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ZINBIEL ALGEBRAS UNDER q-COMMUTATORS

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ABSTRACT. An algebra with the identity $t_1(t_2t_3) = (t_1t_2+t_2t_1)t_3$ is called Zinbiel. For example, $\mathbb{C}[x]$ under the multiplication $a \circ b = b\int_0^x a\,dx$ is Zinbiel. Let $a \circ_q b = a \circ b + q\, b \circ a$ be a q-commutator, where $q \in \mathbb{C}$. We prove that for any Zinbiel algebra A the corresponding algebra under the commutator $A^{(-1)} = (A, \circ_{-1})$ satisfies the identities $t_1t_2 = -t_2t_1$ and $(t_1t_2)(t_3t_4) + (t_1t_4)(t_3t_2) = \mathsf{jac}(t_1, t_2, t_3)t_4 + \mathsf{jac}(t_1, t_4, t_3)t_2$, where $\mathsf{jac}(t_1, t_2, t_3) = (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2$. We find basic identities for q-Zinbiel algebras and prove that they form varieties equivalent to the variety of Zinbiel algebras if $q^2 \neq 1$.

1. Introduction

Zinbiel algebras appear in control theory and in the cohomology theory of Leibniz algebras. An algebra $A = (A, \circ)$ with vector space A and multiplication $\circ$ is called (right)-*Zinbiel* if it satisfies the condition

$$a \circ (b \circ c) = (a \circ b + b \circ a) \circ c$$

for any $a, b, c \in A$ [1, 2, 4, 5]. Sometimes Zinbiel algebras are called *Leibniz dual algebras* or *chronological algebras*.

Example. The algebra $A = \mathbb{C}[x]$ with multiplication $a \circ b = b\int_0^x a\,dx$ or $a \diamond b = \int_0^x \frac{\partial}{\partial x}(a)b\,dx$ is right-Zinbiel.

For simplicity, we assume that all polynomials and vector fields are defined over the field of complex numbers $\mathbb{C}$. In fact, almost all of our results hold for any field K of characteristic $p \neq 2, 3$. For $q \in \mathbb{C}$, we define the *q-commutator* on A by

$$a \circ_q b = a \circ b + q\, b \circ a.$$

For example,

$$a \circ_{-1} b = a \circ b - b \circ a = [a, b] \qquad \text{(the commutator)},$$
$$a \circ_1 b = a \circ b + b \circ a = \{a, b\} \qquad \text{(the anti-commutator)}.$$

Let $A^{(q)}$ be an algebra with vector space A and multiplication $\circ_q$.

Denote by $\mathfrak{Zinbiel}$ the category of Zinbiel algebras. Let $\mathfrak{Zinbiel}^{(q)}$, $q \in \mathbb{C}$, be the category of q-Zinbiel algebras. In other words, the objects of $\mathfrak{Zinbiel}^{(q)}$ are algebras of the form $A^{(q)}$, where $A \in \mathfrak{Zinbiel}$.

The aim of our paper is to find identities for $\mathfrak{Zinbiel}^{(q)}$. The case $q = 1$ was considered in [5]. Namely, it was established that any Zinbiel algebra under an anti-commutator is associative and commutative. In our paper, we prove that any Zinbiel algebra under a commutator satisfies one identity of degree 4 (we denote it $\mathsf{tortkara}$), and this identity is a Lie identity. Here we call an identity *Lie* if it does not follow from the skew-symmetric identity and it is minimal among such identities. For $q \neq \pm 1$, we prove that the class $\mathfrak{Zinbiel}^{(q)}$ forms a variety, and we find its basic identities. Moreover, we establish that this variety is equivalent to the variety $\mathfrak{Zinbiel}$. We show that basic identities for $\mathfrak{Zinbiel}^{(q)}$ differ in the cases $q = -2$ and $q \neq \pm 1, -2$.

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From our results, it follows that the algebra $(\mathbb{C}[x], \circ)$, where

$$a \circ b = a \int_0^x b\, dx,$$

is Zinbiel and the corresponding algebra under a commutator satisfies the identity $\mathsf{tortkara} = 0$.

Another application: the algebra $(\mathbb{C}[x], \diamond)$ with the multiplication

$$a \diamond b = \int_0^x \partial(a) b\, dx,$$

where $\partial = \frac{\partial}{\partial x}$, is also Zinbiel, and the corresponding algebra under the commutator

$$[a, b] = \int_0^x (\partial(a)b - a\partial(b))\, dx$$

satisfies the identity $\mathsf{tortkara} = 0$.

We find identities for the algebra $(\mathbb{C}[x], \star)$, where

$$a \star b = a \int_0^x \left(\int_0^x b\, dx \right) dx.$$

It is not Zinbiel, but the corresponding algebra under a commutator satisfies the identity $\mathsf{tortkara} = 0$.

2. Main Results

Let $f = f(t_1, t_2, \dots, t_k)$ be a nonassociative, noncommutative polynomial. Let $A = (A, \circ)$ be an algebra with vector space A and multiplication $\circ$. We say that $f = 0$ is an *identity* on A if $f(a_1, a_2, \dots, a_k) = 0$ for any substitutions $t_i := a_i \in A$, where $i = 1, 2, \dots, k$. Here we calculate $f(a_1, a_2, \dots a_k)$ in terms of the multiplication $\circ$.

Let, for example,

$$\mathsf{ass}(t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_2)t_3 \qquad \text{(the associator)}.$$

Then any algebra with the identity $\mathsf{ass} = 0$ is associative.

A class of algebras $\mathfrak{L}$ is called a *variety* with generating polynomial identities $f_1 = 0,\ f_2 = 0, \dots$ if any algebra $A \in \mathfrak{L}$ satisfies these identities and, inversely, any algebra with identities $f_1 = 0,\ f_2 = 0, \dots$ belongs to $\mathfrak{L}$.

Let $\mathfrak{L}$ be a category of algebras with some polynomial identity. Denote by $\mathfrak{L}^{(q)}$ the category of algebras $A^{(q)}$, where $A \in \mathfrak{L}$.

If $\mathfrak{Ass}$ is a category of associative algebras, then for any $A \in \mathfrak{Ass}$, $A^{(-1)}$ is a Lie algebra. In other words, $A^{(-1)}$ satisfies the skew-symmetric identity $\mathsf{com} = 0$ and the Jacobi identity $\mathsf{jac} = 0$, where

$$\mathsf{com}(t_1, t_2) = t_1t_2 + t_2t_1,$$
$$\mathsf{jac}(t_1, t_2, t_3) = (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2.$$

The Poincaré–Birkhoff–Witt theorem shows that $\mathfrak{Ass}^{(-1)}$ forms a variety and this variety is generated by the identities $\mathsf{com} = 0$ and $\mathsf{jac} = 0$. Further, for any $A \in \mathfrak{Ass}$ its anti-commutator algebra $A^{(1)}$ is Jordan, i.e., satisfies the commutativity identity $\mathsf{anticom} = 0$ and the Jordan identity $\mathsf{jor} = 0$, where

$$\mathsf{anticom}(t_1, t_2) = t_1t_2 - t_2t_1,$$
$$\mathsf{jor}(t_1, t_2, t_3, t_4) = (t_1t_2)(t_3t_4) + (t_1t_3)(t_4t_2) + (t_1t_4)(t_2t_3) - (t_1(t_3t_4))t_2 - (t_1(t_4t_2))t_3 - (t_1(t_2t_3))t_4.$$

But the category $\mathfrak{Ass}^{(1)}$ does not form a variety [6]. Algebras $A^{(1)}$ satisfy some polynomial identities. The basis of such identities is not known. There exists some algebra that satisfies all identities of $\mathfrak{Ass}^{(1)}$, but it is not isomorphic to any subalgebra of any Jordan algebra of the form $A^{(1)}$, where $A \in \mathfrak{Ass}$.

Let

$$\mathsf{zinbiel}(t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_1 + t_2t_1)t_3$$

be the (right)-*Zinbiel* polynomial. By a Zinbiel polynomial we understand a right-Zinbiel one unless otherwise stated.

Let

$$\begin{aligned}
\mathsf{zinbiel}^{(q)} &= \mathsf{zinbiel}^{(q)}(t_1, t_2, t_3) \\
&= (1 - q - q^2)t_1(t_2t_3 + t_3t_2) - t_2(t_1t_3 + t_3t_1) - t_3(t_1t_2) + (1 + q)t_3(t_2t_1) + (2q + q^2)(t_1t_2)t_3, \\
\mathsf{zinbiel}_1^{(q)} &= \mathsf{zinbiel}_1^{(q)}(t_1, t_2, t_3) = -t_2(t_3t_1) + t_3(t_2t_1) + q(t_1t_2)t_3 - q(t_1t_3)t_2, \\
\mathsf{zinbiel}_2^{(-2)} &= \mathsf{zinbiel}_2^{(-2)}(t_1, t_2, t_3) = -3t_2(t_1t_3 + t_3t_1) - 2t_3(t_1t_2) - t_3(t_2t_1) - (t_1t_2)t_3 + (t_2t_1)t_3.
\end{aligned}$$

We see that

$$\mathsf{zinbiel}^{(0)} = \mathsf{zinbiel}.$$

Let

$$\mathsf{tortkara} = \mathsf{tortkara}(t_1, t_2, t_3, t_4) = (t_1t_2)(t_3t_4) + (t_1t_4)(t_3t_2) - \mathsf{jac}(t_1, t_2, t_3)t_4 - \mathsf{jac}(t_1, t_4, t_3)t_2.$$

Note that

$$\mathsf{tortkara}(t_1, t_2, t_3, t_4) = \mathsf{tortkara}(t_1, t_4, t_3, t_2).$$

If we consider the polynomial $\mathsf{tortkara}$ as a skew-symmetric polynomial, i.e., the identity $\mathsf{com} = 0$ is given, then

$$\mathsf{tortkara}(t_1, t_2, t_3, t_4) = -\mathsf{tortkara}(t_3, t_2, t_1, t_4),$$

$$\mathsf{tortkara}(t_1, t_2, t_3, t_4) + \mathsf{tortkara}(t_1, t_3, t_4, t_2) + \mathsf{tortkara}(t_1, t_4, t_2, t_3) = 0.$$

In our paper, we establish the following results.

