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NILPOTENCY OF ZINBIEL ALGEBRAS

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ABSTRACT. Zinbiel algebras are defined by the identity $(a \circ b) \circ c = a \circ (b \circ c + c \circ b)$. We prove an analog of the Nagata–Higman theorem for Zinbiel algebras. We establish that every finite-dimensional Zinbiel algebra over an algebraically closed field is solvable. Every solvable Zinbiel algebra with solvability length N is a nil-algebra with nil-index 2^N if $p = \operatorname{char} K = 0$ or $p > 2^N - 1$. Conversely, every Zinbiel nil-algebra with nil-index N is solvable with solvability length N if $p = 0$ or $p > N - 1$. Every finite-dimensional Zinbiel algebra over complex numbers is nilpotent, nil, and solvable.

1. INTRODUCTION

Let $A = (A, \circ)$ be an algebra, where A is a vector space over a field K of characteristic $p \geq 0$ and $A \times A \to A$, $(a, b) \mapsto a \circ b$, is a multiplication. Let $f = f(t_1, \ldots, t_k)$ be some noncommutative, nonassociative polynomial with k variables $t_1, \ldots, t_k$. We say that A satisfies an identity $f = 0$ if $f(a_1, \ldots, a_k) = 0$ for any substitutions $t_1 := a_1, \ldots, t_k := a_k$ by elements of A. Here, multiplications are calculated in terms of the multiplication $\circ$.

For example, an algebra with the identity ass $= 0$ is said to be associative if

$$\text{ass} = t_1(t_2t_3) - (t_1t_2)t_3.$$

An algebra with the identity $t^n = 0$ is called a *nil-algebra*. An associative nil-algebra has nil-index n if $a^{n-1} \neq 0$ for some $a \in A$.

Any associative algebra with nil-index n is nilpotent with nilpotency index no greater than $2^n - 1$: for some $N = N(n) \leq 2^n - 1$, the identity

$$t_1 \cdots t_N = 0$$

holds (the Nagata–Higman theorem). In other words,

$$a_1 \circ \cdots \circ a_N = 0$$

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for any $a_1, \ldots, a_N \in A$ [13, 7, 4]. The problem of finding more exact estimates for $N(n)$ remains still difficult. For example, $N(2) = 3$ and $N(3) = 6$.

A similar problem for Lie algebras is also complicated and very interesting. It is related to the Engel theorem and Burnside problems [10].

An algebra with the identity $r\,\mathrm{sym} = 0$ is said to be *right-symmetric*, where

$$r\,\mathrm{sym} = t_1(t_2t_3 - t_3t_2) - (t_1t_2)t_3 + (t_1t_3)t_2$$

(see [5, 18]). In [2], such algebras are called chronological algebras. Later [8], the name "chronological" was used for a different algebra.

Algebras with the identity zinbiel $= 0$, where

$$\mathrm{zinbiel}(t_1, t_2, t_3) = (t_1t_2)t_3 - t_1(t_2t_3) - t_1(t_3t_2),$$

are called *Zinbiel algebras*.

Example. $(\mathbb{C}[x], \star)$, where $(a \star b)(x) = \dfrac{\partial}{\partial x} a(x)b(x)$, is right-symmetric. Moreover, it satisfies also the identity lcom $= 0$, where

$$\mathrm{lcom}(t_1, t_2, t_3) = t_1(t_2t_3) - t_2(t_1t_3).$$

Example. $(\mathbb{C}[x], \circ)$, where $(a \circ b)(x) = a(x) \int_0^x b(t)dt$ is a Zinbiel algebra.

An algebra satisfying the identity leibniz $= 0$, where

$$\mathrm{leibniz}(t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_2)t_3 + (t_1t_3)t_2,$$

is called a *Leibniz* algebra. Such algebras were introduced in [3, 11]. The Koszul dual [6] of the Leibniz operad is defined by the identity zinbiel $= 0$, i.e., by the condition

$$(a \circ b) \circ c = a \circ (b \circ c + c \circ b) \tag{1}$$

for any $a, b, c \in A$. Such algebras are called *Leibniz dual* or *Zinbiel* (read Leibniz in reverse order) algebras [12]. In our paper, we do not follow terminology of [8, 9] and use the term Zinbiel algebras for Leibniz dual algebras. For the history of the name "chronological," see [16].

An algebra A is said to be *solvable* if $A^{(k)} = 0$ for some k, where $A^{(i)}$ are defined by

$$A^{(0)} = A, \quad A^{(i+1)} = A^{(i)} \circ A^{(i)}, \quad i > 0.$$

We say that A has *solvability length* N if $A^{(N)} = 0$, $A^{(N-1)} \neq 0$.

An algebra A is said to be *nilpotent* if there exists N such that the right-bracketed product of any N elements of A vanishes:

$$a_1 \circ (a_2 \circ (\cdots (a_{N-1} \circ a_N) \cdots)) = 0.$$

The minimal N with such property is called the *nilpotency index*. For every nilpotent Zinbiel algebra A, there exists N such that the product of arbitrary N elements of any bracketing type vanishes. It is obvious that any nilpotent algebra is solvable.

We denote by r_a and l_a the right and left multiplication operators on $A = (A, \circ)$:

$$r_a(b) = b \circ a, \quad l_a(b) = a \circ b.$$

The powers $a^{\cdot k}$ and $a^{(\cdot k)}$ are defined by

$$a^{\cdot 1} = a, \qquad a^{\cdot k+1} = l_a^k(a) = a \circ a^{\cdot k},$$
$$a^{(\cdot 1)} = a, \qquad a^{(\cdot k)} = a^{(\cdot k-1)} \circ a^{(\cdot k-1)}.$$

We say that a Zinbiel algebra A is a *nil-algebra* if for every $a \in A$, we have $a^{\cdot k} = 0$ for some $k = k(a)$. Then, given an arbitrary element of a Zinbiel nil-algebra, some power of this element of any bracketing type vanishes. If $a^{\cdot n} = 0$ for all $a \in A$ and $a^{\cdot n-1} \neq 0$ for some $a \in A$, then we say that A is a *nil-algebra with nil-index* n. A Zinbiel algebra is said to be *simple* if it has no proper ideal, i.e., if $I \circ A \subseteq I$, $A \circ I \subseteq I$, then $I = 0$ or $I = A$.

In this paper, we prove the following results.

