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BRIEF COMMUNICATIONS

Hadamard Invertible Matrices, n-Scalar Products, and Determinants

A. S. Dzhumadil′daev

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The *Hadamard product* of two $n\times n$ matrices $A=(a_{i,j})$ and $B=(b_{i,j})$ is an $n\times n$ matrix whose (i,j)-component is $a_{i,j}b_{i,j}$. For example, let $A^{(k)}=(a_{i,j}^k)$ be the Hadamard kth power of A. In particular, for a matrix A with nonzero components, one can define its *Hadamard inverse* as the matrix $A^{(-1)}=(a_{i,j}^{-1})$.

If A is an arbitrary 3×3 orthogonal matrix with nonzero components, then $\det A^{(-1)}=0$. I wish to thank O. Khudoverdyan for pointing out this remarkable property of 3×3 orthogonal matrices and asking me about the analog of this property in the case $n>3$.

Suppose that

$$\operatorname{per} A=\sum_{\sigma\in\mathrm{Sym}_n} a_{1,\sigma(1)}\dots a_{n,\sigma(n)}$$

is the permanent of the $n\times n$ matrix $A=(a_{i,j})$. Cayley [1] found the following relation between the permanents and determinants of 3×3 matrices and of their Hadamard powers:

$$\operatorname{per} A\det A=\det A^{(2)}+2\biggl(\prod_{i,j=1}^{3} a_{i,j}\biggr)\det A^{(-1)}. \tag{1}$$

In particular, for orthogonal matrices $A\in O(3)$, the following relation holds:

$$\operatorname{per} A=\begin{cases}\ \ \det A^{(2)} & \text{if } \det A=1,\\ -\det A^{(2)} & \text{if } \det A=-1.\end{cases}$$

Thus, for $f(x)=x^{(2)}$ we have $\operatorname{per} A=\det f(A)$ if $A\in SO(3)$. In the case $n>3$, we not know whether we can express $\operatorname{per} A$ as a polynomial in the determinants of Hadamard powers (at least, for a reasonable class of matrices, such as $SO(n)$). In the general case, there are no linear mappings $T\colon \mathrm{Mat}_n\to\mathrm{Mat}_n$ such that $\operatorname{per} A=\det T(A)$ for all $n\times n$ matrices A if $n\ge 3$ [2]. In the case $n=2$, such a mapping obviously exists:

$$T\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a & b\\ -c & d\end{pmatrix}.$$

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Definition. Suppose that $k > 1$. Suppose that K is the ground field, K^n is the n-dimensional coordinate vector space over K, and

$$(\,, \dots, \,)_k \colon \underbrace{K^n \times \dots \times K^n}_{k} \to K,$$

is the k-scalar product on K^n defined by the rule

$$(v_1, \dots, v_k)_k = \sum_{i=1}^{n} a_{1,i} a_{2,i} \dots a_{k,i},$$

where $v_i = (a_{i,1}, \dots, a_{i,n}) \in K^n$ is the ith component of $K^n \times \dots \times K^n$ $i = 1, \dots, k$.

For a matrix $A = (a_{i,j})$, by a_i and $\bar{a}_j$ we denote its ith row and jth column. Suppose that $A_{i,j}$ is the matrix obtained from A by crossing out its ith row and jth column.

Theorem 1. *Suppose that $n \ge 3$ and $A = (a_{i,j})$ is an n-quadratic matrix with nonzero components. Then, for all $1 \le j \le n$,*

$$\det A^{(-1)} = \left(\prod_{s=1}^{n} a_{s,j}^{-1} \right) \sum_{i=1}^{n} (-1)^{i+j} (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{n-1} \det A_{i,j}^{(-1)},$$

and, for all $1 \le i \le n$,

$$\det A^{(-1)} = \left(\prod_{s=1}^{n} a_{i,s}^{-1} \right) \sum_{j=1}^{n} (-1)^{i+j} (\bar{a}_1, \dots, \bar{a}_{j-1}, \bar{a}_{j+1}, \dots, \bar{a}_n)_{n-1} \det A_{i,j}^{(-1)}.$$

Proof. Since the transposition of a matrix does not change its determinant, it suffices to consider the first case.