Theorem 2.1. *Let $q^2 \neq 1$. Then $\mathfrak{Zinbiel}^{(q)}$ forms a variety. If $q \neq -2$, then $\mathfrak{Zinbiel}^{(q)}$ is generated by the polynomial identity $\mathsf{zinbiel}^{(q)} = 0$. If $q = -2$, then $\mathfrak{Zinbiel}^{(-2)}$ is generated by the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$. The variety generated by the polynomial identity $\mathsf{zinbiel}^{(q)} = 0$ is equivalent to the variety $\mathfrak{Zinbiel}$ if $q \neq -2$. The variety generated by the polynomial identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$ for $q = -2$ is equivalent to the variety $\mathfrak{Zinbiel}$.*

Theorem 2.2. *For any Zinbiel algebra $(A, \circ)$, its Lie algebra $(A, [\,,\,])$, where $[a, b] = a \circ b - b \circ a$, satisfies the identity $\mathsf{tortkara} = 0$. Any identity of degree 3 of the category $\mathfrak{Zinbiel}^{(-1)}$ follows from the identity $\mathsf{com} = 0$. Any identity of degree 4 for the category $\mathfrak{Zinbiel}^{(-1)}$ follows from the identities $\mathsf{com} = 0$ and $\mathsf{tortkara} = 0$.*

Theorem 2.3. *The algebra $(\mathbb{C}[x], \star)$, where*

$$a \star b = a \int_0^x \left(\int_0^x b \, dx \right) dx,$$

satisfies the right-symmetry identity

$$(t_1t_2)t_3 - (t_1t_3)t_2 = 0 \tag{1}$$

and the identity of degree 4

$$(t_1, t_2, [t_3, t_4]) + (t_1, t_3, [t_4, t_2]) + (t_1, t_4, [t_2, t_3]) = 0. \tag{2}$$

Let A be any algebra with identity (1) *and the identity*

$$(t_1, t_2, [t_3, t_1]) + (t_1, t_3, [t_1, t_2]) + (t_1, t_1, [t_2, t_3]) = 0. \tag{3}$$

Then its Lie algebra $A^{(-1)}$ satisfies the identity $\mathsf{tortkara} = 0$*. In particular, any algebra with identities* (1) *and* (2) *satisfies the identity* $\mathsf{tortkara} = 0$*.*

Thus, for $q^2 \neq 1$ any algebra $L \in \mathfrak{Zinbiel}^{(q)}$ satisfies some polynomial identities and any algebra with these identities can be obtained by some algebra $A \in \mathfrak{Zinbiel}$ as an algebra $A^{(q)}$. More precisely, we establish that $\mathsf{zinbiel} = 0$ (if $q \neq -2$) and $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$ (if $q = -2$) are generating identities for $\mathfrak{Zinbiel}^{(q)}$. We prove that

$$\mathfrak{Zinbiel} = \mathrm{Var}(\mathfrak{Zinbiel}^{(q)}) \qquad \text{if } q \neq \pm 1$$

and

$$\mathrm{Var}(\mathfrak{Zinbiel}^{(q)}) = \langle \mathsf{zinbiel}^{(q)} \rangle \qquad \text{if } q \neq -2, \pm 1,$$
$$\mathrm{Var}(\mathfrak{Zinbiel}^{(q)}) = \langle \mathsf{zinbiel}_1^{(-2)}, \mathsf{zinbiel}_2^{(-2)} \rangle \qquad \text{if } q = -2.$$

Moreover, the categories $\mathfrak{Zinbiel}^{(q)}$ and $\mathfrak{Zinbiel}$ are equivalent if $q^2 \neq 1$. In other words, any algebra with the identity $\mathsf{zinbiel}^{(q)} = 0$ is isomorphic to an algebra of the form $A^{(q)}$ for some $A \in \mathfrak{Zinbiel}$ if $q \neq -2, \pm 1$. If $q = -2$, then any algebra with the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$ can be obtained by some Zinbiel algebra A as $A^{(-2)}$.

It should be interesting to study varieties of algebras with the identities $\mathsf{zinbiel}_1^{(q)} = 0$ or $\mathsf{zinbiel}_2^{(-2)} = 0$. We do not know whether the category $\mathfrak{Zinbiel}^{(-1)}$ forms a variety. It seems that $\mathfrak{Zinbiel}^{(-1)}$ satisfies more identities that do not follow from the identities $\mathsf{com} = 0$ and $\mathsf{tortkara} = 0$. It should be interesting also to construct identities for $(\mathbb{C}[x], \{\ \})$, where $\{a, b\} = a\star b + b\star a$ is the anti-commutator for the multiplication $\star$.

Remark. Let

$$\mathsf{rcom}(t_1, t_2, t_3) = (t_1 t_2)t_3 - (t_1 t_3)t_2$$

be the right-commutativity polynomial and

$$\mathsf{genzinbiel}(t_1, t_2, t_3) = \mathsf{zinbiel}(t_1, t_2, t_3) - \mathsf{zinbiel}(t_3, t_2, t_1)$$

be the *generalized Zinbiel* polynomial. Then any algebra with the identities $\mathsf{rcom} = 0$ and $\mathsf{genzinbiel} = 0$ satisfies the identity (2). Hence such algebras under commutators satisfy the identity $\mathsf{tortkara} = 0$.

Similar identities appear in considering Novikov algebras under an anti-commutator [3]. Let $A = \mathbb{C}[x]$ and $a * b = \partial(a)b$, where $\partial\colon \mathbb{C}[x] \to \mathbb{C}[x]$ is the derivation:

$$\partial x^i = i x^{i-1}.$$

Then $(A, *)$ is a Novikov algebra:

$$a * (b * c - c * b) = (a * b) * c - (a * c) * b,$$
$$a * (b * c) = b * (a * c)$$

for any $a, b, c \in A$. For any Novikov algebra $(A, *)$, its Jordan algebra $(A, \{\ ,\ \})$, where $\{a, b\} = a*b + b*a$, is commutative and satisfies the identity $\mathsf{tortken} = 0$, where

$$\mathsf{tortken}(t_1, t_2, t_3, t_4) = (t_1 t_2)(t_3 t_4) - (t_1 t_4)(t_3 t_2) - (t_1, t_2, t_3)t_4 + (t_1, t_4, t_3)t_2,$$
$$(t_1, t_2, t_3) = t_1(t_2 t_3) - (t_1 t_2)t_3.$$

For example, $K[x]$ under the multiplication $(a, b) \mapsto \partial(ab)$ satisfies these identities.

We do not know whether the algebra $(\mathbb{C}[x], [\,,\,])$ under the commutator $[a,b] = a\star b - b\star a$ is isomorphic to a subalgebra of some algebra of the form $A^{(-1)}$, where A is a Zinbiel algebra. Recall that

$$a \star b = a \int_0^x \int_0^x b\, dx\, dx.$$

In our paper, we study q-identities for right-Zinbiel algebras. One can consider q-identities for left-Zinbiel algebras:

$$(a \circ b) \circ c = a \circ (b \circ c + c \circ b).$$

Note that any right-Zinbiel algebra $(a, \circ)$ under opposite multiplication $a \circ^{\mathrm{op}} b = b \circ a$ can be made left-Zinbiel. Similarly, the algebra $(\mathbb{C}[x], \star^{\mathrm{op}})$ with opposite multiplication

$$a \star^{\mathrm{op}} b = \Big(\iint a\Big) b$$

satisfies the identity

$$([t_1, t_2], t_3, t_4) + ([t_2, t_3], t_1, t_4) + ([t_3, t_1], t_2, t_4) = 0.$$

Since $[\,,\,]^{\mathrm{op}} = -[\,,\,]$, it is clear that its commutator algebra satisfies the identity $\mathsf{tortkara} = 0$.