Theorem 1.1. *Let K be an algebraically closed field of characteristic $p \geq 0$. Then every finite-dimensional Zinbiel algebra is solvable.*

Theorem 1.2. *Let K be a field of characteristic $p \geq 0$ and A be a solvable Zinbiel algebra with solvability length N. If $p = 0$ or $p > 2^N - 1$, then A is a nil-algebra with nil-index no greater than 2^N. Conversely, if A is a Zinbiel nil-algebra with nil-index N and if $p = 0$ or $p > N - 1$, then A is solvable with solvability length N.*

Theorem 1.3. *Let K be a field of characteristic $p \geq 0$. Every Zinbiel nil-algebra is nilpotent. If A is a nil-algbera with nil-index n, then the nilpotency index of A is no greater than $2^n - 1$.*

Corollary 1.4. *Every finite-dimensional, simple Zinbiel algebra over an algebraically closed field of characteristic $p \geq 0$ is isomorphic to the* 1-*dimensional algebra with trivial multiplication.*

Corollary 1.5. *Every finite-dimensional Zinbiel algebra over the field of complex numbers is nilpotent (and, hence solvable and nil). If $p > 0$, then every finite-dimensional Zinbiel algebra over an algebraically closed field of dimension $< \log_2(p+1)$ and characteristic p is nilpotent (and hence solvable and nil).*

Let

$$Z(A) = \{z \in A \mid a \circ z = z \circ a = 0 \ \forall a \in A\}$$

be the *center* of A.

Corollary 1.6. *Let A be a finite-dimensional Zinbiel algebra over the field of complex numbers of dimension n. Then there exists $N < n$ such that the product of any N elements of A in any type of bracketing is equal to* 0*. Moreover, A has the nontrivial center $Z(A) \neq 0$. The same is true for* *any finite-dimensional Zinbiel algebra A over a field of characteristic $p > 0$ if $n = \dim A < \log_2(p+1)$.*

We see that the infinite-dimensional Zinbiel algebra $(\mathbb{C}[x], \circ)$ with the multiplication $(a \circ b)(x) = a(x) \int_0^x b(t)dt$ is not nilpotent and hence not solvable. In other words, Theorem 1.1 and Corollary 1.5 are false in the infinite-dimensional case.

As an application of our results, we classify Zinbiel algebras of dimension ≤ 3 over an algebraically closed field K.

Theorem 1.7. *Let K be an algebraically closed field of any characteristic p.*

Any Zinbiel algebra of dimension 1 is isomorphic to an algebra with trivial multiplication: $A = \langle e_1 \rangle$, $e_1 \circ e_1 = 0$.

Any two-dimensional Zinbiel algebra is isomorphic to the algebra $Q(\beta)$ defined as follows:

$$Q(\alpha) = \langle e_1, e_2 \rangle, \quad \alpha = 0 \textit{ or } 1,$$
$$e_1 \circ e_1 = \alpha e_2, \quad e_1 \circ e_2 = 0, \quad e_2 \circ e_1 = 0, \quad e_2 \circ e_2 = 0.$$

Any three-dimensional Zinbiel algebra with $\dim A \circ A \leq 1$ is isomorphic to the algebra $R(\alpha, \beta, \gamma, \delta)$ defined as follows:

$$R(\alpha, \beta, \gamma, \delta) = \langle e_1, e_2, e_3 \rangle,$$
$$\begin{array}{lll} e_1 \circ e_1 = \alpha e_3, & e_1 \circ e_2 = \beta e_3, & e_1 \circ e_3 = 0, \\ e_2 \circ e_1 = \gamma e_3, & e_2 \circ e_2 = \delta e_3, & e_2 \circ e_3 = 0, \\ e_3 \circ e_1 = 0, & e_3 \circ e_2 = 0, & e_3 \circ e_3 = 0, \end{array}$$

where $\alpha, \beta, \gamma, \delta \in K$.

Any three-dimensional Zinbiel algebra A over an algebraically closed field of characteristic $\neq 2$ with $\dim A \circ A = 2$ is isomorphic to the algebra $W(3)$ defined as follows:

$$\operatorname{char} K \neq 2, \quad W(3) = \langle e_1, e_2, e_3 \rangle,$$
$$\begin{array}{lll} e_1 \circ e_1 = e_2, & e_1 \circ e_2 = \frac{1}{2} e_3, & e_1 \circ e_3 = 0, \\ e_2 \circ e_1 = e_3, & e_2 \circ e_2 = 0, & e_2 \circ e_3 = 0, \\ e_3 \circ e_1 = 0, & e_3 \circ e_2 = 0, & e_3 \circ e_3 = 0. \end{array}$$

There are no 3-dimensional Zinbiel algebras A such that $\dim A \circ A = 3$.

The algebras $R(\alpha, \beta, \gamma, \delta)$ and $W(3)$ are not isomorphic. The following isomorphisms hold:

$$R(\alpha, \beta, \gamma, \delta) \cong R(1, \beta, \gamma, \delta) \quad \textit{if} \quad \alpha \neq 0,$$
$$R(\alpha, \beta, \gamma, \delta) \cong R(\alpha, \beta, \gamma, 1) \quad \textit{if} \quad \delta \neq 0.$$

Therefore, there are two types of nonisomorphic classes of three-dimensional algebras $R(\alpha,\beta,\gamma,\delta)$, where $\alpha,\delta = 0$ or 1 and $\beta,\gamma \in K$, and $W(3)$.

Two-dimensional Zinbiel algebras over complex numbers were also studied in [14].

In our paper, the letter n is used in two senses: sometimes, n denotes the dimension of an algebra, sometimes, we use n as a nil-index. From the context it will be clear in what sense n is used. Note that the nil-index cannot be greater than the dimension of the algebra.

2. Proof of Theorem 1.1

Every Zinbiel algebra is right-commutative:

$$(a \circ b) \circ c = (a \circ c) \circ b.$$

Let

$$C(a) = \{x \in A : a \circ x = 0\}$$

be the *right centralizer* of $a \in A$. An important role in this paper is played by the following property of the right centralizer.

Lemma 2.1. *Let A be a right-commutative algebra. Then for all $a, b \in A$, we have $C(a) \subseteq C(a \circ b)$.*

Proof. If $x \in C(a)$, then $(a \circ b) \circ x = (a \circ x) \circ b = 0$ and $x \in C(a \circ b)$. □

Lemma 2.2. *Let A be a Zinbiel algebra and let $a \in A$. If v is an eigenvector of the linear operator l_a with eigenvalue $\mu \in K$, then $v \circ v$ is an eigenvector with eigenvalue $2^{-1}\mu$.*

Proof. If $a \circ v = \mu v$, then $(a \circ v) \circ v = \mu v \circ v$ and, by the Zinbiel identity, $(a \circ v) \circ v = 2a \circ (v \circ v)$. Therefore, $l_a(v^{\cdot 2}) = 2^{-1}\mu v^{\cdot 2}$. □

Lemma 2.3. *Let A be a Zinbiel algebra of dimension n over an algebraically closed field K of characteristic* $\operatorname{char} K \neq 2$. *Then for every $a \in A$, we have*:

- $l_a^n = 0$, *or*
- *there exists $0 \neq b \in A$ such that $l_a(b) = \lambda b$, $b \circ b = 0$, for some $0 \neq \lambda \in K$.*

Proof. If $l_a \in \operatorname{End} A$ is nil, then by the Hamilton–Cayley theorem $l_a^n = 0$.

