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SPECIAL IDENTITY FOR NOVIKOV–JORDAN ALGEBRAS#

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A commutative algebra with the identity $(a*b)*(c*d)-(a*d)*(c*b)=(a,b,c)*d-(a,d,c)*b$ is called Novikov–Jordan. Example: $K[x]$ under multiplication $a*b=\partial(ab)$ is Novikov–Jordan. A special identity for Novikov–Jordan algebras of degree 5 is constructed. Free Novikov–Jordan algebras with q generators are exceptional for any $q\geq 1$.

Key Words: Jordan algebras; Left-symmetric algebras; Lie algebras; Novikov algebras; q-Commutators; Special identities.

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1. INTRODUCTION

All algebras and vector spaces are considered over a field K of characteristic different from 2. Let $A=(A,\circ)$ be an algebra with a vector space A and a multiplication $A\times A\to A$, $(a,b)\mapsto a\circ b$. If f be some (non)associative polynomial with q variables, then $f=0$ is called an identity on A, if $f(a_1,a_2,\ldots,a_q)=0$ for any substitution $t_i:=a_i\in A$, $i=1,\ldots,q$. Here expressions of the form a_ia_j one understands as $a_i\circ a_j$.

An algebra A with identities $rsym=0$ and $lcom=0$, where

$$rsym(t_1,t_2,t_3)=t_1(t_2t_3)-(t_1t_2)t_3-t_1(t_3t_2)+(t_1t_3)t_2,$$
$$lcom(t_1,t_2,t_3)=t_1(t_2t_3)-t_2(t_1t_3)$$

is called (left) Novikov (Balinskii and Novikov, 1985; Gelfand and Dorfman, 1979; Osborn, 1994; see also Cayley, 1857; Koszul, 1961; Vinberg, 1963). Any Novikov algebra is Lie-admissible: $A^-=(A,[,])$ is Lie if A is Novikov. Here $[a,b]=a\circ b-b\circ a$.

Example. Let $Os_1=(K[x],\circ)$, where $a\circ b=\partial(a)b$ and $\partial=\partial/\partial x$. Then Os_1 is Novikov. Its Lie algebra is isomorphic to a Witt algebra $W_1=\{e_i:[e_i,e_j]=(j-i)e_{i+j},-1\leq i,j\}$.

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Let A be an algebra with a multiplication $\circ$. Let

$$\{a, b\} = a \circ b + b \circ a$$

be its anticommutator and

$$(a, b, c)^+ = \{a, \{b, c\}\} - \{\{a, b\}, c\}$$

be its associator. Denote by A^+ the algebra with the vector space A and the multiplication $\{,\}$.

Algebras with the identity $tortken = 0$, where

$$\begin{aligned} &tortken(t_1, t_2, t_3, t_4) \\ &\quad = -(t_1(t_2t_3))t_4 + ((t_1t_2)t_3)t_4 + (t_1t_2)(t_3t_4) + (t_1(t_4t_3))t_2 - ((t_1t_4)t_3)t_2 - (t_1t_4)(t_2t_3), \end{aligned}$$

were considered in Dzhumadil'daev (2002). Any Novikov algebra is Tortken-admissible: $A^+ = (A, \{\ ,\ \})$ satisfies the Tortken identity if A is Novikov. In terms of associators $(a, b, c)^+$, a Tortken identity looks like

$$\{\{a, b\}, \{c, d\}\} - \{\{a, d\}, \{c, b\}\} = \{(a, b, c)^+, d\} - \{(a, d, c)^+, b\}.$$

Call commutative Tortken algebras *Novikov–Jordan*.

If A is associative, then A^+ satisfies the Jordan identity

$$(t_1t_1)(t_2t_1) - ((t_1t_1)t_2)t_1 = 0.$$

Recall that a Jordan algebra B is called *special* if there exists some associative algebra A such that B is isomorphic to a subalgebra of A^+. Albert has proved that any Jordan algebra with one generator is power-associative (Albert, 1950). Shirshov has proved that any free Jordan algebra with two generators is special (Shirsov, 1956). In case of three or more generators, free Jordan algebras are no longer special. Glennie has constructed a special identity (s-identity for short) of degree 8 (Glennie, 1966). Any identity of a Jordan algebra of degree no more than seven is special. All of these results are well known (see Jacobson, 1968 or Zhevlakov et al., 1976).

By analogy with the correspondence between associative and Jordan algebras, call Novikov–Jordan algebra B *special*, if there exists some Novikov algebra A such that B is isomorphic to a subalgebra of A^+. In this paper we prove that any free Novikov–Jordan algebra is exceptional and we construct one s-identity of degree 5.