3. Why are $\mathsf{zinbiel}^{(q)}$, $\mathsf{zinbiel}_1^{(q)}$, and $\mathsf{zinbiel}_2^{(-2)}$ identities?

Lemma 3.1. *Let $A = (A, \circ)$ be any Zinbiel algebra. Then $\mathsf{zinbiel}^{(q)} = 0$ is an identity on $A^{(q)}$.*

Proof. We have

$$\begin{aligned}
&\mathsf{zinbiel}^{(q)}(a,b,c)\\
&\quad = (1-q-q^2) a \circ_q (b \circ_q c + c \circ_q b) - b \circ_q (a \circ_q c + c \circ_q a) - c \circ_q (a \circ_q b)\\
&\quad + (1+q) c \circ_q (b \circ_q a) + (2q+q^2)(a \circ_q b) \circ_q c\\
&\quad = (1-q-q^2) a \circ (b \circ c + c \circ b) - b \circ (a \circ c + c \circ a) - c \circ (a \circ b) + (1+q) c \circ (b \circ a)\\
&\quad + (2q+q^2)(a \circ b) \circ c + (1-q-q^2) q (b \circ c + c \circ b) \circ a - q(a \circ c + c \circ a) \circ b - q(a \circ b) \circ c\\
&\quad + (1+q) q (b \circ a) \circ c + (2q+q^2) q c \circ (a \circ b) + (1-q-q^2) q a \circ (b \circ c + c \circ b) - q b \circ (c \circ a + a \circ c)\\
&\quad - q c \circ (b \circ a) + (1+q) q c \circ (a \circ b) + (2q+q^2) q (b \circ a) \circ c + (1-q-q^2) q^2 (b \circ c + c \circ b) \circ a\\
&\quad - q^2 (a \circ c + c \circ a) \circ b - q^2 (b \circ a) \circ c + (1+q) q^2 (a \circ b) \circ c + (2q+q^2) q^2 c \circ (b \circ a)\\
&\quad = F_0 + F_1 q + F_2 q^2 + F_3 q^3 + F_4 q^4,
\end{aligned}$$

where

$$\begin{aligned}
F_0 &= a \circ (b \circ c) + a \circ (c \circ b) - b \circ (a \circ c) - b \circ (c \circ a) - c \circ (a \circ b) + c \circ (b \circ a),\\
F_1 &= -b \circ (a \circ c) - b \circ (c \circ a) + c \circ (a \circ b) + (a \circ b) \circ c - (a \circ c) \circ b + (b \circ a) \circ c + (b \circ c) \circ a\\
&\quad - (c \circ a) \circ b + (c \circ b) \circ a,\\
F_2 &= -2a \circ (b \circ c) - 2a \circ (c \circ b) + 3c \circ (a \circ b) + 2(a \circ b) \circ c - (a \circ c) \circ b + 2(b \circ a) \circ c - (c \circ a) \circ b,\\
F_3 &= -a \circ (b \circ c) - a \circ (c \circ b) + c \circ (a \circ b) + 2c \circ (b \circ a) + (a \circ b) \circ c + (b \circ a) \circ c - 2(b \circ c) \circ a\\
&\quad - 2(c \circ b) \circ a,\\
F_4 &= c \circ (b \circ a) - (b \circ c) \circ a - (c \circ b) \circ a.
\end{aligned}$$

By the left-commutativity identity

$$F_0 = 0.$$

By the right-Zinbiel identity

$$F_1 = -b \circ (c \circ a) + (b \circ c) \circ a + (c \circ b) \circ a + c \circ (a \circ b) - (a \circ c) \circ b - (c \circ a) \circ b - b \circ (a \circ c) + (a \circ b) \circ c + (b \circ a) \circ c = 0,$$

$$F_2 = -2a \circ (c \circ b) + 3c \circ (a \circ b) - (a \circ c) \circ b - (c \circ a) \circ b - 2a \circ (b \circ c) + 2(a \circ b) \circ c + 2(b \circ a) \circ c = 0,$$

$$F_3 = 2c \circ (b \circ a) - 2(b \circ c) \circ a - 2(c \circ b) \circ a - a \circ (c \circ b) + c \circ (a \circ b) - a \circ (b \circ c) + (a \circ b) \circ c + (b \circ a) \circ c = 0,$$

$$F_4 = c \circ (b \circ a) - (b \circ c) \circ a - (c \circ b) \circ a = 0.$$

Lemma 3.2. *Let $A = (A, \circ)$ be any Zinbiel algebra. Then $\mathsf{zinbiel}_1^{(q)} = 0$ is an identity on $A^{(q)}$.*

Proof.

$$\begin{aligned}
&\mathsf{zinbiel}_1^{(q)}(a, b, c) \\
&\quad = -b \circ_q (c \circ_q a) + c \circ_q (b \circ_q a) + q(a \circ_q b) \circ_q c - q(a \circ_q c) \circ_q b \\
&\quad = -b \circ (c \circ a) + c \circ (b \circ a) + q(a \circ b) \circ c - q(a \circ c) \circ b - q(c \circ a) \circ b + q(b \circ a) \circ c \\
&\quad + q^2 c \circ (a \circ b) - q^2 b \circ (a \circ c) - qb \circ (a \circ c) + qc \circ (a \circ b) + q^2(b \circ a) \circ c - q^2(c \circ a) \circ b \\
&\quad - q^2(c \circ a) \circ b + q^2(a \circ b) \circ c + q^3 c \circ (b \circ a) - q^3 b \circ (c \circ a) \\
&\quad = G_0 + G_1 q + G_2 q^2 + G_3 q^3,
\end{aligned}$$

where, by the right-Zinbiel identity,

$$\begin{aligned}
G_0 &= -b \circ (c \circ a) + c \circ (b \circ a) = 0, \\
G_1 &= -b \circ (a \circ c) + c \circ (a \circ b) + (a \circ b) \circ c - (a \circ c) \circ b + (b \circ a) \circ c - (c \circ a) \circ b = 0, \\
G_2 &= -b \circ (a \circ c) + c \circ (a \circ b) + (a \circ b) \circ c - (a \circ c) \circ b + (b \circ a) \circ c - (c \circ a) \circ b = 0, \\
G_3 &= -b \circ (c \circ a) + c \circ (b \circ a) = 0.
\end{aligned}$$

Lemma 3.3. *Let $A = (A, \circ)$ be any Zinbiel algebra. Then $\mathsf{zinbiel}_2^{(-2)} = 0$ is an identity on $A^{(-2)}$.*

Proof.

$$\begin{aligned}
&\mathsf{zinbiel}_2^{(-2)}(a, b, c) \\
&\quad = -3b \circ_{-2} (a \circ_{-2} c + c \circ_{-2} a) - 2c \circ_{-2} (a \circ_{-2} b) - c \circ_{-2} (b \circ_{-2} a) - (a \circ_{-2} b) \circ_{-2} c \\
&\quad + (b \circ_{-2} a) \circ_{-2} c \\
&\quad = -3b \circ (a \circ c + c \circ a) - 2c \circ (a \circ b) - c \circ (b \circ a) - (a \circ b) \circ c + (b \circ a) \circ c \\
&\quad + 6(a \circ c + c \circ a) \circ b + 4(a \circ b) \circ c + 2(b \circ a) \circ c + 2c \circ (a \circ b) - 2c \circ (b \circ a) \\
&\quad + 6b \circ (a \circ c + c \circ a) + 4c \circ (b \circ a) + 2c \circ (a \circ b) + 2(b \circ a) \circ c - 2(a \circ b) \circ c \\
&\quad - 12(a \circ c + c \circ a) \circ b - 8(b \circ a) \circ c - 4(a \circ b) \circ c - 4c \circ (b \circ a) + 4c \circ (a \circ b) \\
&\quad = H_1 + H_2 + H_3,
\end{aligned}$$

where, by the right-Zinbiel identity,

$$\begin{aligned}
H_1 &= -3b \circ (c \circ a) - c \circ (b \circ a) - 2c \circ (b \circ a) + 6b \circ (c \circ a) + 4c \circ (b \circ a) - 4c \circ (b \circ a) = 0, \\
H_2 &= -2c \circ (a \circ b) + 6(a \circ c) \circ b + 6(c \circ a) \circ b + 2c \circ (a \circ b) + 2c \circ (a \circ b) \\
&\quad - 12(a \circ c) \circ b - 12(c \circ a) \circ b + 4c \circ (a \circ b) = 6c \circ (a \circ b) - 6(a \circ c + c \circ a) \circ b = 0, \\
H_3 &= -3b \circ (a \circ c) - (a \circ b) \circ c + (b \circ a) \circ c + 4(a \circ b) \circ c + 2(b \circ a) \circ c \\
&\quad + 6b \circ (a \circ c) + 2(b \circ a) \circ c - 2(a \circ b) \circ c - 8(b \circ a) \circ c - 4(a \circ b) \circ c \\
&\quad = 3b \circ (a \circ c) - 3(a \circ b) \circ c - 3(b \circ a) \circ c = 0.
\end{aligned}$$

4. Any q-Identity of Degree 3 Follows from $\mathsf{zinbiel}^{(q)}$ Identities

Lemma 4.1. *Let $q^2 \neq 1$. Any identity of degree 3 for the category $\mathfrak{Zinbiel}^{(q)}$ follows from the identity $\mathsf{zinbiel}^{(q)} = 0$ if $q \neq -2$ and from the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$ if $q = -2$.*

Proof. Let

$$X(t_1, t_2, t_3) = \lambda_1 t_1(t_2 t_3) + \lambda_2 t_2(t_3 t_1) + \lambda_3 t_3(t_1 t_2) + \lambda_4 t_2(t_1 t_3) + \lambda_5 t_3(t_2 t_1) + \lambda_6 t_1(t_3 t_2) + \lambda_7 (t_1 t_2) t_3 + \lambda_8 (t_2 t_3) t_1 + \lambda_9 (t_3 t_1) t_2 + \lambda_{10} (t_2 t_1) t_3 + \lambda_{11} (t_3 t_2) t_1 + \lambda_{12} (t_1 t_3) t_2$$

be a generic, noncommutative, nonassociative polynomial of degree 3. Substitute here, instead of parameters t_1, t_2, and t_3, elements a, b, and c of a Zinbiel algebra $(A, \circ)$ and calculate $X(a, b, c)$ in terms of the multiplication $\circ$. Suppose that A is a free Zinbiel algebra with generators a, b, and c.