If l_a is not nil, then by Hamilton–Cayley theorem l_a, as an operator over an algebraically closed field, has a nontrivial eigenvalue $0 \neq \mu \in K$. Let $v \in A$ be an eigenvector of l_a with the eigenvalue μ. By Lemma 2.2, $l_a(v^{(\cdot k)}) = 2^{-k}\mu v^{(\cdot k)}$ for all k. Therefore, if $\mu \neq 0$, then there exists $N \leq n = \dim A$ such that $v^{(\cdot N-1)} \neq 0$, $v^{(\cdot N)} = 0$.

Therefore, if l_a is not nil, then there exists a nonzero eigenvalue $\mu \in K$ and $l_a(b) = \lambda b$, $b \circ b = 0$, for $b = v^{(\cdot N-1)}$, $\lambda = 2^{-N+1}\mu \neq 0$. □

Lemma 2.4. *For every finite-dimensional Zinbiel algebra A over an algebraically closed field, there exists $x \neq 0$ such that $C(x) = A$.*

Proof. Prove that there exists $a_0 \neq 0$ such that $C(a_0) = C(a_0 \circ b)$ for all $0 \neq b \in A$.

Take any nonzero element $a_1 \in A$ as a_0. If $C(a_1) = A$, then there is nothing to prove: $C(a_0) = A = C(a_0 \circ b)$ for any $b \in A$. Assume that $C(a_1) \neq A$.

If $C(a_1) \neq C(a_1 \circ a_2)$ for some $a_2 \in A$, then by Lemma 2.1,

$$C(a_1) \subset C(a_1 \circ a_2).$$

Now take $a_1 \circ a_2$ as a_0 and repeat the procedure. If

$$C(a_1 \circ a_2) \neq C((a_1 \circ a_2) \circ a_3)$$

for some $a_3 \in A$, then take $(a_1 \circ a_2) \circ a_3$ as a_0, and so on. Finally, we obtain a sequence of nonzero elements $a_1, a_2, \dots, a_k \in A$ such that

$$C(a_1) \subset C(a_1 \circ a_2) \subset C((\cdots(a_1 \circ a_2) \cdots a_{k-1}) \circ a_k) \subseteq A.$$

Since A is finite-dimensional, this sequence terminates at some k. In other words,

$$C((\cdots(a_1 \circ a_2) \cdots a_{k-1}) \circ a_k) = C(((\cdots(a_1 \circ a_2) \cdots a_{k-1}) \circ a_k) \circ a_{k+1})$$

for any $0 \neq a_{k+1} \in A$. Now take $a_0 = (\cdots(a_1 \circ a_2) \cdots) \circ a_k$.

Therefore, we have proved that there exists $a_0 \neq 0$ such that $C(a_0) = C(a_0 \circ b)$ for all $b \neq 0$.

Now prove that $C(a_0) = A$.

If $l_{a_0} = 0$, then $C(a_0) = A$. Assume that $l_{a_0} \neq 0$ and N is the nilpotency index of l_{a_0}, i.e., $1 < N \leq n$, $l_{a_0}^{N-1} \neq 0$, and $l_{a_0}^N = 0$.

If $\operatorname{char} K = 2$, then by the Zinbiel identity, $(a_0 \circ b) \circ b = 2(a_0 \circ (b \circ b)) = 0$ for all $b \in A$. Therefore, $b \in C(a_0 \circ b) = C(a_0)$ for all $0 \neq b \in A$. In other words, $C(a_0) = A$.

If $\operatorname{char} K \neq 2$, then by Lemma 2.3 $l_{a_0}^n = 0$ for $n = \dim A$ or there exists $0 \neq b \in A$ such that $a_0 \circ b = \lambda b$, where $\lambda \neq 0$ and $b \circ b = 0$. The second case is not possible:

$$b \in C(a_0 \circ b) = C(a_0) \Rightarrow a_0 \circ b = 0 \Rightarrow \lambda = 0,$$

a contradiction. Therefore, $l_{a_0}^n = 0$. Let N be the nilpotency index of l_{a_0}: $l_{a_0}^{N-1} \neq 0$, $l_{a_0}^N = 0$ for $1 < N \leq n = \dim A$. There exists $c \in A$ such that $b = l_{a_0}^{N-1}(c) \neq 0$. Then $a_0 \circ b = l_{a_0}^N(c) = 0$ and, by the definition of a_0,

$$C(a_0) = C(a_0 \circ b) = C(0) = A.$$

Therefore, in all cases we can take $x = a_0$. □

Lemma 2.5. *Every finite-dimensional Zinbiel algebra over an algebraically closed field of dimension >1 has a proper ideal.*

Proof. By Lemma 2.4, there exists $0 \neq x \in A$ such that $C(x) = A$. Prove that $I = A \circ x = \{y \circ x : y \in A\}$ is an ideal of A. For every $a \in A$, we have

$$(y \circ x) \circ a = (y \circ a) \circ x \in I.$$

Since $C(x) = A$, we have $x \circ y = 0$ for any $y \in A$. Therefore,

$$a \circ (y \circ x) = a \circ (y \circ x + x \circ y) = (a \circ y) \circ x \in I$$

for all $a \in A$. Therefore, I is a two-sided ideal of A.

Prove that $\dim I < n = \dim A$. Take a basis $\{e_1, \ldots, e_n\}$ for A with $e_1 = x$. Then $x \circ x = 0$ since $C(x) = A$. Therefore, I is the linear span of the vectors $e_1 \circ x = x \circ x = 0$, $e_2 \circ x, \ldots, e_n \circ x$. Therefore, $\dim I < n = \dim A$. If $I \neq 0$, then we can take I as a proper ideal of A.

If $I = A \circ x = 0$, then we can take as a proper ideal the one-dimensional ideal generated by x. □

Proof of Theorem 1.1. We use induction on $n = \dim A$.

