big)\big)$$

(by left-commutativity rule)

$$= deg\big(A_n \circ \big(\cdots \big(A_2 \circ \big(B_1 \circ \big(A'_i \circ a_{1,n_1}\big)\big)\big) \cdots\big)\big)$$

(by (5))

$$= deg\Big(\sum_f \lambda_f A_n \circ \big(\cdots \big(A_2 \circ (B_1 \circ e_{\alpha,f})\big) \cdots\big) + \dots$$
$$+ \sum_g \lambda_g A_n \circ \big(\cdots \big(A_2 \circ (B_1 \circ e_{\beta,g})\big) \cdots\big)\Big)$$

(by (7) and Proposition 4.1)

$$< deg\big(A_n \circ \big(\cdots \big(A_i \circ \big(\cdots (A_2 \circ A_1) \cdots\big)\big) \cdots\big)\big).$$

Lemma 5.2 is proved. □

For $x, y \in F_{n+1}$ we will write $x \equiv y \pmod{\mathcal{F}_\beta}$ if $x - y \in \mathcal{F}_\beta$.

Lemma 5.3. *Suppose that a permutation $\sigma \in S_{n+1}$ acts on $e_{\alpha,f}$ such that each element in $e_{\alpha,f}$ remains in its own block. Then*

$$\sigma e_{\alpha,f} \equiv e_{\alpha,f} \pmod{\mathcal{F}_\beta},$$

otherwise,

$$\sigma e_{\alpha,f} \equiv e_{\alpha,g} \pmod{\mathcal{F}_\beta},$$

where $\alpha \succ \beta$ and $f \neq g$.

Corollary 5.4. *$\mathcal{F}_\alpha$ and F_α are S_{n+1}-modules.*

Proof of Lemma 5.3. Let $\alpha = (n_1, \dots, n_l) \vdash n$ and $Y_{\tilde{\alpha},f}$ be a Novikov tableau

$$\begin{matrix} f_{1,1} & f_{1,2} & \cdots & \cdots & \cdots & f_{1,n_1} & f_{1,n_1+1} \\ f_{2,1} & f_{2,2} & \cdots & \cdots & f_{2,n_2} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ f_{l,1} & f_{l,2} & \cdots & f_{l,n_l} & & & \end{matrix}$$

where $f_{i,j} \in \Omega$, and $e_{\alpha,f}$ is the corresponding base element. Let $B_{0,f}, B_{1,f}, B_{2,f}, \dots$ be block sequences of $e_{\alpha,f}$.

Since any permutation is a composition of transpositions, it is enough to consider the case when σ is a transposition. Suppose that $\sigma = (i, j)$ is transposition that moves i to j and j to i. There might be two cases: a_i and a_j are in one block or in different blocks of $e_{\alpha,f}$.

Let us consider the case when a_i, a_j are in one block, say, $B_{q,f}$. This case we divide into two subcases: $q = 0$ and $q > 0$.

Case $q = 0$. Then a_s and a_t might be in one row or in two different rows of B_0. If they are in one row, in order to obtain $e_{\alpha,f}$ from $\sigma e_{\alpha,f}$, we use right-symmetric rule for $\sigma e_{\alpha,f}$ and by Lemma 5.2 we obtain

$$\sigma e_{\alpha,f} \equiv e_{\alpha,f} \pmod{\mathcal{F}_\beta}.$$

If they are in two different rows, then by left-commutativity rule we can assume that these rows are top first and second rows of $e_{\alpha,f}$. Therefore, in this case we can assume that the base element $e_{\alpha,f}$ has a form $A_2 \circ A_1$, with A_1 and A_2 as in Lemma 5.2.