A free Zinbiel algebra in degree 3 has a 6-dimensional polylinear part with a basis

$$\{(a \circ b) \circ c,\ (b \circ c) \circ a,\ (c \circ a) \circ b,\ (b \circ a) \circ c,\ (c \circ b) \circ a,\ (a \circ c) \circ b\}.$$

By the (right)-Zinbiel identity, other 6 right-bracketed elements are linear combinations of basic elements:

$$\begin{aligned}
a \circ (b \circ c) &= (a \circ b) \circ c + (b \circ a) \circ c, \\
b \circ (c \circ a) &= (b \circ c) \circ a + (c \circ b) \circ a, \\
c \circ (a \circ b) &= (c \circ a) \circ b + (a \circ c) \circ b, \\
a \circ (c \circ b) &= (a \circ c) \circ b + (c \circ a) \circ c, \\
b \circ (a \circ c) &= (b \circ a) \circ c + (a \circ b) \circ c, \\
c \circ (b \circ a) &= (c \circ b) \circ a + (b \circ c) \circ a.
\end{aligned}$$

We obtain that

$$\begin{aligned}
&X(a, b, c) \\
&\quad = (\lambda_1 + q\lambda_2 + q\lambda_3 + \lambda_4 + q^2\lambda_5 + q\lambda_6 + \lambda_7 + q\lambda_8 + q^2\lambda_9 + q\lambda_{10} + q^2\lambda_{11} + q\lambda_{12})(a \circ b) \circ c \\
&\quad + (q\lambda_1 + q^2\lambda_2 + \lambda_3 + q\lambda_4 + q\lambda_5 + \lambda_6 + q\lambda_7 + q^2\lambda_8 + q\lambda_9 + q^2\lambda_{10} + q\lambda_{11} + \lambda_{12})(a \circ c) \circ b \\
&\quad + (\lambda_1 + q\lambda_2 + q^2\lambda_3 + \lambda_4 + q\lambda_5 + q\lambda_6 + q\lambda_7 + q\lambda_8 + q^2\lambda_9 + \lambda_{10} + q^2\lambda_{11} + q\lambda_{12})(b \circ a) \circ c \\
&\quad + (q\lambda_1 + \lambda_2 + q\lambda_3 + q\lambda_4 + \lambda_5 + q^2\lambda_6 + q^2\lambda_7 + \lambda_8 + q\lambda_9 + q\lambda_{10} + q\lambda_{11} + q^2\lambda_{12})(b \circ c) \circ a \\
&\quad + (q\lambda_1 + q\lambda_2 + \lambda_3 + q^2\lambda_4 + q\lambda_5 + \lambda_6 + q\lambda_7 + q^2\lambda_8 + \lambda_9 + q^2\lambda_{10} + q\lambda_{11} + q\lambda_{12})(c \circ a) \circ b \\
&\quad + (q^2\lambda_1 + \lambda_2 + q\lambda_3 + q\lambda_4 + \lambda_5 + q\lambda_6 + q^2\lambda_7 + q\lambda_8 + q\lambda_9 + q\lambda_{10} + \lambda_{11} + q^2\lambda_{12})(c \circ b) \circ a.
\end{aligned}$$

Therefore, $X = 0$ is an identity for $\mathfrak{Zinbiel}^{(q)}$ if and only if

$$\begin{aligned}
\lambda_1 + q\lambda_2 + q\lambda_3 + \lambda_4 + q^2\lambda_5 + q\lambda_6 + \lambda_7 + q\lambda_8 + q^2\lambda_9 + q\lambda_{10} + q^2\lambda_{11} + q\lambda_{12} &= 0, \\
q\lambda_1 + q^2\lambda_2 + \lambda_3 + q\lambda_4 + q\lambda_5 + \lambda_6 + q\lambda_7 + q^2\lambda_8 + q\lambda_9 + q^2\lambda_{10} + q\lambda_{11} + \lambda_{12} &= 0, \\
\lambda_1 + q\lambda_2 + q^2\lambda_3 + \lambda_4 + q\lambda_5 + q\lambda_6 + q\lambda_7 + q\lambda_8 + q^2\lambda_9 + \lambda_{10} + q^2\lambda_{11} + q\lambda_{12} &= 0, \\
q\lambda_1 + \lambda_2 + q\lambda_3 + q\lambda_4 + \lambda_5 + q^2\lambda_6 + q^2\lambda_7 + \lambda_8 + q\lambda_9 + q\lambda_{10} + q\lambda_{11} + q^2\lambda_{12} &= 0, \\
q\lambda_1 + q\lambda_2 + \lambda_3 + q^2\lambda_4 + q\lambda_5 + \lambda_6 + q\lambda_7 + q^2\lambda_8 + \lambda_9 + q^2\lambda_{10} + q\lambda_{11} + q\lambda_{12} &= 0, \\
q^2\lambda_1 + \lambda_2 + q\lambda_3 + q\lambda_4 + \lambda_5 + q\lambda_6 + q^2\lambda_7 + q\lambda_8 + q\lambda_9 + q\lambda_{10} + \lambda_{11} + q^2\lambda_{12} &= 0.
\end{aligned}$$

We solve this system. It has 12 unknowns λ_i, $i = 1, 2, \ldots, 12$.

Suppose that $q \neq 0, \pm 1$.

If $q \neq -2$, then we can take the parameters λ_i, $7 \leq i \leq 12$, as free and express the other parameters as linear combinations of the free parameters:

$$\lambda_1 = -\frac{1}{q(2+q)}((-1+q+q^2)\lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} - (q+1)\lambda_{11} + (-1+q+q^2)\lambda_{12}),$$
$$\lambda_2 = -\frac{1}{q(2+q)}(\lambda_7 + (-1+q+q^2)\lambda_8 + \lambda_9 + (-1+q+q^2)\lambda_{10} + \lambda_{11} - (1+q)\lambda_{12}),$$
$$\lambda_3 = -\frac{1}{q(2+q)}(\lambda_7 + \lambda_8 + (-1+q+q^2)\lambda_9 - (q+1)\lambda_{10} + (-1+q+q^2)\lambda_{11} + \lambda_{12}),$$
$$\lambda_4 = -\frac{1}{q(2+q)}(\lambda_7 + (-1+q+q^2)\lambda_8 - (1+q)\lambda_9 + (-1+q+q^2)\lambda_{10} + \lambda_{11} + \lambda_{12}),$$
$$\lambda_5 = -\frac{1}{q(2+q)}(-(1+q)\lambda_7 + \lambda_8 + (-1+q+q^2)\lambda_9 + \lambda_{10} + (-1+q+q^2)\lambda_{11} + \lambda_{12}),$$
$$\lambda_6 = -\frac{1}{q(2+q)}((-1+q+q^2)\lambda_7 - (1+q)\lambda_8 + \lambda_9 + \lambda_{10} + \lambda_{11} + (-1+q+q^2)\lambda_{12}).$$

Suppose now that $q = -2$. We can choose λ_i, $i = 6, 8, 9, 10, 11, 12$, as free parameters, and we can find

$$\lambda_1 = \frac{1}{2}(2\lambda_6 + \lambda_8 - \lambda_{11}),$$
$$\lambda_2 = \frac{1}{2}(2\lambda_6 - 5\lambda_8 - 3\lambda_9 - 6\lambda_{10} - 3\lambda_{11} - \lambda_{12}),$$
$$\lambda_3 = \lambda_6 - \lambda_8 - 3\lambda_9 - 2\lambda_{10} - 3\lambda_{11},$$
$$\lambda_4 = \lambda_6 - \frac{5}{2}\lambda_8 - 2\lambda_9 - 3\lambda_{10} - \frac{3}{2}\lambda_{11},$$
$$\lambda_5 = \frac{1}{2}(2\lambda_6 - \lambda_8 - 5\lambda_9 - 2\lambda_{10} - 5\lambda_{11} + \lambda_{12}),$$
$$\lambda_7 = -\lambda_8 - \lambda_9 - \lambda_{10} - \lambda_{11} - \lambda_{12}.$$