Assume that $n = 1$. Prove that any one-dimensional Zinbiel algebra is isomorphic to an algebra with trivial multiplication. If $\dim A = 1$ and A is generated by the basis element e_1, then $e_1 \circ e_1 = \alpha e_1$ for some $\alpha \in K$. By the Zinbiel identity,

$$\mathrm{zinbiel}(e_1, e_1, e_1) = 0 \quad \Rightarrow \quad \alpha^2 e_1 = 0 \quad \Rightarrow \quad \alpha = 0.$$

Therefore, any 1-dimensional algebra A is solvable.

Assume that $n > 1$ and our statement is true for $n - 1$. By Lemma 2.5, A has some proper ideal J. Since $\dim J < n$ and $\dim A/J < n$, by the induction hypothesis J and A/J are solvable. Therefore, A is solvable. This completes the proof. □

3. Proof of Theorem 1.2

In this section, n is a positive integer, not necessarily equal to $\dim A$.

Lemma 3.1. *For arbitrary elements $a_1, \ldots, a_{k+s}$ of a Zinbiel algebra A, the product*

$$(a_1 \circ (a_2 \circ (\cdots (a_{k-1} \circ a_k) \cdots))) \circ (a_{k+1} \circ (a_{k+2} \circ (\cdots (a_{k+s-1} \circ a_{k+s}) \cdots)))$$

is the sum of $\binom{k+s-1}{s}$ elements of the form

$$a_{\sigma(1)} \circ (a_{\sigma(2)} \circ (\cdots (a_{\sigma(k+s-1)} \circ a_{\sigma(k+s)}) \cdots)),$$

where $\sigma \in \mathrm{Sym}_{k+s}$ runs through all permutations such that

$$\sigma(i) < \sigma(j) \leq k \quad \Rightarrow \quad i < j,$$
$$k < \sigma(i) < \sigma(j) \leq k + s \quad \Rightarrow \quad i < j.$$

Proof. This is easy induction on $k + s$ that uses the Zinbiel identity. □

Corollary 3.2. *Let A be a Zinbiel algebra. Then for every $a \in A$ and all $i, j \in \mathbb{Z}_+$, we have*

$$a^{\cdot i} \circ a^{\cdot j} = \binom{i+j-1}{j} a^{\cdot i+j}.$$

Given $a_1, a_2, \ldots, a_n \in A$, denote by $S_n(a_1, a_2, \ldots, a_n)$ the sum of $n!$ right-bracketed products formed by taking $a_1, a_2, \ldots, a_n$ in all possible orders. Let $a \star b = a \circ b + b \circ a$ be the Jordan product in $A = (A, \circ)$. The product $(a, b) \mapsto a \star b$ is also known as the shuffle product [15, 17]. If A is a Zinbiel algebra, then $(A, \star)$ is associative and commutative [12].

Let I_n be the ideal of A generated by the right-bracketed nth powers $a^{\cdot n}$, $a \in A$.

Lemma 3.3. *For arbitrary elements $a_1, \ldots, a_{k+r}$ of a Zinbiel algebra A, we have*

$$S_k(a_1, \ldots, a_k) \star S_r(a_{k+1}, \ldots, a_{k+r}) = S_{k+r}(a_1, \ldots, a_{k+r}).$$

Proof. We use induction on k. Let $k = 1$. By Lemma 3.1,

$$\begin{aligned} a_1 \star S_{r-1}(a_2, \ldots, a_r) &= a_1 \circ S_{r-1}(a_2, \ldots, a_r) + S_{r-1}(a_2, \ldots, a_r) \circ a_1 \\ &= S_r(a_1, \ldots, a_r). \end{aligned}$$

Assume that our statement is true for $k - 1$. Since

$$S_k(a_1, \ldots, a_k) = S_1(a_1) \star S_{k-1}(a_2, \ldots, a_k),$$

by the result of [12], we have

$$\begin{aligned} &S_k(a_1, \ldots, a_k) \star S_r(a_{k+1}, \ldots, a_{k+r}) \\ &\qquad = (S_1(a_1) \star S_{k-1}(a_2, \ldots, a_k)) \star S_r(a_{k+1}, \ldots, a_{k+r}) \end{aligned}$$

(the associativity of $\star$)

$$= S_1(a_1) \star (S_{k-1}(a_2, \ldots, a_k) \star S_r(a_{k+1}, \ldots, a_{k+r}))$$

(the induction assumption)

$$= S_1(a_1) \star S_{k+r-1}(a_2, \ldots, a_{k+r})$$

(the induction assumption)

$$= S_{k+r}(a_1, \ldots, a_{k+r}).$$

The lemma is proved. □

Lemma 3.4. *Let A be a Zinbiel algebra. Then for arbitrary $a_1, \ldots, a_n$, $u \in A$,*

$$S_n(a_1, \ldots, a_n) \circ u = \sum_{i=1}^{n} a_i \circ S_n(a_1, \ldots, \hat{a}_i, \ldots, a_n, u),$$

where $\hat{a}_i$ means that the element a_i is omitted.

Proof. By Lemma 3.3,

$$
\begin{aligned}
&S_n(a_1,\dots,a_n)\circ u=(a_1\star S_{n-1}(a_2,\dots,a_n))\circ u\\
&=a_1\circ(S_{n-1}(a_2,\dots,a_n)\star u)+S_{n-1}(a_2,\dots,a_n)\circ(a_1\star u)\\
&=a_1\circ S_n(a_2,\dots,a_n,u)+S_{n-1}(a_2,\dots,a_n)\circ S_2(a_1,u)\\
&=a_1\circ S_n(a_2,\dots,a_n,u)+(a_2\star S_{n-2}(a_3,\dots,a_n))\circ S_2(a_1,u)\\
&=a_1\circ S_n(a_2,\dots,a_n,u)+a_2\circ(S_{n-2}(a_3,\dots,a_n)\star S_2(a_1,u))\\
&\quad+S_{n-2}(a_3,\dots,a_n)\circ(a_2\star S_2(a_1,u))\\
&=a_1\circ S_n(a_2,\dots,a_n,u)+a_2\circ(S_n(a_1,a_3,\dots,a_n,u))\\
&\quad+S_{n-2}(a_3,\dots,a_n)\circ S_3(a_1,a_2,u)\\
&=\dots=a_1\circ S_n(a_2,\dots,a_n,u)+a_2\circ(S_n(a_1,a_3,\dots,a_n,u))+\dots\\
&\quad+S_1(a_n)\circ S_n(a_1,a_2,\dots,a_{n-1},u).
\end{aligned}
$$

The lemma is proved. □

Lemma 3.5. *Let A be a Zinbiel algebra, n be an integer, and $n<p$ or $p=0$. Then the ideal $I_n=\langle a^{\cdot n}: a\in A\rangle$ generated by nth right-bracketed powers is, as a vector space, the linear span of elements of the form $a_1\circ(a_2\circ(\cdots(a_k\circ S_n(a_{k+1},\dots,a_{k+n}))\cdots))$, where $a_1,\dots,a_{k+n}$ are any elements of A.*

Proof. Denote by J_n the linear span of elements of the form $X=a_1\circ(a_2\circ(\cdots(a_k\circ S_n(a_{k+1},\dots,a_{k+n}))\cdots))$. We will prove that $I_n=J_n$.