Let us adopt the following notation

$$\tilde{f}_{i,k} = \big(\cdots(f_{i,1} \circ f_{i,2})\cdots\big) \circ f_{i,k}.$$

Suppose that σ permutes the elements $f_{1,s} = a_i$ and $f_{2,t} = a_j$. Then

$$\begin{aligned}
\sigma e_{\alpha,f} &= \sigma(A_2 \circ A_1) \\
&= \sigma\Big(\Big(\big(\cdots\big((\tilde{f}_{2,t-1} \circ a_j) \circ f_{2,t+1}\big)\cdots\big) \circ f_{2,n_2}\Big) \\
&\quad \circ \Big(\big(\cdots\big((\tilde{f}_{1,s-1} \circ a_i) \circ f_{1,s+1}\big)\cdots\big) \circ f_{1,n_1}\Big)\Big) \\
&= \Big(\big(\cdots\big((\tilde{f}_{2,t-1} \circ a_i) \circ f_{2,t+1}\big)\cdots\big) \circ f_{2,n_2}\Big) \\
&\quad \circ \Big(\big(\cdots\big((\tilde{f}_{1,s-1} \circ a_j) \circ f_{1,s+1}\big)\cdots\big) \circ f_{1,n_1}\Big)
\end{aligned}$$

(by right-symmetric rule applied for the first and the second row elements and by Lemma 5.2)

$$\begin{aligned}
&\equiv \Big(\Big(\big(\cdots(\tilde{f}_{2,t-1} \circ f_{2,t+1})\cdots\big) \circ f_{2,n_2}\Big) \circ a_i\Big) \\
&\quad \circ \Big(\Big(\big(\cdots(\tilde{f}_{1,s-1} \circ f_{1,s+1})\cdots\big) \circ f_{1,n_1}\Big) \circ a_j\Big) \pmod{\mathcal{F}_\beta}
\end{aligned}$$

(by left-commutative rule)

$$\begin{aligned}
&= \Big(\big(\cdots(\tilde{f}_{1,s-1} \circ f_{1,s+1})\cdots\big) \circ f_{1,n_1}\Big) \\
&\quad \circ \Big(\Big(\Big(\big(\cdots(\tilde{f}_{2,t-1} \circ f_{2,t+1})\cdots\big) \circ f_{2,n_2}\Big) \circ a_i\Big) \circ a_j\Big) \pmod{\mathcal{F}_\beta}
\end{aligned}$$

(by right-symmetric rule and by Lemma 5.2)

$$\begin{aligned}
&\equiv \Big(\big(\cdots(\tilde{f}_{1,s-1} \circ f_{1,s+1})\cdots\big) \circ f_{1,n_1}\Big) \\
&\quad \circ \Big(\Big(\Big(\big(\cdots(\tilde{f}_{2,t-1} \circ f_{2,t+1})\cdots\big) \circ f_{2,n_2}\Big) \circ a_j\Big) \circ a_i\Big) \pmod{\mathcal{F}_\beta}
\end{aligned}$$

(by left-commutative rule)

$$\begin{aligned}
&= \Big(\Big(\big(\cdots(\tilde{f}_{2,t-1} \circ f_{2,t+1})\cdots\big) \circ f_{2,n_2}\Big) \circ a_j\Big) \\
&\quad \circ \Big(\Big(\big(\cdots(\tilde{f}_{1,s-1} \circ a_{1,s+1})\cdots\big) \circ f_{1,n_1}\Big) \circ a_i\Big) \pmod{\mathcal{F}_\beta}
\end{aligned}$$

(by right-symmetric rule applied for the first and the second row elements and by Lemma 5.2)

$$\begin{aligned}&\equiv \big(\big(\cdots\big((\tilde{f}_{2,t-1}\circ a_j)\circ f_{2,t+1}\big)\cdots\big)\circ f_{2,n_2}\big)\\&\quad\circ\big(\big(\cdots\big((\tilde{f}_{1,s-1}\circ a_i)\circ f_{1,s+1}\big)\cdots\big)\circ f_{1,n_1}\big) \pmod{\mathcal{F}_\beta}\\&= e_{\alpha,f} \pmod{\mathcal{F}_\beta}.\end{aligned}$$

So, permutation of any two elements of $e_{\alpha,f}$ in $B_{0,f}$ again gives $e_{\alpha,f}$ modulo $\mathcal{F}_\beta$.