Let $F(q)$ be a free Novikov–Jordan algebra with q generators $t_1, t_2, \ldots, t_q$. Define a polynomial *besken* of degree 5 with two variables by

$$\begin{aligned} besken(t_1, t_2) = {} & (((t_1t_1)t_1)t_2)t_2 + (((t_1t_2)t_2)t_1)t_1 + 2(((t_1t_1)t_2)t_2)t_1 \\ & + 2(((t_1t_2)t_1)t_1)t_2 - 3(((t_1t_1)t_2)t_1)t_2 - 3(((t_1t_2)t_1)t_2)t_1. \end{aligned}$$

The aim of our paper is to prove that $besken = 0$ is an identity for any Novikov–Jordan algebra of the form A^+, where A is Novikov, but *besken* is not an identity on the free Novikov–Jordan algebra $F(q)$ for any $q \geq 1$. More exactly, we establish that it has a consequence of degree 7 that is not an identity on $F(1)$, and therefore $besken = 0$ is not an identity on any free Novikov–Jordan algebra.

Theorem 1.1. *$besken = 0$ is an s-identity for Novikov–Jordan algebras. It is not an identity for any free Novikov–Jordan algebra $F(q)$, $q \geq 1$.*

Remark. If char $K = 2$, then $A^- = A^+$, and a Novikov–Jordan algebra satisfies the Jacobi identity $jac = 0$, where $jac = jac(t_1, t_2, t_3) = (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2$. In this case,

$$jac = 0 \not\Rightarrow tortken = 0,$$
$$jac = 0 \Rightarrow besken = 0.$$

But the identities of $jac = 0$ and $tortken = 0$ have a common consequence:

$$jac = 0 \Rightarrow tortken' = 0,$$
$$tortken = 0 \Rightarrow tortken' = 0,$$

where

$$\begin{aligned} tortken' &= tortken(t_1, t_2) \\ &= (t_1t_1)(t_2t_2) - (t_1t_2)(t_2t_1) - (t_1, t_1, t_2)t_2 + (t_1, t_2, t_2)t_1. \end{aligned}$$

For any Novikov algebra A and its Novikov–Jordan algebra A^+,

$$besken(a, b) = 2(a, (a, a, b)^+, b)^+ + \{(a, \{a, a\}, b)^+, b\} + \{(b, \{b, a\}, a)^+, a\}.$$

Another formulation of the identity $besken(a, b) = 0$:

$$(a, (a, a, b)^+, b)^+ + \{(a, \{a, a\}, b)^+, b\} = -(b, (b, a, a)^+, a)^+ - \{(b, \{b, a\}, a)^+, a\}.$$

Define a multilinear commutative polynomial $besken'$ of degree 5 by

$$\begin{aligned} &besken'(t_1, t_2, t_3, t_4, t_5) \\ &\quad = -t_4t_1t_3t_5t_2 + t_4t_2t_3t_5t_1 + t_5t_1t_2t_3t_4 - t_5t_1t_3t_4t_2 - t_5t_2t_1t_3t_4 \\ &\qquad + t_5t_2t_3t_4t_1 - 2t_3t_1t_4t_2t_5 + 2t_3t_1t_4t_5t_2 + 2t_3t_2t_4t_1t_5 - 2t_3t_2t_4t_5t_1 + 2t_4t_1t_5t_2t_3 \\ &\qquad - 2t_4t_2t_5t_1t_3 + 3t_4t_1t_2t_3t_5 - 3t_4t_2t_1t_3t_5 + 4t_4t_2t_1t_5t_3 - 4t_4t_1t_2t_5t_3. \end{aligned}$$

Here we use left-normed bracketing. For instance, $t_1t_2t_3t_4t_5$ means $(((t_1t_2)t_3)t_4)t_5$. One can establish that $besken' = 0$ is also an s-identity of a Novikov–Jordan algebras and is equivalent (up to the identities $tortken = 0$ and $com = 0$) to the identity $besken = 0$. For example,

$$besken'(t_2, t_1, t_1, t_1, t_2) = besken(t_1, t_2).$$

2. SPECIAL IDENTITY

Lemma 2.1. *For any Novikov algebra* $(A, \circ)$ *and for any* $a, b, c \in A$ *we have the following relations*