Substitute these expressions for λ's in $X(t_1, t_2, t_3)$. We obtain that, if $X = 0$ is an identity on $\mathfrak{Zinbiel}^{(q)}$, then the polynomial $X(t_1, t_2, t_3)$ should be a linear combination of six polynomials:

$$f_1 = -(-1+q+q^2)t_1(t_2t_3) - (-1+q+q^2)t_1(t_3t_2) - t_2(t_1t_3) - t_2(t_3t_1) - t_3(t_1t_2) + t_3(t_2t_1) + qt_3(t_2t_1) + 2q(t_1t_2)t_3 + q^2(t_1t_2)t_3,$$
$$f_2 = -(-1+q+q^2)t_1(t_2t_3) - (-1+q+q^2)t_1(t_3t_2) - t_2(t_1t_3) - t_2(t_3t_1) - t_3(t_1t_2) + t_3(t_2t_1) + qt_3(t_2t_1) + 2q(t_1t_2)t_3 + q^2(t_1t_2)t_3,$$
$$f_3 = -(-1+q+q^2)t_1(t_2t_3) - (-1+q+q^2)t_1(t_3t_2) - t_2(t_1t_3) - t_2(t_3t_1) - t_3(t_1t_2) + t_3(t_2t_1) + qt_3(t_2t_1) + 2q(t_1t_2)t_3 + q^2(t_1t_2)t_3,$$
$$f_4 = -(-1+q+q^2)t_1(t_2t_3) - (-1+q+q^2)t_1(t_3t_2) - t_2(t_1t_3) - t_2(t_3t_1) - t_3(t_1t_2) + t_3(t_2t_1) + qt_3(t_2t_1) + 2q(t_1t_2)t_3 + q^2(t_1t_2)t_3,$$
$$f_5 = -(-1+q+q^2)t_1(t_2t_3) - (-1+q+q^2)t_1(t_3t_2) - t_2(t_1t_3) - t_2(t_3t_1) - t_3(t_1t_2) + t_3(t_2t_1) + qt_3(t_2t_1) + 2q(t_1t_2)t_3 + q^2(t_1t_2)t_3,$$
$$f_6 = -(-1+q+q^2)t_1(t_2t_3) - (-1+q+q^2)t_1(t_3t_2) - t_2(t_1t_3) - t_2(t_3t_1) - t_3(t_1t_2) + t_3(t_2t_1) + qt_3(t_2t_1) + 2q(t_1t_2)t_3 + q^2(t_1t_2)t_3$$

if $q \neq -2$ and

$$\begin{aligned}
f_1 &= t_1(t_2t_3) + t_1(t_3t_2) + t_2(t_1t_3) + t_2(t_3t_1) + t_3(t_1t_2) + t_3(t_2t_1),\\
f_2 &= -\frac{1}{2}t_2(t_3t_1) + \frac{1}{2}t_3(t_2t_1) - (t_1t_2)t_3 + (t_1t_3)t_2,\\
f_3 &= -3t_2(t_1t_3) - 3t_2(t_3t_1) - 2t_3(t_1t_2) - t_3(t_2t_1) - (t_1t_2)t_3 + (t_2t_1)t_3,\\
f_4 &= \frac{1}{2}t_1(t_2t_3) - \frac{5}{2}t_2(t_1t_3) - \frac{5}{2}t_2(t_3t_1) - t_3(t_1t_2) - \frac{1}{2}t_3(t_2t_1) - (t_1t_2)t_3 + (t_2t_3)t_1,\\
f_5 &= -2t_2(t_1t_3) - \frac{3}{2}t_2(t_3t_1) - 3t_3(t_1t_2) - \frac{5}{2}t_3(t_2t_1) - (t_1t_2)t_3 + (t_3t_1)t_2,\\
f_6 &= -\frac{1}{2}t_1(t_2t_3) - \frac{3}{2}t_2(t_1t_3) - \frac{3}{2}t_2(t_3t_1) - 3t_3(t_1t_2) - \frac{5}{2}t_3(t_2t_1) - (t_1t_2)t_3 + (t_3t_2)t_1
\end{aligned}$$

if $q = -2$.

We see that

$$\begin{aligned}
f_1 &= \mathsf{zinbiel}^{(q)},\\
f_2 &= \mathsf{zinbiel}^{(q)}(t_1, t_3, t_2),\\
f_3 &= \mathsf{zinbiel}^{(q)}(t_2, t_1, t_3),\\
f_4 &= \mathsf{zinbiel}^{(q)}(t_2, t_3, t_1),\\
f_5 &= \mathsf{zinbiel}^{(q)}(t_3, t_1, t_2),\\
f_6 &= \mathsf{zinbiel}^{(q)}(t_3, t_2, t_1)
\end{aligned}$$

if $q \neq -2$ and

$$\begin{aligned}
f_1 &= -\mathsf{zinbiel}^{(-2)}(t_1, t_2, t_3),\\
f_2 &= \frac{1}{2}\mathsf{zinbiel}_1^{(-2)})(t_1, t_2, t_3),\\
f_3 &= \mathsf{zinbiel}_2^{(-2)}(t_1, t_2, t_3),\\
f_4 &= -\frac{1}{2}\mathsf{zinbiel}^{(-2)}(t_1, t_2, t_3) - \frac{1}{2}\mathsf{zinbiel}_1^{(-2)}(t_2, t_3, t_1) + \mathsf{zinbiel}_2^{(-2)}(t_1, t_2, t_3),\\
f_5 &= 3\mathsf{zinbiel}^{(-2)}(t_1, t_2, t_3) + \frac{1}{2}\mathsf{zinbiel}_1^{(-2)}(t_1, t_2, t_3) - \mathsf{zinbiel}_2^{(-2)}(t_3, t_1, t_2),\\
f_6 &= -\frac{3}{2}\mathsf{zinbiel}^{(-2)}(t_1, t_2, t_3) - \frac{1}{2}\mathsf{zinbiel}_1^{(-2)}(t_2, t_3, t_1) + \mathsf{zinbiel}_2^{(-2)}(t_1, t_2, t_3) + \mathsf{zinbiel}_2^{(-2)}(t_2, t_3, t_1)
\end{aligned}$$

if $q = -2$.

Now we establish that the identity $\mathsf{zinbiel}^{(-2)} = 0$ is a consequence of the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$.

We have

$$\begin{aligned}
&-\mathsf{zinbiel}_1^{(-2)}(t_1, t_2, t_3) + \mathsf{zinbiel}_2^{(-2)}(t_1, t_2, t_3)\\
&\quad = t_2(t_3t_1) - t_3(t_2t_1) + 2(t_1t_2)t_3 - 2(t_1t_3)t_2 - 3t_2(t_1t_3 + t_3t_1) - 2t_3(t_1t_2) - t_3(t_2t_1) - (t_1t_2)t_3 + (t_2t_1)t_3\\
&\quad = -3t_2(t_1t_3) - \frac{5}{2}t_2(t_3t_1) - 2t_3(t_1t_2) - \frac{3}{2}t_3(t_2t_1) - (t_1t_3)t_2 + (t_2t_1)t_3.
\end{aligned}$$

Thus,

$$\begin{aligned}
&-\mathsf{zinbiel}_1^{(-2)}(t_1, t_2, t_3) + \mathsf{zinbiel}_2^{(-2)}(t_1, t_2, t_3) - \mathsf{zinbiel}_1^{(-2)}(t_2, t_3, t_1)\\
&\quad + \mathsf{zinbiel}_2^{(-2)}(t_2, t_3, t_1) - \mathsf{zinbiel}_1^{(-2)}(t_3, t_1, t_2) + \mathsf{zinbiel}_2^{(-2)}(t_3, t_1, t_2)\\
&\quad = \frac{9}{2}(-t_1(t_2t_3 + t_3t_2) - t_2(t_3t_1 + t_1t_3) - t_3(t_1t_2 + t_2t_1)).
\end{aligned}$$

In other words,

$$\mathsf{zinbiel}^{(-2)}(t_1,t_2,t_3) = \frac{2}{9}\{-\mathsf{zinbiel}_1^{(-2)}(t_1,t_2,t_3) + \mathsf{zinbiel}_2^{(-2)}(t_1,t_2,t_3) - \mathsf{zinbiel}_1^{(-2)}(t_2,t_3,t_1) + \mathsf{zinbiel}_2^{(-2)}(t_2,t_3,t_1) - \mathsf{zinbiel}_1^{(-2)}(t_3,t_1,t_2) + \mathsf{zinbiel}_2^{(-2)}(t_3,t_1,t_2)\}.$$

Thus, any polylinear identity of degree 3 follows from the identity $\mathsf{zinbiel}^{(q)} = 0$ if $q \neq -2$. If $q = -2$, then any polylinear identity of degree 3 follows from the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$.