Case $q>0$. Let $a_i, a_j \in B_{q,f}$ for $q>0$. Recall that σ permutes $f_{s,1}=a_i$ and $f_{t,1}=a_j$. Let us show calculations in terms of Novikov tableau. We have

$$\sigma Y_{\tilde{\alpha},f} = \sigma \begin{array}{ccccccc} f_{1,1} & f_{1,2} & \cdots & \cdots & \cdots & f_{1,n_1} & f_{1,n_1+1}\\ \vdots & \vdots & \cdots & \vdots & & & \\ a_i & f_{s,2} & \cdots & \cdots & f_{s,n_s} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ a_j & f_{t,2} & \cdots & \cdots & f_{t,n_t} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ f_{l,1} & f_{l,2} & \cdots & f_{l,n_l} & & & \end{array}$$

(since $q>0$, here in fact $n_s=n_t$)

$$= \begin{array}{ccccccc} f_{1,1} & f_{1,2} & \cdots & \cdots & \cdots & f_{1,n_1} & f_{1,n_1+1}\\ \vdots & \vdots & \cdots & \vdots & & & \\ a_j & f_{s,2} & \cdots & \cdots & f_{s,n_s} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ a_i & f_{t,2} & \cdots & \cdots & f_{t,n_t} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ f_{l,1} & f_{l,2} & \cdots & f_{l,n_l} & & & \end{array}$$

(by left-commutative rule)

$$= \begin{array}{ccccccc} f_{1,1} & f_{1,2} & \cdots & \cdots & \cdots & f_{1,n_1} & f_{1,n_1+1}\\ \vdots & \vdots & \cdots & \vdots & & & \\ a_i & f_{t,2} & \cdots & \cdots & f_{t,n_t} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ a_j & f_{s,2} & \cdots & \cdots & f_{s,n_s} & & \\ \vdots & \vdots & \cdots & \vdots & & & \\ f_{l,1} & f_{l,2} & \cdots & f_{l,n_l} & & & \end{array}$$

(by case $q = 0$)

$$
\equiv \begin{array}{cccccccc}
f_{1,1} & f_{1,2} & \cdots & \cdots & \cdots & f_{1,n_1} & f_{1,n_1+1} \\
\vdots & \vdots & \cdots & \vdots & & & \\
a_i & f_{s,2} & \cdots & \cdots & f_{s,n_s} & & \\
\vdots & \vdots & \cdots & \vdots & & & \\
a_j & f_{t,2} & \cdots & \cdots & f_{t,n_t} & & \\
\vdots & \vdots & \cdots & \vdots & & & \\
f_{l,1} & f_{l,2} & \cdots & f_{l,n_l} & & &
\end{array} \pmod{\mathcal{F}_\beta}
$$
$$
\equiv Y_{\widetilde{\alpha},f} \pmod{\mathcal{F}_\beta}.
$$

So, we have proved the first part of our lemma.

Now, suppose that $a_i \in B_{p,f}$ and $a_j \in B_{q,f}$ for $p \neq q$. Then p-th and q-th block sequences of $\sigma e_{\alpha,f}$ become $B_{p,g}$ and $B_{q,g}$ for some filling g such that $g \neq f$. The filling g might not be Novikov. If g is not Novikov, then take a permutation τ that sorts block sequences $B_{p,g}$ and $B_{q,g}$ in increasing order, so

$$
\sigma e_{\alpha,f} = \tau^{-1} e_{\alpha,g} \equiv e_{\alpha,g} \pmod{\mathcal{F}_\beta}
$$

where τ^{-1} acts on $e_{\alpha,g}$ such that all elements remain in their own blocks and $f \neq g$. By the first part of our lemma