We have

$$S_n(a_1,a_2,\dots,a_n)=\sum(-1)^{n-r}(a_{i_1}+a_{i_2}+\dots+a_{i_r})^{\cdot n},$$

where the summation is taken over all nonempty subsets $\{i_1,i_2,\dots,i_r\}\subseteq\{1,2,\dots,n\}$ and all products are right-bracketed. Therefore,

$$S_n(a_1,\dots,a_n)\in I_n.$$

Hence,

$$a_1\circ(a_2\circ(\cdots(a_k\circ S_n(a_{k+1},\dots,a_{k+n}))\cdots))\in I_n$$

for all k. In other words, $J_n\subseteq I_n$.

Now we prove that $I_n\subseteq J_n$. It is clear that J_n is a left ideal and $A\circ J_n\subseteq J_n$. If $p=0$ or $n<p$, then

$$a^{\cdot n}=(n!)^{-1}S_n(a,\dots,a).$$

Therefore, in the case of $p=0$ or $n<p$, we can choose generators for I_n of the form $S_n(x_1,\dots,x_n)$. Therefore, to establish that $I_n=J_n$, it suffices to prove that $X\circ u\in J_n$ for all $X\in J_n$ and $u\in A$.

By induction on $k=0,1,2,\dots$, we prove that $X\circ u\in J_n$ for all $u\in A$, where $X\in J_n$ has the form $a_1\circ(a_2\circ(\cdots(a_k\circ S_n(a_{k+1},\dots,a_{k+n}))\cdots))$.

Let $k = 0$. Then by Lemma 3.3,

$$X = S_n(a_1, \dots, a_n) = a_1 \star S_{n-1}(a_2, \dots, a_n)$$

and by Lemma 3.4,

$$X \circ u = \sum_{i=1}^{n} a_i \circ S_n(a_1, \dots, \hat{a_i}, \dots, a_n, u) \in I_n.$$

Therefore, our statement is true for $k = 0$.

Assume that our statement is true for $k - 1$. In other words, for all $Y = a_2 \circ (\cdots (a_k \circ S_n(a_{k+1}, \dots, a_{k+n})) \cdots) \in J_n$ and $u \in A$ we have $Y \circ u \in J_n$. We know that $a_1 \circ (u \circ Y) \in J_n$. Then for $X = a_1 \circ Y$, by the Zinbiel identity,

$$X \circ u = a_1 \circ (Y \circ u + u \circ Y) = a_1 \circ (Y \circ u) + a_1 \circ (u \circ Y) \in J_n.$$

Hence, our statement is proved for k. □

Lemma 3.6. *Let $(A, \circ)$ be a Zinbiel algebra. Then for any $a_1, \dots, a_k \in A$, we have*

$$(\cdots (a_1 \circ a_2) \cdots \circ a_{k-1}) \circ a_k = a_1 \circ S_{k-1}(a_2, \dots, a_k).$$

Proof. By the Zinbiel identity, our statement is true for $k = 3$. Assume that it is true for $k - 1$. Then

$$\begin{aligned}(\cdots (a_1 \circ a_2) \cdots \circ a_{k-1}) \circ a_k &= (a_1 \circ S_{k-2}(a_2, \dots, a_{k-1})) \circ a_k \\ &= a_1 \circ (S_{k-2}(a_2, \dots, a_{k-1}) \star a_k).\end{aligned}$$

By Lemma 3.3,

$$S_{k-2}(a_2, \dots, a_{k-1}) \star a_k = S_{k-1}(a_2, \dots, a_{k-1}, a_k).$$

Therefore, our statement is true for k. The lemma is proved. □

Lemma 3.7. *Let n be an integer and $n - 2 < p$ or $p = 0$. Then $S_{n-1}(a_1, \dots, a_{n-1}) \circ S_2(b_1, b_2) \in I_n$ for any $a_1, \dots, a_{n-1}, b_1, b_2 \in A$.*

Proof. By Lemma 3.6, $(n-1)$th right-bracketed product of a by $b \circ b$ is

$$\begin{aligned}(a^{\cdot\, n-1}) \circ (b \circ b) &= (1/(n-2)!)(1/2)((((a \circ a) \circ \cdots) \circ a) \circ b) \circ b \\ &= (1/(n-2)!)(1/2) a \circ S_n(a, a, \cdots, a, b, b) \in I_n.\end{aligned}$$

Since $S_k(a_1, \dots, a_k)$ is the sum of elements of the form $a^{\cdot k}$, the proof is complete. □

Lemma 3.8.

$$a \circ (b \circ c) = S_2(a, b \circ c) - b \circ S_2(a, c).$$

Proof. We have

$$S_2(a, b \circ c) - b \circ S_2(a, c) = a \circ (b \circ c) + (b \circ c) \circ a - b \circ (a \circ c) - b \circ (c \circ a)$$
$$= (a \circ b) \circ c.$$

The lemma is proved. □

Lemma 3.9. *Let A be a Zinbiel algebra, $n \geq 3$, and let $p = 0$ or $n - 1 < p$. Then*

$$I_{n-1} \circ I_{n-1} \subseteq I_n.$$

Proof. By Lemma 3.5, every two elements $u, v \in I_{n-1}$ can be represented in the form

$$u = a_1 \circ (a_2 \circ (\cdots (a_k \circ S_{n-1}(a_{k+1}, \ldots, a_{k+n-1})) \cdots)),$$
$$v = b_1 \circ (b_2 \circ (\cdots (b_r \circ S_{n-1}(b_{r+1}, \ldots, b_{r+n-1})) \cdots))$$

for some $a_i, b_j \in A$.