$$
\begin{aligned}
&(a, (a, a, b)^+, b)^+\\
&\quad = a \circ (a \circ ((b \circ a) \circ b)) + a \circ (b \circ ((a \circ a) \circ b)) - a \circ (b \circ ((b \circ a) \circ a))\\
&\qquad + a \circ (((a \circ a) \circ b) \circ b) - a \circ (((b \circ a) \circ a) \circ b) - b \circ (b \circ ((a \circ a) \circ a))\\
&\qquad - b \circ (((a \circ a) \circ a) \circ b) + b \circ (((b \circ a) \circ a) \circ a) + (a \circ a) \circ ((a \circ b) \circ b)\\
&\qquad - 2(a \circ a) \circ ((b \circ a) \circ b) + (b \circ a) \circ ((b \circ a) \circ a),
\end{aligned}
$$

$$
\begin{aligned}
&\{(a, \{a, a\}, b)^+, b\}\\
&\quad = 2a \circ (a \circ (a \circ (b \circ b))) - 2a \circ (a \circ (b \circ (b \circ a))) + 6a \circ (a \circ ((a \circ b) \circ b))\\
&\qquad - 4a \circ (a \circ ((b \circ a) \circ b)) - 6a \circ (b \circ ((a \circ a) \circ b)) + 2a \circ (b \circ ((b \circ a) \circ a))\\
&\qquad - 4a \circ (((a \circ a) \circ b) \circ b) + 2a \circ (((b \circ a) \circ a) \circ b) + 2b \circ (b \circ ((a \circ a) \circ a))\\
&\qquad + 2b \circ (((a \circ a) \circ a) \circ b) - 4(a \circ a)((a \circ b) \circ b) + 4(a \circ a) \circ ((b \circ a) \circ b),
\end{aligned}
$$

$$
\begin{aligned}
&\{(b, \{b, a\}, a)^+, a\}\\
&\quad = -2a \circ (a \circ (a \circ (b \circ b))) + 2a \circ (a \circ (b \circ (b \circ a))) - 6a \circ (a \circ ((a \circ b) \circ b))\\
&\qquad + 2a \circ (a \circ ((b \circ a) \circ b)) + 4a \circ (b \circ ((a \circ a) \circ b)) + 2a \circ (((a \circ a) \circ b) \circ b)\\
&\qquad - 2b \circ (((b \circ a) \circ a) \circ a) + 2(a \circ a)((a \circ b) \circ b) - 2(b \circ a) \circ ((b \circ a) \circ a).
\end{aligned}
$$

Proof. We show calculations for $(a, (a, a, b)^+, b)^+$. Other calculations for $\{(a, \{a, a\}, b)^+, b\}$ and $\{(b, \{b, a\}, a)^+, a\}$ are similar.

By Lemma 3.4 (Dzhumadil'daev, 2002)

$$
\begin{aligned}
&(a, (a, a, b)^+, b)^+\\
&\quad = (a, a \circ [b, a] + [b, a] \circ a, b)^+\\
&\quad = (a \circ [b, a]) \circ [b, a] + [b, a] \circ (a \circ [b, a]) + ([b, a] \circ a) \circ [b, a]\\
&\qquad + [b, a] \circ ([b, a] \circ a).
\end{aligned}
$$

We have

$$
\begin{aligned}
[a, b] \circ [a, b] &= (a \circ b - b \circ a) \circ (a \circ b - b \circ a)\\
&= \text{(by left-commutativity identity)}
\end{aligned}
$$

$$a \circ ((a \circ b) \circ b) - a \circ ((b \circ a) \circ b) - b \circ ((a \circ b) \circ a) + b \circ ((b \circ a) \circ a).$$

By right-symmetry and left-commutativity conditions

$$\begin{aligned}(a \circ b) \circ a &= (a \circ a) \circ b + a \circ (b \circ a) - a \circ (a \circ b)\\ &= (a \circ a) \circ b + a \circ (b \circ a) - a \circ (a \circ b)\\ &= (a \circ a) \circ b + b \circ (a \circ a) - a \circ (a \circ b).\end{aligned}$$

Therefore

$$\begin{aligned}[a, b] \circ [a, b] = a \circ ((a \circ b) \circ b) - a \circ ((b \circ a) \circ b) - b \circ ((a \circ a) \circ b)\\ - a \circ (b \circ (b \circ a)) + a \circ (a \circ (b \circ b)) + b \circ ((b \circ a) \circ a).\end{aligned}$$