$$
\sigma e_{\alpha,f} \equiv e_{\alpha,g} \pmod{\mathcal{F}_\beta}.
$$

Our lemma is proved completely. □

6. Permutation submodules of Novikov algebras

We recall briefly some facts about permutation module (details see in [8,10]). Suppose that $\lambda \vdash n$. A Young tableau t of shape λ, is a Young diagram of λ with boxes filled by numbers $1, 2, \ldots, n$, such that each number occurs exactly once. In this case, we say that t is a λ-tableau. For example, the list of tableaux of shape $(2, 1)$ is

$$
\begin{array}{ll} 1 & 2, \\ 3 & \end{array} \qquad
\begin{array}{ll} 2 & 1, \\ 3 & \end{array} \qquad
\begin{array}{ll} 1 & 3, \\ 2 & \end{array} \qquad
\begin{array}{ll} 3 & 1, \\ 2 & \end{array} \qquad
\begin{array}{ll} 2 & 3, \\ 1 & \end{array} \qquad
\begin{array}{ll} 3 & 2. \\ 1 & \end{array}
$$

Two λ-tableaux t_1 and t_2 are row equivalent, denoted $t_1 \sim t_2$, if the corresponding rows of the two tableaux contain the same elements. The λ-tabloid is an equivalence class,

$$
\{t\} = \{t_1 \mid t_1 \sim t\}.
$$

For example,

$$\begin{matrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & \\ 8 & & & \end{matrix} \sim \begin{matrix} 1 & 5 & 2 & 3 \\ 7 & 4 & 6 & \\ 8 & & & \end{matrix} \nsim \begin{matrix} 7 & 5 & 2 & 1 \\ 3 & 4 & 6 & \\ 8 & & & \end{matrix}$$

Recall that $\sigma \in S_n$ acts on a tableau $t = (t_{i,j})$ of a shape $\lambda \vdash n$ as follows:

$$\sigma t = \{\sigma(t_{i,j})\}.$$

This action of S_n on tableaux induces its action on tabloids.

Definition. Suppose $\lambda \vdash n$. Let $M^\lambda = \mathbf{C}\{\{t_1\}, \{t_2\}, \ldots, \{t_k\}\}$ where $\{t_1\}, \{t_2\}, \ldots, \{t_k\}$ is a complete list of λ-tabloids. Then M^λ is called the permutation module of shape λ.

Recall that for any partition $\alpha \vdash n$ its weight $w(\alpha)$ gives us a partition of $n+1$. Therefore, we are able to consider $M^{w(\alpha)}$ as an S_{n+1}-module.

Lemma 6.1. *Let $\alpha \vdash n$. The following isomorphism of S_{n+1}-modules takes place*

$$F_\alpha \cong M^{w(\alpha)}.$$

Proof. Let $e_{\alpha,f} \in F_\alpha$. Then for any $\sigma \in S_{n+1}$

$$\sigma e_{\alpha,f} \equiv e_{\alpha,g} \pmod{\mathcal{F}_\alpha}$$

where f, g are some Novikov fillings. By Lemma 5.3, $f = g$ if σ keeps the elements of $e_{\alpha,f}$ in their blocks, otherwise we obtain $g \neq f$. Then we can consider each block of $e_{\alpha,f}$ as a row of Young tabloid. Suppose the block sequences of $e_{\alpha,f}$ are $B_{0,f}, B_{1,f}, \ldots, B_{s,f}$. In order to get one-to-one correspondence between $e_{\alpha,f}$ and Young tabloids of shape $w(\alpha)$, we put all block sequences in one column and sort them first by their lengths, and then sort them by indices of blocks if two blocks have the same lengths.

$$sort \begin{cases} B_{0,f} \\ B_{1,f} \\ \vdots \\ B_{s,f} \end{cases}$$

This construction shows that F_α is isomorphic to permutation module $M^{w(\alpha)}$. $\square$