5. Restoring of Zinbiel Algebras by q-Zinbiel Algebras

Lemma 5.1. *Suppose that an algebra $\mathcal{A} = (A, \star)$ satisfies the identity $\mathsf{zinbiel}^{(q)} = 0$ if $q \neq -2$ and the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$ if $q = -2$. Then $\mathcal{A}$ is isomorphic to some algebra of the form $A^{(q)}$, where $A = (A, \circ)$ is an algebra with vector space A and Zinbiel multiplication $\circ$.*

Proof. If $q \neq -2$, then

$$(1-q^2)^{-1}\mathsf{zinbiel}(t_1,t_2,t_3) = ((q^3-q)(2+q))^{-1}((q-1)\mathsf{zinbiel}^{(q)}(t_1,t_2,t_3) + (q-1)\mathsf{zinbiel}^{(q)}(t_2,t_1,t_3) - q\,\mathsf{zinbiel}^{(q)}(t_2,t_3,t_1) + q^2\mathsf{zinbiel}^{(q)}(t_3,t_2,t_1)).$$

If $q = -2$, then

$$-\frac{1}{3}\mathsf{zinbiel}(t_1,t_2,t_3) = -\frac{1}{3}\mathsf{zinbiel}_1^{(-2)}(t_1,t_2,t_3) + \frac{2}{3}\mathsf{zinbiel}_1^{(-2)}(t_2,t_3,t_1) - \frac{1}{3}\mathsf{zinbiel}_1^{(-2)}(t_3,t_1,t_2) - \frac{1}{3}\mathsf{zinbiel}_2^{(-2)}(t_1,t_2,t_3) - \frac{2}{3}\mathsf{zinbiel}_2^{(-2)}(t_2,t_3,t_1) + \frac{2}{3}\mathsf{zinbiel}_2^{(-2)}(t_3,t_1,t_2).$$

Let $(A, \star)$ be any algebra with the identity $\mathsf{zinbiel}^{(q)} = 0$ if $q \neq -2$ or satisfy the identities $\mathsf{zinbiel}_1^{(-2)} = 0$ and $\mathsf{zinbiel}_2^{(-2)} = 0$ if $q = -2$.

We have proved that the algebra $(A, \circ)$, where

$$a \circ b = (1-q^2)^{-1}(a \star b - qb \star a),$$

is Zinbiel. It is easy to see that

$$a \circ_q b = a \circ b + qb \circ a = a \star b.$$

Thus, the q-algebra of $(A, \star)$ is isomorphic to $\mathcal{A} = (A, \circ)$.

6. Zinbiel Algebra under a -1-Commutator

Let

$$\mathsf{tortkara}(t_1,t_2,t_3,t_4) = (t_1t_2)(t_3t_4) + (t_1t_4)(t_3t_2) - \mathsf{jac}(t_1,t_2,t_3)t_4 - \mathsf{jac}(t_1,t_4,t_3)t_2,$$

where

$$\mathsf{jac}(t_1,t_2,t_3) = (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2.$$

Lemma 6.1. *Let $(A, \circ)$ be a (right or left) Zinbiel algebra. Then $(A, [\,,\,])$ satisfies the identity $\mathsf{tortkara} = 0$, where $[a,b] = a \circ b - b \circ a$.*

Proof. Assume, for simplicity, that A is right-Zinbiel.

Since

$$(a,b,c) = a \circ (b \circ c) - (a \circ b) \circ c = -(b \circ a) \circ c,$$

we have

$$\mathsf{jac}(a,b,c) = (a,b,c) + (b,c,a) + (c,a,b) - (b,a,c) - (c,b,a) - (a,c,b)$$
$$= -(b \circ a) \circ c - (c \circ b) \circ a - (a \circ c) \circ b + (a \circ b) \circ c + (b \circ c) \circ a + (c \circ a) \circ b = [a,b] \circ c + [b,c] \circ a + [c,a] \circ b.$$

Thus,

$$\begin{aligned}
[\mathsf{jac}(a,b,c),d] &= \mathsf{jac}(a,b,c)\circ d - d\circ \mathsf{jac}(a,b,c)\\
&= ([a,b]\circ c)\circ d + ([b,c]\circ a)\circ d + ([c,a]\circ b)\circ d - d\circ([a,b]\circ c) - d\circ([b,c]\circ a) - d\circ([c,a]\circ b)\\
&= ([a,b]\circ c)\circ d + ([b,c]\circ a)\circ d + ([c,a]\circ b)\circ d - [a,b]\circ(d\circ c) - [b,c]\circ(d\circ a) - [c,a]\circ(d\circ b).
\end{aligned}$$

In a similar way, one calculates $[\mathsf{jac}(a,d,c),b]$ and

$$\begin{aligned}
&[\mathsf{jac}(a,b,c),d] + [\mathsf{jac}(a,d,c),b]\\
&\quad= ([a,b]\circ c)\circ d + ([b,c]\circ a)\circ d + ([c,a]\circ b)\circ d - [a,b]\circ(d\circ c) - [b,c]\circ(d\circ a) - [c,a]\circ(d\circ b)\\
&\quad+ ([a,d]\circ c)\circ b + ([d,c]\circ a)\circ b + ([c,a]\circ d)\circ b - [a,d]\circ(b\circ c) - [d,c]\circ(b\circ a) - [c,a]\circ(b\circ d).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathsf{tortkara}(a,b,c,d) = [[a,b],[c,d]] + [[a,d],[c,b]]\\
&\quad- ([a,b]\circ c)\circ d - ([b,c]\circ a)\circ d - ([c,a]\circ b)\circ d + [a,b]\circ(d\circ c) + [b,c]\circ(d\circ a) + [c,a]\circ(d\circ b)\\
&\quad- ([a,d]\circ c)\circ b - ([d,c]\circ a)\circ b - ([c,a]\circ d)\circ b + [a,d]\circ(b\circ c) + [d,c]\circ(b\circ a) + [c,a]\circ(b\circ d)\\
&\quad= [a,b]\circ(c\circ d) + [a,d]\circ(c\circ b) + [d,c]\circ(a\circ b) + [b,c]\circ(a\circ d) + [c,a]\circ(d\circ b) + [c,a]\circ(b\circ d)\\
&\quad- ([a,b]\circ c)\circ d - ([b,c]\circ a)\circ d - ([c,a]\circ b)\circ d - ([a,d]\circ c)\circ b - ([d,c]\circ a)\circ b - ([c,a]\circ d)\circ b\\
&\quad= X + Y,
\end{aligned}$$

where

$$\begin{aligned}
X &= [a,d]\circ(c\circ b) + [d,c]\circ(a\circ b) + [c,a]\circ(d\circ b) - ([a,d]\circ c)\circ b - ([d,c]\circ a)\circ b - ([c,a]\circ d)\circ b,\\
Y &= [a,b]\circ(c\circ d) + [b,c]\circ(a\circ d) + [c,a]\circ(b\circ d) - ([a,b]\circ c)\circ d - ([b,c]\circ a)\circ d - ([c,a]\circ b)\circ d.
\end{aligned}$$

By the right-Zinbiel identity,

$$X = (c\circ[a,d] + a\circ[d,c] + d\circ[c,a])\circ b.$$

By the left-commutativity identity,

$$c\circ[a,d] + a\circ[d,c] + d\circ[c,a] = 0.$$

Thus, $X = 0$. Similarly, $Y = 0$. Thus, $\mathsf{tortkara}(a,b,c,d) = 0$ for any $a,b,c,d\in A$ if A is right-Zinbiel.

Lemma 6.2. *The identity* $\mathsf{tortkara} = 0$ *is minimal for* $\mathfrak{Zinbiel}^{(-1)}$*, i.e., any identity of degree* 3 *for* $\mathfrak{Zinbiel}^{(-1)}$ *follows from the skew-symmetric identity and any identity of degree* 4 *for* $\mathfrak{Zinbiel}^{(-1)}$ *follows from the identity* $\mathsf{tortkara} = 0$.

Proof. Let

$$X_3(t_1,t_2,t_3) = \lambda_1(t_1t_2)t_3 + \lambda_2(t_1t_3)t_2 + \lambda_3(t_2t_3)t_1$$

be a generic anti-commutative polynomial, i.e., an element of the free anti-commutative algebra of degree 3. Let

$$\begin{aligned}
&X_4(t_1,t_2,t_3,t_4)\\
&\quad= \lambda_1(t_1t_2)(t_3t_4) + \lambda_2(t_1t_3)(t_2t_4) + \lambda_3(t_2t_3)(t_1t_4) + \lambda_4((t_1t_2)t_3)t_4 + \lambda_5((t_1t_3)t_2)t_4\\
&\quad+ \lambda_6((t_1t_4)t_2)t_3 + \lambda_7((t_2t_3)t_1)t_4 + \lambda_8((t_2t_4)t_1)t_3 + \lambda_9((t_3t_4)t_1)t_2 + \lambda_{10}((t_1t_2)t_4)t_3\\
&\quad+ \lambda_{11}((t_1t_3)t_4)t_2 + \lambda_{12}((t_1t_4)t_3)t_2 + \lambda_{13}((t_2t_3)t_4)t_1 + \lambda_{14}((t_2t_4)t_3)t_1 + \lambda_{15}((t_3t_4)t_2)t_1
\end{aligned}$$

be a generic anti-commutative polynomial of degree 4.