We use induction on $k + r$ and prove that $u \circ v \in I_n$. Assume that $k + r = 0$. Then

$$u = S_{n-1}(a_1, \cdots, a_{n-1}), \quad v = S_{n-1}(b_1, \cdots, b_{n-1}).$$

By Lemma 3.3, $v = S_2(b_1, S_{n-1}(b_2, \ldots, b_{n-1}))$. Therefore, by the Zinbiel identity,

$$u \circ v = S_{n-1}(a_1, \cdots, a_{n-1}) \circ S_2(b_1, S_{n-2}(b_2, \cdots, b_{n-1}))$$
$$= (S_{n-1}(a_1, \ldots, a_{n-1}) \circ b_1) \circ S_{n-2}(b_2, \ldots, b_{n-1})$$

(see Lemma 3.4)

$$= \sum_{i=1}^{n-1} (a_i \circ S_{n-1}(a_1, \ldots, \hat{a_i}, \ldots, a_{n-1}, b_1)) \circ S_{n-2}(b_2, \ldots, b_{n-1})$$
$$= \sum_{i=1}^{n-1} a_i \circ (S_{n-1}(a_1, \ldots, \hat{a_i}, \ldots, a_{n-1}, b_1) \star S_{n-2}(b_2, \ldots, b_{n-1}))$$

(Lemma 3.3)

$$= \sum_{i=1}^{n-1} a_i \circ S_{2n-3}(a_1, \ldots, \hat{a_i}, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}).$$

Therefore, in view of

$$S_{2n-3}(x_1, \ldots, x_{2n-3}) \in I_{2n-3} \subseteq I_n,$$

we see that $u \circ v \in I_n$.

Assume that for $k + r - 1$ our statement is true. Consider two cases: $k > 0$ and $r > 0$.

If $k > 0$, then $u = a_1 \circ u_1$ for

$$u_1 = a_2 \circ (\cdots (a_k \circ S_{n-1}(a_{k+1}, \ldots, a_{k+n-1})) \cdots).$$

By the induction assumption, $u_1 \circ v \in I_n$ and $v \circ u_1 \in I_n$. Therefore,

$$u \circ v = a_1 \circ (u_1 \circ v + v \circ u_1) \in I_n.$$

If $r > 0$, then $v = b_1 \circ v_1$, where

$$v_1 = b_2 \circ (\cdots (b_r \circ S_{n-1}(b_{r+1}, \ldots, b_{n+r-1})) \cdots).$$

As we have verified above,

$$v \circ u = (b_1 \circ v_1) \circ u = b_1 \circ (v_1 \circ u + u \circ v_1) \in I_n,$$

and by Lemma 3.3,

$$u \star v \in I_{2n-2} \subseteq I_n.$$

Hence

$$u \circ v = u \star v - v \circ u \in I_n.$$

Therefore, our statement is true for $k + r$. The lemma is proved. □

Lemma 3.10. *Let A be a Zinbiel algebra and $p = 0$ or $n - 1 < p$. Then*

$$A^{(n)} \subseteq I_n.$$

Proof. We use induction on $n \geq 2$. It is easy to see that

$$a \circ (b \circ c) = S_2(a, b \circ c) - b \circ S_2(a, c).$$

Thus,

$$(x \circ y) \circ (b \circ c) = S_2(x \circ y, b \circ c) - b \circ S_2(x \circ y, c).$$

Therefore,

$$(x \circ y) \circ (b \circ c) \in I_2.$$

In other words, $A^{(2)} \subseteq I_2$.

Now assume that $A^{(n-1)} \subseteq I_{n-1}$. Then by Lemma 3.9,

$$A^{(n)} = A^{(n-1)} \circ A^{(n-1)} \subseteq I_{n-1} \circ I_{n-1} \subseteq I_n.$$

□

Proof of Theorem 1.2. Let A be a solvable Zinbiel algebra with solvability length N and let $p = 0$ or $p > 2^N - 1$. Prove that A is nil with nil-index 2^N by induction on N. For $N = 1$, the statement is obvious. Assume that the condition $A^{(N-1)} = 0$ implies $a^{\cdot 2^{N-1}} = 0$ for every $a \in A$.

Now assume that $A^{(N)} = 0$. Then $\bar{A}^{(N-1)} = 0$ for $\bar{A} = A/A^{(N-1)}$. Therefore, by the induction hypothesis, $a^{\cdot 2^{N-1}} \in A^{(N-1)}$ for all $a \in A$. Thus,

$$a^{\cdot 2^{N-1}} \circ a^{\cdot 2^{N-1}} \in A^{(N)} = 0.$$

By Corollary 3.2,

$$a^{\cdot 2^{N-1}} \circ a^{\cdot 2^{N-1}} = \binom{2^N - 1}{2^{N-1}} a^{\cdot 2^N}.$$

Therefore, if $p > 2^N - 1$, then A is nil with nil-index 2^N.

Now we prove that A is solvable with solvability length N if A is nil with nil-index N. If $N = 2$, then

$$(d \circ e) \circ (b \circ c) = S_2(d \circ e, b \circ c) - b \circ S_2(d \circ e, c).$$

Thus, by Lemma 3.5, $A^{(2)} \subseteq I_2$. By Lemma 3.10, $A^{(n)} \subseteq I_n$ if $p = 0$ or $n > p - 1$. Hence, A is solvable with solvability length N if $I_N = 0$ and $p = 0$ or $N - 1 < p$. □

4. Proof of theorem 1.3

Let n be any positive integer. For $x, y \in A$, write $x \equiv y$ if $x - y \in I_n$. Note that for any $a \in A$,

$$a \circ x \equiv a \circ y, \quad x \circ a \equiv y \circ a$$

if $x \equiv y$.

Lemma 4.1. *For any* $a_1, \ldots, a_{n-1}, b \in A$,

$$S_{n-1}(a_1, \ldots, a_{n-1}) \circ b \equiv -b \circ S_{n-1}(a_1, \ldots, a_{n-1}).$$

Proof. By Lemma 3.3,

$$b \circ S_{n-1}(a_1, a_2, \ldots, a_{n-1}) + S_{n-1}(a_1, a_2, \ldots, a_{n-1}) \circ b$$
$$= S_n(a_1, a_2, \ldots, a_{n-1}, b) \in I_n.$$

The lemma is proved. □

Lemma 4.2. *For any* $x_1, \ldots, x_{n-1}, f, e \in A$,

$$f \circ (S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e) \equiv S_2(f, S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e),$$
$$f \circ (e \circ S_{n-1}(x_1, x_2, \ldots, x_{n-1})) \equiv S_2(f, e \circ S_{n-1}(x_1, x_2, \ldots, x_{n-1})).$$

Proof. By Lemma 4.1,

$$f \circ (S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e) = -f \circ (e \circ S_{n-1}(x_1, x_2, \ldots, x_{n-1})) + y_1,$$

where

$$y_1 = f \circ S_n(x_1, \ldots, x_{n-1}, e) \in I_n.$$

Therefore, by Lemma 3.8,

$$f \circ (S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e) = S_2(f, S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e + y_2,$$

where by Lemma 3.7

$$y_2 = -S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ S_2(f, e) \in I_n.$$