Example. Let us consider Novikov tableau of the following form

$$Y_{(5,\widetilde{4^2,2},1^3),f} = \begin{matrix} a_1 & a_{10} & a_{12} & a_{13} & a_{14} & a_{16} \\ a_{11} & a_6 & a_8 & a_9 & & \\ a_7 & a_3 & a_4 & a_5 & & \\ a_{15} & a_2 & & & & \\ a_{17} & & & & & \\ a_{18} & & & & & \\ a_{19} & & & & & \end{matrix}$$

Its block sequences are $B_{0,f} = a_2a_3a_4a_5a_6a_8a_9a_{10}a_{12}a_{13}a_{14}a_{16}$, $B_{1,f} = a_1$, $B_{2,f} = a_7a_{11}$, $B_{3,f} = a_{15}$, $B_{4,f} = a_{17}a_{18}a_{19}$.

To $Y_{(5,\widetilde{4^2,2},1^3),f}$ corresponds the following Young tableau

$$sort \begin{cases} B_{0,f} & a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_8 \quad a_9 \quad a_{10} \quad a_{12} \quad a_{13} \quad a_{14} \quad a_{16} \\ B_{1,f} & a_{17} \quad a_{18} \quad a_{19} \\ B_{2,f} = & a_7 \quad a_{11} \\ B_{3,f} & a_1 \\ B_{4,f} & a_{15} \end{cases}$$

We have got the Young tabloid of shape $(12, 3, 2, 1^2)$.

7. Proof of Theorem 2.1 parts a and b

Let $\alpha \vdash n$. Recall that induced module $Ind_{S_\alpha}^{S_n}(\mathbf{1})$ is isomorphic to the permutation module M^α (details see in [7], [8] or [10]). Then by Lemma 6.1

$$F_\alpha \cong Ind_{S_{w(\alpha)}}^{S_n}(\mathbf{1}).$$

Since as S_{n+1}-modules,

$$F_{n+1} \cong \bigoplus_{\alpha \vdash n} F_\alpha$$

we obtain the following isomorphism of S_{n+1}-modules

$$F_{n+1} \cong \bigoplus_{\alpha \vdash n} Ind_{S_{w(\alpha)}}^{S_{n+1}}(\mathbf{1}).$$

By Young's rule,

$$Ind_{S_{w(\alpha)}}^{S_n}(\mathbf{1}) = \bigoplus_{\beta \trianglerighteq w(\alpha)} K_{\beta w(\alpha)} S^\beta.$$

Hence,

$$F_{n+1} \cong \bigoplus_{\alpha\vdash n}\bigoplus_{\beta\trianglerighteq w(\alpha)} K_{\beta w(\alpha)} S^{\beta},$$

and

$$F_{n+1} \cong \bigoplus_{\beta\vdash n+1}\left(\sum_{\alpha\vdash n} K_{\beta w(\alpha)}\right) S^{\beta}.$$

8. Proof of Theorem 2.1 part c

By Theorem 2.1 part b, $\beta = (\beta_1, \dots, \beta_k) \vdash n+1$ is Novikov admissible if and only if $\sum_{\alpha\vdash n} K_{\beta w(\alpha)} \neq 0$. Note that $\sum_{\alpha\vdash n} K_{\beta w(\alpha)} \neq 0$ if and only if there is an $\alpha \vdash n$ so that $\beta \trianglerighteq w(\alpha)$. In this section we prove that for $\beta = (\beta_1, \dots, \beta_k) \vdash n+1$ there exists $\alpha \vdash n$ so that $\beta \trianglerighteq w(\alpha)$ if and only if

$$\beta_1 - 1 \geq \beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k.$$

To prove this statement we need some preliminary facts.

Lemma 8.1. *Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 1$. Then*

$$\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n \leq \lambda_{\sigma(1)} + 2\lambda_{\sigma(2)} + \ldots + n\lambda_{\sigma(n)}$$

for any $\sigma \in S_n$.