So, by left-commutativity identity,

$$\begin{aligned}&[a, b] \circ (a \circ [a, b])\\ &\quad = a \circ (a \circ (a \circ (b \circ b))) - a \circ (a \circ (b \circ (b \circ a))) + a \circ (a \circ ((a \circ b) \circ b))\\ &\qquad - a \circ (a \circ ((b \circ a) \circ b)) - a \circ (b \circ ((a \circ a) \circ b)) + a \circ (b \circ ((b \circ a) \circ a)).\end{aligned}$$

Analogous calculations show that

$$\begin{aligned}&(a \circ [a, b]) \circ [a, b]\\ &\quad = 2a \circ (a \circ (a \circ (b \circ b))) - 2a \circ (a \circ (b \circ (b \circ a))) + 2a \circ (a \circ ((a \circ b) \circ b))\\ &\qquad - a \circ (a \circ ((b \circ a) \circ b)) - 3a \circ (b \circ ((a \circ a) \circ b))\\ &\qquad + a \circ (b \circ ((b \circ a) \circ a)) + b \circ (b \circ ((a \circ a) \circ a)),\end{aligned}$$

$$\begin{aligned}&([a, b] \circ a) \circ [a, b]\\ &\quad = -2a \circ (a \circ (a \circ (b \circ b))) + 2a \circ (a \circ (b \circ (b \circ a))) - 2a \circ (a \circ ((a \circ b) \circ b))\\ &\qquad + a \circ (a \circ ((b \circ a) \circ b)) + 4a \circ (b \circ ((a \circ a) \circ b))\\ &\qquad - a \circ (b \circ ((b \circ a) \circ a)) + a \circ (((a \circ a) \circ b) \circ b) - a \circ (((b \circ a) \circ a) \circ b)\\ &\qquad - 2b \circ (b \circ ((a \circ a) \circ a)) - b \circ (((a \circ a) \circ a) \circ b) + b \circ (((b \circ a) \circ a) \circ a),\end{aligned}$$

$$\begin{aligned}&[a, b] \circ ([a, b] \circ a)\\ &\quad = -a \circ (a \circ (a \circ (b \circ b))) + a \circ (a \circ (b \circ (b \circ a))) - a \circ (a \circ ((a \circ b) \circ b))\\ &\qquad + 2a \circ (a \circ ((b \circ a) \circ b)) + a \circ (b \circ ((a \circ a) \circ b)) - 2a \circ (b \circ ((b \circ a) \circ a))\\ &\qquad + (a \circ a) \circ ((a \circ b) \circ b) - 2(a \circ a) \circ ((b \circ a) \circ b) + (b \circ a) \circ ((b \circ a) \circ a).\end{aligned}$$

Therefore

$$\begin{aligned}&(a, (a, a, b)^+, b)^+\\ &\quad = a \circ (a \circ ((b \circ a) \circ b)) + a \circ (b \circ ((a \circ a) \circ b)) - a \circ (b \circ ((b \circ a) \circ a))\\ &\qquad + a \circ (((a \circ a) \circ b) \circ b) - a \circ (((b \circ a) \circ a) \circ b) - b \circ (b \circ ((a \circ a) \circ a))\\ &\qquad - b \circ (((a \circ a) \circ a) \circ b) + b \circ (((b \circ a) \circ a) \circ a) + (a \circ a) \circ ((a \circ b) \circ b)\\ &\qquad - 2(a \circ a) \circ ((b \circ a) \circ b) + (b \circ a) \circ ((b \circ a) \circ a).\end{aligned}$$

Corollary 2.2. *Let A be any Novikov algebra. Then for any $a, b \in A$,*

$$2(a, (a, a, b)^+, b)^+ + \{(a, \{a, a\}, b)^+, b\} + \{(b, \{b, a\}, a)^+, a\} = 0.$$

In particular besken $= 0$ is an identity for any Novikov–Jordan algebra. In other words, it is an identity for any special Novikov–Jordan algebra.

Define the polynomial *jetken* of degree 7 with one variable by the rule

$$\begin{aligned} jetken(t) = &-((((tt)t)(tt))t)t - (((tt)t)(tt))(tt) - 2(((tt)(tt))(tt))t \\ &- 2((((tt)t)t)t)(tt) + 3((((tt)t)t)(tt))t + 3(((tt)(tt))t)(tt). \end{aligned}$$

Corollary 2.3. *jetken $= 0$ is an identity for any special Novikov–Jordan algebra.*

Proof. Notice that $jetken(t) = besken(t, tt)$. □

3. FREE BASIS OF NOVIKOV–JORDAN ALGEBRA OF DEGREE ≤ 7

In this section we construct a basis of a free Novikov–Jordan algebra with one generator of degree no more than 7.