Further,

$$f \circ (e \circ S_{n-1}(x_1, x_2, \ldots, x_{n-1})) \equiv -f \circ (S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e)$$
$$\equiv -S_2(f, S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e) \equiv S_2(f, e \circ S_{n-1}(x_1, x_2, \ldots, x_{n-1})).$$

The lemma is proved. □

Lemma 4.3. *For any* $a, b, x_1, \dots, x_{n-1}, e \in A$,

$$a \circ (b \circ (S_{n-1}(x_1, x_2, \dots, x_{n-1}) \circ e)) \equiv (a \circ b) \circ (S_{n-1}(x_1, x_2, \dots, x_{n-1}) \circ e),$$
$$a \circ (b \circ (e \circ S_{n-1}(x_1, x_2, \dots, x_{n-1}))) \equiv (a \circ b) \circ (e \circ S_{n-1}(x_1, x_2, \dots, x_{n-1})).$$

Proof. By Lemma 4.2,

$$a \circ (b \circ (S_{n-1}(x_1, x_2, \dots, x_{n-1}) \circ e)) \equiv a \circ S_2(b, S_{n-1}(x_1, \dots, x_{n-1}) \circ e)$$

(Zinbiel identity)

$$\equiv (a \circ b) \circ (S_{n-1}(x_1, x_2, \cdots x_{n-1}) \circ e).$$

Therefore,

$$a \circ (b \circ (e \circ S_{n-1}(x_1, x_2, \cdots x_{n-1})))$$

(see Lemma 3.3)

$$\equiv a \circ (b \circ (-S_{n-1}(x_1, x_2, \cdots x_{n-1}) \circ e))$$
$$\equiv -(a \circ b) \circ (S_{n-1}(x_1, x_2, \cdots x_{n-1}) \circ e)$$

(see Lemma 3.3)

$$\equiv (a \circ b) \circ (e \circ S_{n-1}(x_1, x_2, \cdots x_{n-1})).$$

The lemma is proved. □

Proof of Theorem 1.3. Use induction on the nil-index n. Let $n = 2$. By Lemma 3.8, any Zinbiel algebra with identity $a^{\cdot 2} = 0$ is nilpotent with nil-index 3: $a \circ (b \circ c) = 0$ for any $a, b, c \in A$.

Assume that for any $a_1, \dots, a_k \in A$,

$$a_1 \circ (a_2 \circ (\cdots (a_{k-1} \circ a_k))) \in I_{n-1}$$

for some $k \leq 2^{n-1} - 1$. Prove that for any $a_1, \dots, a_{2k+1} \in A$,

$$a_1 \circ (a_2 \circ (\cdots (a_{2k} \circ a_{2k+1}))) \in I_n.$$

By the induction assumption,

$$a_{k+2} \circ (\cdots (a_{2k} \circ a_{2k+1})) \in I_{n-1}.$$

Therefore, by Lemma 3.5,

$$a_{k+2} \circ (\cdots (a_{2k} \circ a_{2k+1})) = S_{n-1}(x_1, \dots, x_{n-1})$$

or

$$a_{k+2} \circ (\cdots (a_{2k} \circ a_{2k+1})) = y_1 \circ (y_2 \circ (\cdots (y_s \circ S_{n-1}(x_1, \dots, x_{n-1}))))$$

for some $x_1, \dots, x_{n-1}, y_1, \dots, y_s \in A$.

In the first case, by Lemma 4.3,

$$a_1 \circ (a_2 \circ (\cdots (a_k \circ (a_{k+1} \circ S_{n-1}(x_1, x_2, \cdots, x_{n-1})))))$$
$$\equiv (a_1 \circ (a_2 \circ (\cdots (a_{k-1} \circ a_k)))) \circ (a_{k+1} \circ S_{n-1}(x_1, \dots, x_{n-1})).$$

By the induction assumption,

$$(a_1 \circ (a_2 \circ (\cdots (a_{k-1} \circ a_k)))) \in I_{n-1}.$$

Therefore, by Lemma 3.9,

$$a_1 \circ (a_2 \circ (\cdots (a_{2k} \circ a_{2k+1}))) \in I_{n-1} \circ I_{n-1} \subseteq I_n.$$

In the second case, by Lemma 4.3,

$$\begin{gathered} a_k \circ (a_{k+1} \circ (\cdots (a_{2k} \circ a_{2k+1}))) \\ \equiv a_k \circ (a_{k+1} \circ (y_1 \circ (y_2 \circ (\cdots (y_s \circ S_{n-1}(x_1, \ldots, x_{n-1})))))) \\ \equiv b \circ (S_{n-1}(x_1, \ldots, x_{n-1}) \circ e), \end{gathered}$$

where

$$b = -a_k \circ (a_{k+1} \circ (y_1 \circ (\cdots (y_{s-2} \circ y_{s-1})))) \in A, \quad e = y_s \in A.$$

Thus, by Lemma 4.3,

$$\begin{gathered} a_1 \circ (\cdots \circ (a_{k-1} \circ (b \circ (S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e)))) \\ \equiv (a_1 \circ (\cdots (a_{k-1} \circ b))) \circ (S_{n-1}(x_1, x_2, \ldots, x_{n-1}) \circ e) \end{gathered}$$

By the induction assumption,

$$a_1 \circ (\cdots (a_{k-1} \circ b)) \in I_{n-1}.$$

Therefore, by Lemma 3.9,

$$a_1 \circ (\cdots (a_{2k} \circ a_{2k+1})) \in I_{n-1} \circ I_{n-1} \subseteq I_n.$$

We obtain that the right-bracketed product of any $2k + 1 \leq 2^n - 1$ elements of A belongs to I_n. In other words, any Zinbiel nil-algebra is nilpotent.

Any solvable algebra with solvability index N is nil if $p = 0$ or $p > 2^N - 1$. Any nil-algebra, as we have proved above, is nilpotent. Any nilpotent algebra is solvable. □

5. Proof of Theorem 1.7

Before giving the proof, recall some facts about central extensions of algebras.