By Laplace's theorem, we have

$$\sum_{i=1}^{n} (-1)^i a_{i,s}^{-1} \det A_{i,j}^{(-1)} = 0$$

for all $s \ne j$. Hence

$$\sum_{s \ne j} \prod_{q=1}^{n} (a_{q,s} a_{q,j}^{-1}) \sum_{i=1}^{n} (-1)^i a_{i,s}^{-1} \det A_{i,j}^{(-1)} = 0.$$

In other words,

$$\begin{aligned} 0 &= \sum_{s \ne j} \sum_{i=1}^{n} (-1)^i a_{i,j}^{-1} \prod_{q \ne i} a_{q,s} a_{q,j}^{-1} \det A_{i,j}^{(-1)} = \sum_{i=1}^{n} (-1)^i a_{i,j}^{-1} \left(\sum_{s \ne j} \prod_{q \ne i} a_{q,s} a_{q,j}^{-1} \right) \det A_{i,j}^{(-1)} \\ &= \sum_{i=1}^{n} (-1)^i a_{i,j}^{-1} \left(1 - \sum_{s=1}^{n} \prod_{q \ne i} a_{q,s} a_{q,j}^{-1} \right) \det A_{i,j}^{(-1)} \\ &= \sum_{i=1}^{n} (-1)^i a_{i,j}^{-1} \det A_{i,j}^{(-1)} - \sum_{i=1}^{n} (-1)^i a_{i,j}^{-1} (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{n-1} \prod_{q \ne i} a_{q,j}^{-1} \det A_{i,j}^{(-1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{n} (-1)^i (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{n-1} \prod_{q=1}^{n} a_{q,j}^{-1} \det A_{i,j}^{(-1)} \\ &\qquad = \sum_{i=1}^{n} (-1)^i a_{i,j}^{-1} \det A_{i,j}^{(-1)} = (-1)^j \det A^{(-1)}. \quad \square \end{aligned}$$

Corollary 1. *Suppose that, for vectors* $a_1, \dots, a_n \in K^n$, $n > 2$, *any* $n-1$ *of them are orthogonal in the sense of* $(n-1)$*-scalar products*:

$$(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{n-1} = 0$$

for all $i = 1, \dots, n$. *Assume that* $a_{i,j} \neq 0$ *for all* i, j, *where* $a_i = (a_{i,1}, \dots, a_{i,n})$. *Suppose that* A *is a matrix with rows (columns)* $a_1, \dots, a_n$. *Then the matrix* $A^{(-1)} = (a_{i,j}^{-1})$ *is singular*: $\det A^{(-1)} = 0$.

In the case $n = 3$, we obtain the property of 3×3 orthogonal matrices given above.

Let us present another version of Theorem 1. Along with the well-known formula

$$\det A = \sum_{i,j=1}^{n} a_{i,j} \Delta_{i,j},$$

where $\Delta_{i,j} = (-1)^{i+j} \det A_{i,j}$ is a cofactor, the following "complication" of this formula is valid.

Theorem 2. *Assume that all the components of the matrix* $A = (a_{i,j}) \in \mathrm{Mat}_n$, $n \geq 3$, *are distinct from zero and* $a_i^{-1} = (a_{i,1}^{-1}, \dots, a_{i,n}^{-1})$ *and* $\bar{a}_j^{-1} = (a_{1,j}^{-1}, \dots, a_{n,j}^{-1})$ *are the* i*th row and the* j*th column of the matrix* $A^{(-1)}$. *Then, for any* $1 \leq j \leq n$,