Proof. It is sufficient to prove this statement for a transposition $\sigma = (i, j) \in S_n$, where $1 \leq i < j \leq n$. We have

$$\begin{aligned}
&\lambda_1 + \ldots + i\lambda_{\sigma(i)} + \ldots + j\lambda_{\sigma(j)} + \ldots + n\lambda_n \\
&\quad = \lambda_1 + \ldots + i\lambda_j + \ldots + j\lambda_i + \ldots + n\lambda_n \\
&\quad = \lambda_1 + \ldots + j\lambda_j - (j-i)\lambda_j + \ldots + i\lambda_i + (j-i)\lambda_i + \ldots + n\lambda_n \\
&\quad = \lambda_1 + \ldots + i\lambda_i + \ldots + j\lambda_j + \ldots + n\lambda_n + (j-i)(\lambda_i - \lambda_j) \\
&\quad \geq \lambda_1 + \ldots + i\lambda_i + \ldots + j\lambda_j + \ldots + n\lambda_n. \qquad \square
\end{aligned}$$

Lemma 8.2. *Let $\beta = (\beta_1, \dots, \beta_k) \vdash n+1$. If $\beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k > \beta_1 - 1$ then $\beta \notin Im\, w$.*

Proof. Let $\beta = (\beta_1, \dots, \beta_k) \vdash n+1$ and assume that $\beta \in Im\, w$. Then, there exists a partition $\alpha = (n^{i_n}, \dots, 2^{i_2}, 1^{i_1}) \vdash n$ such that

$$w(\alpha) = sort\left(n+1 - \sum_{j=1}^{n} i_j, i_1, i_2, \dots, i_n\right) = (\beta_1, \dots, \beta_k)$$

and

$$\beta_1 = max\left\{i_1, n+1-\sum_{j=1}^{n} i_j\right\}.$$

Set $i_2 = \beta_3, \ldots, i_{k-1} = \beta_k, i_k = 0, \ldots, i_n = 0$, then

$$n+1-\sum_{j=1}^{n} i_j = 1+\beta_3+2\beta_4+\ldots+(k-2)\beta_k.$$

By Lemma 8.1 we obtain the minimal value of $n+1-\sum_{j=1}^{n} i_j$. So, we have a contradiction to the assumption of our lemma

$$n+1-\sum_{j=1}^{n} i_j > \beta_1. \qquad \square$$

Lemma 8.3. *Let* $\beta = (\beta_1, \ldots, \beta_k) \vdash n+1$. *There is a* $\gamma \in Im\, w$ *such that* $\beta \trianglerighteq \gamma$ *if and only if*

$$\beta_1 - 1 \geq \beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k.$$

Proof. Suppose that $\gamma = (\gamma_1, \ldots, \gamma_l) \in Im\, w$. Then by Lemma 8.2

$$\gamma_1 - 1 \geq \gamma_3 + 2\gamma_4 + \ldots + (l-2)\gamma_l.$$

We show that if $\beta \trianglerighteq \gamma$, then

$$\beta_1 - 1 \geq \beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k.$$

Since $k \leq l$,

$$\begin{aligned}
\beta_1 - 1 \geq \gamma_1 - 1 &\geq \gamma_3 + 2\gamma_4 + \ldots + (l-2)\gamma_l \\
&= (\gamma_3 + \ldots + \gamma_l) + (\gamma_4 + \ldots + \gamma_l) + \ldots + \gamma_l \\
&= (n+1-\gamma_1-\gamma_2) + (n+1-\gamma_1-\gamma_2-\gamma_3) + \ldots \\
&\quad + (n+1-\gamma_1-\gamma_2-\ldots-\gamma_{l-1}) \\
&\geq (k-2)(n+1) - (\beta_1+\beta_2) - (\beta_1+\beta_2+\beta_3) - \ldots - (\beta_1+\beta_2+\ldots+\beta_{k-1}) \\
&= (k-2)(\beta_1+\ldots+\beta_k) - (\beta_1+\beta_2) - \ldots - (\beta_1+\beta_2+\ldots+\beta_{k-1}) \\
&= \beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k.
\end{aligned}$$