Denote by $\{,\}$ a multiplication in a free commutative algebra. A free commutative algebra with one generator a has a base with $1, 1, 1, 2, 3, 6, 11$ elements correspondingly until degree 7. Namely, one can get the following basic elements

$$a,$$
$$\{a, a\},$$
$$\{\{a, a\}, a\},$$
$$\{\{\{a, a\}, a\}, a\}, \quad \{\{a, a\}, \{a, a\}\},$$
$$\{\{\{\{a, a\}, a\}, a\}, a\}, \quad \{\{\{a, a\}, \{a, a\}\}, a\}, \quad \{\{\{a, a\}, a\}, \{a, a\}\},$$
$$\begin{gathered} \{\{\{\{\{a, a\}, a\}, a\}, a\}, a\}, \quad \{\{\{\{a, a\}, \{a, a\}\}, a\}, a\}, \quad \{\{\{\{a, a\}, a\}, \{a, a\}\}, a\}, \\ \{\{\{\{a, a\}, a\}, a\}, \{a, a\}\}, \quad \{\{\{a, a\}, \{a, a\}\}, \{a, a\}\}, \quad \{\{\{a, a\}, a\}, \{\{a, a\}, a\}\}, \end{gathered}$$

$$\begin{gathered} \{\{\{\{\{\{a, a\}, a\}, a\}, a\}, a\}, a\}, \quad \{\{\{\{\{a, a\}, \{a, a\}\}, a\}, a\}, a\}, \\ \{\{\{\{\{a, a\}, a\}, \{a, a\}\}, a\}, a\}, \quad \{\{\{\{\{a, a\}, a\}, a\}, \{a, a\}\}, a\}, \\ \{\{\{\{a, a\}, \{a, a\}\}, \{a, a\}\}, a\}, \quad \{\{\{\{a, a\}, a\}, \{\{a, a\}, a\}\}, a\}, \\ \{\{\{\{\{a, a\}, a\}, a\}, a\}, \{a, a\}\}, \quad \{\{\{\{a, a\}, \{a, a\}\}, a\}, \{a, a\}\}, \\ \{\{\{\{a, a\}, a\}, \{a, a\}\}, \{a, a\}\}, \quad \{\{\{\{a, a\}, a\}, a\}, \{\{a, a\}, a\}\}, \\ \{\{\{a, a\}, \{a, a\}\}, \{\{a, a\}, a\}\}. \end{gathered}$$

Suppose now that the multiplication $\{,\}$ is not only commutative, but also Tortken. The condition $tortken(t_1, t_2, t_3, t_4) = 0$ is trivial if at least three parameters among t_1, t_2, t_3, t_4 are equal. Therefore for a commutative Tortken algebra with one generator a, the Tortken identity gives nontrivial conditions if its degree is at least 6. We have one condition in degree 6,

$$tortken(a, a, \{a, a\}, \{a, a\}) = 0,$$

and three conditions in degree 7,

$$tortken(a, a, \{a, a\}, \{\{a, a\}, a\}) = 0,$$
$$tortken(a, a, \{\{a, a\}, a\}, \{a, a\}) = 0,$$

and the consequence of the condition in degree 6,

$$\{a, tortken(a, a, \{a, a\}, \{a, a\})\} = 0.$$

For example, $\{\{\{a, a\}, a\}, \{\{a, a\}, a\}\}$ is a linear combination of other elements of degree 6.

We see that a base for a free Novikov–Jordan algebra with one generator a until degree 7 has one, one, one, two, three, five, eight elements, correspondingly. One can choose basic elements of free commutative Tortken algebras just the same as in the commutative case until degree 5 and the five elements in degree 6:

$$\{\{\{\{\{a, a\}, a\}, a\}, a\}, a\}, \quad \{\{\{\{a, a\}, \{a, a\}\}, a\}, a\},$$
$$\{\{\{\{a, a\}, a\}, \{a, a\}\}, a\}, \quad \{\{\{\{a, a\}, a\}, a\}, \{a, a\}\},$$
$$\{\{\{a, a\}, \{a, a\}\}, \{a, a\}\},$$

and the following eight elements in degree 7:

$$X_1 = \{\{\{\{a, a\}, a\}, \{a, a\}\}, \{a, a\}\},$$
$$X_2 = \{\{\{\{a, a\}, \{a, a\}\}, a\}, \{a, a\}\},$$
$$X_3 = \{\{\{\{\{a, a\}, a\}, a\}, a\}, \{a, a\}\},$$
$$X_4 = \{\{\{\{a, a\}, \{a, a\}\}, \{a, a\}\}, a\},$$
$$X_5 = \{\{\{\{\{a, a\}, a\}, a\}, \{a, a\}\}, a\},$$
$$X_6 = \{\{\{\{\{a, a\}, a\}, \{a, a\}\}, a\}, a\},$$
$$X_7 = \{\{\{\{\{a, a\}, \{a, a\}\}, a\}, a\}, a\},$$
$$X_8 = \{\{\{\{\{\{a, a\}, a\}, a\}, a\}, a\}, a\}.$$