$$\det A = \sum_{i=1}^{n} \lambda_{i,j} a_{i,j} \Delta_{i,j},$$

and, for any $1 \leq i \leq n$,

$$\det A = \sum_{j=1}^{n} \bar{\lambda}_{i,j} a_{i,j} \Delta_{i,j},$$

where

$$\lambda_{i,j} = \Big(\prod_{s \neq i} a_{s,j}\Big) (a_1^{-1}, \dots, a_{i-1}^{-1}, a_{i+1}^{-1}, \dots, a_n^{-1})_{n-1},$$

$$\bar{\lambda}_{i,j} = \Big(\prod_{s \neq j} a_{i,s}\Big) (\bar{a}_1^{-1}, \dots, \bar{a}_{j-1}^{-1}, \bar{a}_{j+1}^{-1}, \dots, \bar{a}_n^{-1})_{n-1}.$$

Remark 1. If any $n-1$ rows of an $n \times n$ matrix are $(n-1)$-orthogonal and $n > 3$, then any $n-1$ columns of this matrix need not also be $(n-1)$-orthogonal. Also, note that it is not necessary that K be a field. It is sufficient that it be a commutative ring.

The Hadamard inverse admits some of the usual matrix operations. The mapping $A \mapsto \det A^{(-1)}$ is skew-symmetric with respect to permutations of the rows (columns), but the addition of some row (column) to another row (column) may change the determinant $\det A^{(-1)}$.

Corollary 2. *Suppose that* $A = (a_{i,j})$ *is a matrix with invertible components* $a_{i,j}$ *and* B *is a matrix obtained from* A *by multiplying some row (column) by a number* $\epsilon \in K$, *where* $\epsilon^{n-1} = 1$. *Then* $\det A^{(-1)} = \det B^{(-1)}$.

The condition $a_{ij} \neq 0$ in the statement of Theorem 1 can be dropped if we consider Cayley determinants defined as follows. To any $n \times n$ matrix A we assign an element of K denoted by $\mathrm{Ca}(A)$ according to the rule

$$\mathrm{Ca}(A) = \sum_{\sigma \in \mathrm{Sym}_n} \operatorname{sign} \sigma\, a_{1,1} \cdots \widehat{a_{1,\sigma(1)}} \cdots a_{1,n} a_{2,1} \cdots \widehat{a_{2,\sigma(2)}} \cdots a_{2,n} \cdots a_{n,1} \cdots \widehat{a_{n,\sigma(n)}} \cdots a_{n,n}.$$

If all the $a_{i,j}$ are not zeros, then

$$\mathrm{Ca}(A) = \Big(\prod_{i,j=1}^{n} a_{i,j}\Big) A^{(-1)}.$$

Below is another version of Theorem 1.

Theorem 3. *Suppose that* $A = (a_{i,j})$ *is an* $n \times n$ *matrix and* $n \geq 3$. *Then, for all* $1 \leq j \leq n$,

$$\mathrm{Ca}(A) = \sum_{i=1}^{n} \left(\prod_{s \neq j}^{n} a_{i,s} \right) (-1)^{i+j} (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)_{n-1} \, \mathrm{Ca}(A_{i,j}),$$

and, for all $1 \leq i \leq n$,

$$\mathrm{Ca}(A) = \sum_{j=1}^{n} \left(\prod_{s \neq i} a_{s,j} \right) (-1)^{i+j} (\bar{a}_1, \dots, \bar{a}_{j-1}, \bar{a}_{j+1}, \dots, \bar{a}_n)_{n-1} \, \mathrm{Ca}(A_{i,j}).$$

Corollary 3. *If each row* (*column*) *contains, at least, two zeros, then* $\mathrm{Ca}(A) = 0$.

Remark 2. There is another generalization of Cayley's relation (1) for matrices of fourth order. Suppose that A is a 4×4 matrix. Then

$$\operatorname{per} A \det A = \det A^{(2)} + 2 \sum_{i,j=1}^{4} (-1)^{i+j} a_{i,j}^2 \, \mathrm{Ca}(A_{i,j}).$$

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S. Demirel University, Kazakhstan
E-mail: askar@math.kz