Let F be the free Zinbiel algebra with generators a, b, c, and d and with multiplication $(x,y)\mapsto xy$. We calculate $X_3(a,b,c)\in F$ in terms of the commutator $[x,y] = xy - yx$. We have

$$\begin{aligned}
X_3(a,b,c) &= \lambda_1[[a,b],c] + \lambda_2[[a,c],b] + \lambda_3[[b,c],a] = (\lambda_1-\lambda_2-\lambda_3)(ab)c + (-\lambda_1+\lambda_2+\lambda_3)(ac)b\\
&\quad+ (-\lambda_1-\lambda_2-\lambda_3)(ba)c + (\lambda_1+\lambda_2+\lambda_3)(bc)a + (-\lambda_1-\lambda_2+\lambda_3)(ca)b + (\lambda_1+\lambda_2-\lambda_3)(cb)a.
\end{aligned}$$

Since the set of elements

$$\{(ab)c,\ (ba)c,\ (ca)b,\ (ac)b,\ (bc)a,\ (cb)a\}$$

forms a basis of the polylinear part of the free Zinbiel algebra in degree 3, we see that $X_3(a,b,c)=0$ if and only if $\lambda_1=0$, $\lambda_2=0$, and $\lambda_3=0$. This means that any identity of degree 3 for $\mathfrak{Zinbiel}^{(-1)}$ follows from the anti-commutativity identity $t_1t_2+t_2t_1=0$.

We calculate $X_4(a,b,c,d)$ in terms of the commutator. We have

$$\begin{aligned}
&X_4(a,b,c,d)\\
&\quad=(\lambda_1+\lambda_2+\lambda_3+\lambda_4-\lambda_5+\lambda_6-\lambda_7+\lambda_8+\lambda_9-\lambda_{10}-\lambda_{11}+\lambda_{12}-\lambda_{13}+\lambda_{14}+\lambda_{15})((ab)c)d\\
&\quad+(-\lambda_1-\lambda_2-\lambda_3-\lambda_4+\lambda_5-\lambda_6+\lambda_7-\lambda_8-\lambda_9+\lambda_{10}+\lambda_{11}-\lambda_{12}+\lambda_{13}-\lambda_{14}-\lambda_{15})((ab)d)c\\
&\quad+(\lambda_1+\lambda_2-\lambda_3-\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8+\lambda_9-\lambda_{10}-\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15})((ac)b)d\\
&\quad+(-\lambda_1-\lambda_2+\lambda_3+\lambda_4-\lambda_5-\lambda_6-\lambda_7-\lambda_8-\lambda_9+\lambda_{10}+\lambda_{11}-\lambda_{12}-\lambda_{13}-\lambda_{14}-\lambda_{15})((ac)d)b\\
&\quad+(-\lambda_1+\lambda_2-\lambda_3-\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8-\lambda_9-\lambda_{10}+\lambda_{11}-\lambda_{12}+\lambda_{13}+\lambda_{14}-\lambda_{15})((ad)b)c\\
&\quad+(\lambda_1-\lambda_2+\lambda_3+\lambda_4-\lambda_5-\lambda_6-\lambda_7-\lambda_8+\lambda_9+\lambda_{10}-\lambda_{11}+\lambda_{12}-\lambda_{13}-\lambda_{14}+\lambda_{15})((ad)c)b\\
&\quad+(-\lambda_1+\lambda_2+\lambda_3-\lambda_4-\lambda_5+\lambda_6-\lambda_7+\lambda_8+\lambda_9+\lambda_{10}-\lambda_{11}+\lambda_{12}-\lambda_{13}+\lambda_{14}+\lambda_{15})((ba)c)d\\
&\quad+(\lambda_1-\lambda_2-\lambda_3+\lambda_4+\lambda_5-\lambda_6+\lambda_7-\lambda_8-\lambda_9-\lambda_{10}+\lambda_{11}-\lambda_{12}+\lambda_{13}-\lambda_{14}-\lambda_{15})((ba)d)c\\
&\quad+(-\lambda_1-\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8+\lambda_9+\lambda_{10}+\lambda_{11}+\lambda_{12}-\lambda_{13}+\lambda_{14}+\lambda_{15})((bc)a)d\\
&\quad+(\lambda_1+\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7-\lambda_8-\lambda_9-\lambda_{10}-\lambda_{11}-\lambda_{12}+\lambda_{13}-\lambda_{14}-\lambda_{15})((bc)d)a\\
&\quad+(\lambda_1-\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8-\lambda_9+\lambda_{10}+\lambda_{11}+\lambda_{12}+\lambda_{13}-\lambda_{14}-\lambda_{15})((bd)a)c\\
&\quad+(-\lambda_1+\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7-\lambda_8+\lambda_9-\lambda_{10}-\lambda_{11}-\lambda_{12}-\lambda_{13}+\lambda_{14}+\lambda_{15})((bd)c)a\\
&\quad+(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5+\lambda_6+\lambda_7+\lambda_8+\lambda_9-\lambda_{10}+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15})((ca)b)d\\
&\quad+(-\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5-\lambda_6-\lambda_7-\lambda_8-\lambda_9+\lambda_{10}-\lambda_{11}-\lambda_{12}-\lambda_{13}-\lambda_{14}-\lambda_{15})((ca)d)b\\
&\quad+(-\lambda_1-\lambda_2-\lambda_3+\lambda_4+\lambda_5+\lambda_6-\lambda_7+\lambda_8+\lambda_9+\lambda_{10}+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15})((cb)a)d\\
&\quad+(\lambda_1+\lambda_2+\lambda_3-\lambda_4-\lambda_5-\lambda_6+\lambda_7-\lambda_8-\lambda_9-\lambda_{10}-\lambda_{11}-\lambda_{12}-\lambda_{13}-\lambda_{14}-\lambda_{15})((cb)d)a\\
&\quad+(-\lambda_1+\lambda_2-\lambda_3+\lambda_4+\lambda_5+\lambda_6-\lambda_7-\lambda_8+\lambda_9+\lambda_{10}+\lambda_{11}+\lambda_{12}-\lambda_{13}-\lambda_{14}-\lambda_{15})((cd)a)b\\
&\quad+(\lambda_1-\lambda_2+\lambda_3-\lambda_4-\lambda_5-\lambda_6+\lambda_7+\lambda_8-\lambda_9-\lambda_{10}-\lambda_{11}-\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15})((cd)b)a\\
&\quad+(-\lambda_1+\lambda_2+\lambda_3-\lambda_4+\lambda_5-\lambda_6+\lambda_7+\lambda_8-\lambda_9-\lambda_{10}+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}-\lambda_{15})((da)b)c\\
&\quad+(\lambda_1-\lambda_2-\lambda_3+\lambda_4-\lambda_5+\lambda_6-\lambda_7-\lambda_8+\lambda_9+\lambda_{10}-\lambda_{11}-\lambda_{12}-\lambda_{13}-\lambda_{14}+\lambda_{15})((da)c)b\\
&\quad+(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7-\lambda_8-\lambda_9+\lambda_{10}+\lambda_{11}+\lambda_{12}+\lambda_{13}+\lambda_{14}-\lambda_{15})((db)a)c\\
&\quad+(-\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7+\lambda_8+\lambda_9-\lambda_{10}-\lambda_{11}-\lambda_{12}-\lambda_{13}-\lambda_{14}+\lambda_{15})((db)c)a\\
&\quad+(\lambda_1+\lambda_2-\lambda_3+\lambda_4+\lambda_5+\lambda_6-\lambda_7-\lambda_8-\lambda_9+\lambda_{10}+\lambda_{11}+\lambda_{12}-\lambda_{13}-\lambda_{14}+\lambda_{15})((dc)a)b\\
&\quad+(-\lambda_1-\lambda_2+\lambda_3-\lambda_4-\lambda_5-\lambda_6+\lambda_7+\lambda_8+\lambda_9-\lambda_{10}-\lambda_{11}-\lambda_{12}+\lambda_{13}+\lambda_{14}-\lambda_{15})((dc)b)a.
\end{aligned}$$

Since the set of elements

$$\begin{aligned}
\{&((ab)c)d,\ ((bc)a)d,\ ((ca)b)d),\ ((ba)c)d,\ ((cb)a)d,\ ((ac)b)d,\\
&((ab)d)c,\ ((bd)a)c,\ ((da)b)c,\ ((ba)d)c,\ ((db)a)c,\ ((ad)b)c,\\
&((ad)c)b,\ ((dc)a)b,\ ((ca)d)b,\ ((da)c)b,\ ((cd)a)b,\ ((ac)d)b,\\
&((db)c)a,\ ((bc)d)a,\ ((cd)b)a,\ ((bd)c)a,\ ((cb)d)a,\ ((dc)b)a\}
\end{aligned}$$

forms a basis of the degree-4 polylinear part of the free Zinbiel algebra, the condition $X_4(a,b,c,d)=0$ gives us a system of 24 linear equations with 15 unknowns λ_i, $i=1,\dots,15$. We see that this system has rank 12, and we can take λ_{10}, λ_{12}, and λ_{15} as free parameters and express the other parameters in the following way:

$$\begin{aligned}
\lambda_1 &= -\lambda_{12} - \lambda_{15}, \\
\lambda_2 &= \lambda_{10} + \lambda_{15}, \\
\lambda_3 &= -\lambda_{10} - \lambda_{12}, \\
\lambda_4 &= \lambda_{10} + \lambda_{12} + \lambda_{15}, \\
\lambda_5 &= -\lambda_{10} - \lambda_{12} - \lambda_{15}, \\
\lambda_6 &= -\lambda_{10}, \\
\lambda_7 &= \lambda_{10} + \lambda_{12} + \lambda_{15}, \\
\lambda_8 &= \lambda_{10}, \\
\lambda_9 &= -\lambda_{12}, \\
\lambda_{11} &= -\lambda_{12}, \\
\lambda_{13} &= \lambda_{15}, \\
\lambda_{14} &= -\lambda_{15}.
\end{aligned}$$