Let A be a Zinbiel algebra, $C^1(A, K)$ be a space of linear forms $f : A \to K$, $C^2(A, K)$ be a space of bilinear forms $\psi : A \times A \to K$, and $C^3(A, K)$ be a space of trilinear forms $\phi : A \times A \times A \to K$. Recall the definitions of coboundary operators for small degrees:

$$d : C^1(A, K) \to C^2(A, K)$$

is given by

$$df(a, b) = -f(a \circ b)$$

and

$$d : C^2(A, K) \to C^3(A, K)$$

is given by

$$d\psi(a,b,c) = \psi(a \circ b, c) - \psi(a, b \circ c) - \psi(a, c \circ b).$$

Then $B^2(A,K)$ is a space of bilinear forms of the form df, where $f \in C^1(A,K)$, and $Z^2(A,K)$ is a space of bilinear forms ψ such that $d\psi = 0$. It is easy to verify that $d^2 f = 0$ for any linear form $f : A \to K$. Therefore, for any Zinbiel algebra A,

$$B^2(A,K) \subseteq Z^2(A,K).$$

The second cohomology space is defined as follows:

$$H^2(A,K) = Z^2(A,K)/B^2(A,K).$$

Standard homological arguments show that $H^2(A,K)$ can be interpreted as a space of central extensions of A:

$$0 \to Z \to \tilde{A} \to A \to 0.$$

In other words, any algebra $\tilde{A}$ with abelian ideal Z is equal as a vector space to the direct sum $A \oplus Z$ and the multiplication in $\tilde{A}$ is given by

$$(a + z) \circ (a_1 + z_1) = a \circ a_1 + \eta(a, a_1),$$

where a bilinear mapping $\eta : A \times A \to Z$ satisfies the relation

$$\eta(a \circ b, c) - \eta(a, b \circ c) - \eta(a, c \circ b) = 0 \quad \forall a, b, c \in A.$$

If for some linear mapping $\omega : A \to Z$,

$$\eta(a,b) = -\omega(a \circ b) \quad \forall a, b \in A,$$

then the algebra $\tilde{A}$ under this multiplication is isomorphic to the direct sum of the algebras $A \oplus Z$.

This interpretation of the second cohomology spaces will be used in describing algebras of small dimensions.

We will use one more result. Assume that A is ableian: $a \circ b = 0$ for any $a, b \in A$. Then $B^2(A,K) = 0$. Therefore, for any abelian algebra A of dimension n, the second cohomology space is isomorphic to n^2-dimensional matrix space:

$$H^2(A,K) = Z^2(A,K) \cong \mathrm{Mat}_n\,.$$

Proof of Theorem 1.7. It is easy to verify that all algebras mentioned in Theorem 1.7 are Zinbiel.

If $\dim A = 1$ and A is generated by the basis element e_1, then $e_1 \circ e_1 = \alpha e_1$ for some $\alpha \in K$. By the Zinbiel identity,

$$\mathrm{zinbiel}(e_1, e_1, e_1) = 0 \Rightarrow \alpha^2 e_1 = 0 \Rightarrow \alpha_1 = 0.$$

By Corollary 1.6, for any Zinbiel algebra A over an algebraically closed field of characteristic 0 or $p > 7$, there exists the nontrivial center $Z(A)$ and an exact extension of Zinbiel algebras

$$0 \to Z(A) \to A \to \bar{A} \to 0$$

holds. In other words, $A/Z(A) \cong \bar{A}$. Therefore, the classification of algebras A is equivalent to the problem of calculation of second cohomology group $H^2(\bar{A}, K)$.

Let $A = \langle e_1 \rangle$ be a one-dimensional Zinbiel algebra. Since any one-dimensional algebra is abelian, $H^2(A, K)$ is one-dimensional and is generated by a cocycle

$$\psi(e_1, e_1) = 1.$$

Therefore, any 2-dimensional Zinbiel algebra $\tilde{A} = \langle e_1, e_2 \rangle$ with the central element e_2 has the following multiplication table:

$$e_1 \circ e_1 = \beta e_2, \quad e_1 \circ e_2 = 0, \quad e_2 \circ e_1 = 0, \quad e_2 \circ e_2 = 0.$$

If $\beta = 0$, then we obtain the algebra $Q(0)$. If $\beta \neq 0$ under the new basis $\{1/\sqrt{\beta} e_1, e_2\}$, then we obtain the algebra $Q(1)$.

Since $Q(0)$ is abelian and two-dimensional, $H^2(Q(0), K)$ is four-dimensional and is generated by four cocycles ψ_i, $i = 1, 2, 3, 4$, such that

$$\psi_1(e_1, e_1) = 1, \quad \psi_2(e_1, e_2) = 1, \quad \psi_3(e_2, e_1) = 1, \quad \psi_4(e_2, e_2) = 1$$

(non-written components are 0). Therefore, any three-dimensional extension of $Q(0)$ by the one-dimensional center is equivalent to $R(\alpha, \beta, \gamma, \delta)$. Take a new basis in $R(\alpha, \beta, \gamma, \delta)$. Under the basis $\{1/\sqrt{\alpha} e_1, e_2, e_3\}$, we obtain the algebra $R(1, \beta, \gamma, \delta)$ if $\alpha \neq 0$. Similarly, the new basis $\{e_1, 1/\sqrt{\delta} e_2, e_3\}$ gives us the algebra $R(\alpha, \beta, \gamma, 1)$ if $\delta \neq 0$.

Now we calculate the second cohomology of $Q(1)$. Note that there are six cocyclicity conditions $d\psi(e_i, e_j, e_s) = 0$, where $i, j, s = 1, 2$, $j \leq s$. They give us the following three nontrivial relations:

$$\psi(e_1, e_1) = 1, \quad 2\psi(e_1, e_2) = \psi(e_2, e_1), \quad \psi(e_2, e_2) = 0.$$

Therefore, $Z^2(Q(1), K)$ is two-dimensional and is generated by the cocycles ψ_1 and ψ_2 such that

$$\psi_1(e_1, e_1) = 1, \quad \psi_2(e_2, e_1) = 1, \quad \psi_2(e_1, e_2) = 1/2$$

(non-written components are 0). Note that $\psi_1 = d\omega$ for $\omega \in C^1(Q(1), K)$ given by $\omega(e_2) = -1$. Therefore, $H^2(Q(1), K)$ is one-dimensional and is generated by a class of the cocycle ψ_2. The corresponding central extension is equivalent to the algebra $\tilde{A} = Q(1) + K$ with the following multiplication table:

$$e_1 \circ e_1 = e_2, \quad e_1 \circ e_2 = \frac{\alpha}{2} e_3, \quad e_2 \circ e_1 = \alpha e_3, \quad e_2 \circ e_2 = 0,$$

where $Q(1) = \langle e_1, e_2 \rangle$ and one-dimensional center element is denoted by e_3. Note that in $\tilde{A}$, one can obtain the new basis $\{e_1, e_2, 1/\sqrt{\alpha} e_3\}$ if $\alpha \neq 0$. Under this basis, we obtain the algebra $W(3)$. If $\alpha = 0$, then we obtain the algebra $R(1, 0, 0, 0)$.

A direct calculation shows that Theorem 1.7 is true also for cases $p = 2, 3, 5, 7$. □

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