Proof of Theorem 1.1. We see that

$$\begin{aligned} &jetken(a) \\ &\quad = -\{\{\{\{\{a, a\}, a\}, \{a, a\}\}, a\}, a\} - \{\{\{\{a, a\}, a\}, \{a, a\}\}, \{a, a\}\} \\ &\qquad - 2\{\{\{\{a, a\}, \{a, a\}\}, \{a, a\}\}, a\} - 2\{\{\{\{\{a, a\}, a\}, a\}, a\}, \{a, a\}\} \\ &\qquad + 3\{\{\{\{\{a, a\}, a\}, a\}, \{a, a\}\}, a\} + 3\{\{\{\{a, a\}, \{a, a\}\}, a\}, \{a, a\}\}. \end{aligned}$$

By Corollary 2.3, $jetken = 0$ is an identity for any special Novikov–Jordan algebra. Thus the condition $jetken(a) = 0$ gives us a linear dependence condition for the basic elements X_i, $i = 1, 2, \ldots, 7$, of free Novikov–Jordan algebras with one generator. So $jetken = 0$ is not an identity for a free Novikov algebra with one generator. Therefore *besken* $= 0$ is not an identity for a free Novikov algebra with q generators for any $q > 0$. □

Remark. One can check that the algebra $(K[x], \square_k)$ constructed in Dzhumadil'daev (2002),

$$a\square_k b = \partial(\partial^{p^k-1}(a)\partial^{p^k-1}(b)),$$

satisfies the identity *besken* $= 0$.

Question 1. There exists an exceptional simple Jordan algebra. It is the algebra of 3×3 hermitian matrices with entries in the Octonian algebra. What is the analog of such a simple exceptional algebra in the Novikov–Jordan case?

Question 2. Is it true that all identities of a Novikov–Jordan algebra Os_1^+ follow from identities *com* $= 0$, *tortken* $= 0$, and *besken* $= 0$?

We have checked that all identities of degree ≤ 5 of the algebra Os_1^+ follow from these three identities. We have checked also that any identity of a Novikov–Jordan algebra with one generator of degree ≤ 7 follows from identities *com* $= 0$, *tortken* $= 0$, and *besken* $= 0$.

Remark. If A is a Novikov algebra, then *tortken* $= 0$ and *besken* $= 0$ are identities of A^+. One can ask about identities of A under the q-commutator $A^{(q)} = (A, \circ_q)$, where $a \circ_q b = a \circ b + qb \circ a$. We have proved that $A^{(q)}$ satisfies the two identities

$$t_1(t_2t_3 - t_3t_2) + (q-1)((t_1t_2)t_3 - (t_1t_3)t_2) = 0, \tag{1}$$

$$-(q-1)(t_1(t_2t_3) - t_2(t_1t_3)) + q(t_1t_2 - t_2t_1)t_3 = 0. \tag{2}$$

These identities are minimal for $q \neq \pm 1$. If $q = -1$, $p \neq 2$, then these identities are reduced to one:

$$t_1(t_2t_3) + t_2(t_3t_1) + t_3(t_1t_2) = 0 \qquad \text{(Jacobi identity)}.$$

Moreover, if $q^3 \neq -1$, then any algebra with identities (1) and (2) is isomorphic to some algebra of the form $A^{(q)}$, where A is Novikov. If $q^3 = -1$, then this is not true.

The Lie algebra $A^{(-1)}$ satisfies the following skew-symmetric identity of degree 5:

$$\sum_{\sigma \in Sym_4} \text{sign}\ \sigma(((t_0t_{\sigma(1)})t_{\sigma(2)})t_{\sigma(3)})t_{\sigma(4)} = 0.$$

We do not know whether these identities form a base of the T-ideal of identities of $A^{(-1)}$ where $A = O_1(m)$. Recall that $O_1(m)$ is a divided power algebra of dimension p^m if $p > 0$.

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