Conversely, suppose that $\beta = (\beta_1, \ldots, \beta_k) \vdash n+1$ satisfies

$$\beta_1 - 1 \geq \beta_3 + 2\beta_4 + \ldots + (k-2)\beta_k.$$

Let us denote

$$m = \beta_1 - 1 - \beta_3 - 2\beta_4 - \ldots - (k-2)\beta_k.$$

Suppose that $m = 0$. Then set

$$i_1 = \beta_2, \qquad i_2 = \beta_3, \quad \ldots, \quad i_{k-1} = \beta_k, \qquad i_k = 0, \quad \ldots, \quad i_n = 0.$$

So, we obtain

$$\alpha = \big((k-1)^{\beta_k}, \ldots, 2^{\beta_3}, 1^{\beta_2}\big) \vdash n,$$

and its weight is

$$w(\alpha) = \beta.$$

Since $\beta \trianglerighteq \beta$,

$$\gamma = \beta.$$

Now suppose that $m \geq 1$. Then for any j, set $i_j = 0$, except

$$i_1 = \beta_2, \quad \ldots, \quad i_{k-2} = \beta_{k-1}, \qquad i_{k-1} = \beta_k - 1, \qquad i_{k+m-1} = 1.$$

So, we obtain

$$\alpha = \big(k+m-1, (k-1)^{\beta_k - 1}, \ldots, 2^{\beta_3}, 1^{\beta_2}\big) \vdash n,$$

and its weight is

$$w(\alpha) = (\beta_1, \ldots, \beta_{k-1}, \beta_k - 1, 1).$$

Since $\beta \trianglerighteq (\beta_1, \ldots, \beta_{k-1}, \beta_k - 1, 1)$,

$$\gamma = (\beta_1, \ldots, \beta_{k-1}, \beta_k - 1, 1). \qquad \square$$

9. Proof of Corollary 2.2

In order to get $GL(V)$-module for N_{n+1}, we write

$$N_{n+1} \cong V^{\otimes(n+1)} \otimes_{K[S_{n+1}]} F_{n+1}.$$

Recall that for permutation S_{n+1}-module $Ind_{S_\mu}^{S_{n+1}}(\mathbf{1})$ as $GL(V)$-module (formula (6.25) in [7]) we have

$$V^{\otimes(n+1)} \otimes_{K[S_{n+1}]} Ind_{S_\mu}^{S_{n+1}}(\mathbf{1}) \cong \bigotimes_{\mu_i \in \mu} Sym^{\mu_i} V$$

where $\mu = (\mu_1, \dots, \mu_k) \vdash n+1$ and $Sym^{\mu_i} V$ is the symmetric power of V. By part a of Theorem 2.1, we obtain

$$N_{n+1} \cong \bigoplus_{\alpha \vdash n} \bigotimes_{w_i \in w(\alpha)} Sym^{w_i} V$$

as $GL(V)$-module and by part b of Theorem 2.1, we obtain

$$N_{n+1} \cong V^{\otimes(n+1)} \otimes_{K[S_{n+1}]} F_{n+1} \cong \bigoplus_{\beta \vdash n+1} \left(\sum_{\alpha \vdash n} K_{\beta w(\alpha)} \right) \left(V^{\otimes(n+1)} \otimes_{K[S_{n+1}]} S^\beta \right)$$
$$= \bigoplus_{\beta \vdash n+1} \left(\sum_{\alpha \vdash n} K_{\beta w(\alpha)} \right) S_\beta(V).$$

10. Proof of Corollary 2.4

a. Since $K_{(n+1)\beta} = 1$ for any $\beta \vdash n+1$,

$$\sum_{\alpha \vdash n} K_{(n+1)w(\alpha)} = p(n).$$

b. Let $\beta = (n+1-l, 1^l) \vdash n+1$. If $w(\alpha)$ has $k+1$ parts, then

$$K_{\beta w(\alpha)} = \binom{k}{l}.$$

Note that by definition of weight we can say that $w(\alpha)$ has $k+1$ parts if and only if α has k different parts. Therefore,

$$\sum_{\alpha \vdash n} K_{\beta w(\alpha)} = \sum_{k \geq l} p(n,k) \binom{k}{l}$$

where $p(n,k)$ is the number of partitions of n into k different parts [2].

c. Let $\beta = (\beta_1, \beta_2) \vdash n+1$. Then $\beta_i = 0$, if $k > 2$. Note that $\beta_1 - 1 \geq 0$. By Theorem 2.1c β is Novikov admissible.

d. Let $\beta = (l^k)$ and $k > 2$. Then $\beta_1 = l$, $\beta_2 = l$, $\dots$, $\beta_k = l$. Note that $\beta_1 - 1 < \beta_3 + \dots + (k-2)\beta_k$. By Theorem 2.1c β is not Novikov admissible.

11. Proof of Corollary 2.5

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $\mu = (\mu_1, \mu_2, \dots, \mu_m) \vdash n$, the symmetric polynomial $p_\lambda(x_1, x_2, \dots, x_m)$ can be seen as a generating function for character value on a fixed conjugacy class K_λ of M^μ (for details see [10]). Therefore,

$$\chi(\lambda) = \sum_{\alpha \vdash n} C\big(\lambda, w(\alpha)\big).$$

12. S_n-module structures for $n \le 10$

In the following table all Novikov admissible partitions β and multiplicities m_β of corresponding irreducible modules to β in F_n for $1 \le n \le 10$ are given. Recall that F_n is a multilinear part of a free Novikov algebra with n generators.

β	m_β	β	m_β	β	m_β	β	m_β	β	m_β
(1)	1	$(5,1)$	12	$(7,1)$	30	$(6,3)$	31	$(7,2,1)$	52
(2)	1	$(4,2)$	8	$(6,2)$	27	$(6,2,1)$	29	$(7,1^3)$	10
(1^2)	1	$(4,1^2)$	5	$(6,1^2)$	17	$(6,1^3)$	5	$(6,4)$	30
(3)	2	(3^2)	3	$(5,3)$	17	$(5,4)$	15	$(6,3,1)$	34
$(2,1)$	2	$(3,2,1)$	3	$(5,2,1)$	15	$(5,3,1)$	16	$(6,2^2)$	12
(4)	3	(7)	11	$(5,1^3)$	2	$(5,2^2)$	6	$(6,2,1^2)$	8
$(3,1)$	4	$(6,1)$	19	(4^2)	5	$(5,2,1^2)$	3	(5^2)	10
(2^2)	1	$(5,2)$	16	$(4,3,1)$	7	$(4^2,1)$	5	$(5,4,1)$	15
$(2,1^2)$	1	$(5,1^2)$	9	$(4,2^2)$	2	$(4,3,2)$	3	$(5,3,2)$	8
(5)	5	$(4,3)$	8	$(4,2,1^2)$	1	$(4,3,1^2)$	1	$(5,3,1^2)$	3
$(4,1)$	7	$(4,2,1)$	7	$(3^2,2)$	1	(10)	30	$(5,2^2,1)$	1
$(3,2)$	4	$(4,1^3)$	1	(9)	22	$(9,1)$	67	$(4^2,2)$	2
$(3,1^2)$	2	$(3^2,1)$	2	$(8,1)$	45	$(8,2)$	72	$(4^2,1^2)$	1
$(2^2,1)$	1	$(3,2^2)$	1	$(7,2)$	46	$(8,1^2)$	47	$(4,3^2)$	1
(6)	7	(8)	15	$(7,1^2)$	28	$(7,3)$	55		

Example.

$$F_4 = 3S^{(4)} \oplus 4S^{(3,1)} \oplus S^{(2^2)} \oplus S^{(2,1^2)}.$$

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