Therefore,

$$X_4 = X_4(t_1, t_2, t_3, t_4) = \lambda_{10} f_1 + \lambda_{12} f_2 + \lambda_{15} f_3,$$

where

$$\begin{aligned}
f_1 &= (t_1t_3)(t_2t_4) - (t_2t_3)(t_1t_4) + ((t_1t_2)t_3)t_4 + ((t_1t_2)t_4)t_3 - ((t_1t_3)t_2)t_4 - ((t_1t_4)t_2)t_3 \\
&\quad + ((t_2t_3)t_1)t_4 + ((t_2t_4)t_1)t_3, \\
f_2 &= -(t_1t_2)(t_3t_4) - (t_2t_3)(t_1t_4) + ((t_1t_2)t_3)t_4 - ((t_1t_3)t_2)t_4 - ((t_1t_3)t_4)t_2 + ((t_1t_4)t_3)t_2 \\
&\quad + ((t_2t_3)t_1)t_4 - ((t_3t_4)t_1)t_2, \\
f_3 &= -(t_1t_2)(t_3t_4) + (t_1t_3)(t_2t_4) + ((t_1t_2)t_3)t_4 - ((t_1t_3)t_2)t_4 + ((t_2t_3)t_1)t_4 + ((t_2t_3)t_4)t_1 \\
&\quad - ((t_2t_4)t_3)t_1 + ((t_3t_4)t_2)t_1.
\end{aligned}$$

We see that the following equalities of skew-symmetric polynomials hold:

$$\begin{aligned}
f_1 &= (t_1t_3)(t_2t_4) + (t_1t_4)(t_2t_3) - ((t_2t_1)t_4 + (t_1t_4)t_2 + (t_4t_2)t_1)t_3 - ((t_2t_1)t_3 + (t_1t_3)t_2 + (t_3t_2)t_1)t_4 \\
&= \mathsf{tortkara}(t_1, t_3, t_2, t_4), \\
f_2 &= -(t_1t_2)(t_3t_4) - (t_1t_4)(t_3t_2) + ((t_1t_2)t_3 - (t_1t_3)t_2 + (t_2t_3)t_1)t_4 + ((t_3t_1)t_4 + (t_1t_4)t_3 + (t_4t_3)t_1)t_2 \\
&= -\mathsf{tortkara}(t_1, t_2, t_3, t_4), \\
f_3 &= (t_2t_1)(t_3t_4) + (t_2t_4)(t_3t_1) - ((t_2t_1)t_3 + (t_1t_3)t_2 + (t_3t_2)t_1)t_4 - ((t_3t_2)t_4 + (t_2t_4)t_3 + (t_4t_3)t_2)t_1 \\
&= \mathsf{tortkara}(t_2, t_1, t_3, t_4).
\end{aligned}$$

These mean that any identity of degree 3 of the category $\mathfrak{Zinbiel}^{(-1)}$ follows from the identity $\mathsf{tortkara} = 0$.

7. Zinbiel–Jordan Algebras

In [5], it is proved that $(A, \{\ ,\ \})$ is commutative and associative if $(A, \circ)$ is Zinbiel, where $\{a, b\} = a \circ b + b \circ a$.

8. Algebras with Multiplication $a \star b = a \iint b$

Let $A = \mathbb{C}[x]$ and $\partial = \partial/\partial x$ and $\int = \int_0^x$ be the differentiation and integration endomorphisms. We set here $\int 0 = 0$. For example, $\partial(x^4) = 4x^3$ and $\int x^4 = x^5/5$. We write $\iint a$ instead of $\int_0^x \Big(\int_0^x a\, dx \Big)\, dx$.

Endow $\mathbb{C}[x]$ by another multiplication:

$$a \star b = a \iint b.$$

For example, $x^2 \star x^3 = 1/20x^7$.

Lemma 8.1. *Let*

$$G = \left(\int a\right)\int\left(b\iint c\right) + \left(\int b\right)\int\left(c\iint a\right) + \left(\int c\right)\int\left(a\iint b\right) - \left(\int b\right)\int\left(a\iint c\right) - \left(\int c\right)\int\left(b\iint a\right) - \left(\int a\right)\int\left(c\iint b\right).$$

Then $G = 0$.

Proof. We have

$$\partial G = G_1 + G_2,$$

where

$$G_1 = a\int\left(b\iint c\right) + b\int\left(c\iint a\right) + c\int\left(a\iint b\right) - b\int\left(a\iint c\right) - c\int\left(b\iint a\right) - a\int\left(c\iint b\right),$$

$$G_2 = \left(\int a\right)\left(b\iint c\right) + \left(\int b\right)\left(c\iint a\right) + \left(\int c\right)\left(a\iint b\right) - \left(\int b\right)\left(a\iint c\right) - \left(\int c\right)\left(b\iint a\right) - \left(\int a\right)\left(c\iint b\right).$$

Then

$$\partial G_1 = G_{1,1} + G_{1,2},$$

where

$$G_{1,1} = \partial a\int\left(b\iint c\right) + \partial b\int\left(c\iint a\right) + \partial c\int\left(a\iint b\right) - \partial b\int\left(a\iint c\right) - \partial c\int\left(b\iint a\right) - \partial a\int\left(c\iint b\right),$$

$$G_{1,2} = a\left(b\iint c\right) + b\left(c\iint a\right) + c\left(a\iint b\right) - b\left(a\iint c\right) - c\left(b\iint a\right) - a\left(c\iint b\right) = 0.$$

Further,

$$\partial G_2 = G_{2,1} + G_{2,2} + G_{2,3},$$

where

$$G_{2,1} = a\left(b\iint c\right) + b\left(c\iint a\right) + c\left(a\iint b\right) - b\left(a\iint c\right) - c\left(b\iint a\right) - a\left(c\iint b\right) = 0,$$

$$G_{2,2} = \left(\int a\right)\partial b\left(\iint c\right) + \left(\int b\right)\partial c\left(\iint a\right) + \left(\int c\right)\partial a\left(\iint b\right) - \left(\int b\right)\partial a\left(\iint c\right) - \left(\int c\right)\partial b\left(\iint a\right) - \left(\int a\right)\partial c\left(\iint b\right),$$

$$G_{2,3} = \left(\int a\right)\left(b\int c\right) + \left(\int b\right)\left(c\int a\right) + \left(\int c\right)\left(a\int b\right)$$
$$- \left(\int b\right)\left(a\int c\right) - \left(\int c\right)\left(b\int a\right) - \left(\int a\right)\left(c\int b\right) = 0.$$

By the formula of integration by parts,

$$\int\left(a\iint b\right) - \int\left(b\iint a\right) = -\left(\int b\right)\left(\iint a\right) + \left(\int a\right)\left(\iint b\right).$$

Therefore, $G_{1,1} + G_{2,2} = 0$. Thus, we obtain a functional identity, $\partial^2 G = 0$. Thus, $G = 0$.

Lemma 8.2. *For any* $a, b, c \in K[x]$,

$$\iint\left(a\iint\left(b\iint c\right)\right) + \iint\left(b\iint\left(c\iint a\right)\right) + \iint\left(c\iint\left(a\iint b\right)\right)$$
$$- \iint\left(b\iint\left(a\iint c\right)\right) - \iint\left(c\iint\left(b\iint a\right)\right) - \iint\left(a\iint\left(c\iint b\right)\right)$$
$$=\left(\iint a\right)\iint\left(b\iint c\right) + \left(\iint b\right)\iint\left(c\iint a\right) + \left(\iint c\right)\iint\left(a\iint b\right)$$
$$- \left(\iint b\right)\iint\left(a\iint c\right) - \left(\iint c\right)\iint\left(b\iint a\right) - \left(\iint a\right)\iint\left(c\iint b\right).$$

Proof. Denote by R and Q the left-hand and right-hand expressions of this identity. We see that

$$\partial^2 R = R_1,$$

where

$$R_1 = a\iint\left(b\iint c\right) + b\iint\left(c\iint a\right) + c\iint\left(a\iint b\right)$$
$$- b\iint\left(a\iint c\right) - c\iint\left(b\iint a\right) - a\iint\left(c\iint b\right).$$

Further,

$$\partial^2 Q = Q_1 + 2Q_2 + Q_3,$$

where

$$Q_1 = R_1,$$
$$Q_2 = G,$$
$$Q_3 = \left(\iint a\right)b\iint c + \left(\iint b\right)c\iint a + \left(\iint c\right)a\iint b$$
$$- \left(\iint b\right)a\iint c - \left(\iint c\right)b\iint a - \left(\iint a\right)c\iint b = 0$$

(for the definition of G see Lemma 8.1). By Lemma 8.1, $Q_2 = 0$. Thus, we obtain a functional identity $\partial^2 R = \partial^2 Q$. Thus, $R = Q$.

Lemma 8.3. *For any* $a, b, c \in \mathbb{C}[x]$,

$$(a \star b) \star c = (a \star c) \star b,$$
$$(a, b, [c, d]) + (a, c, [d, b]) + (a, d, [b, c]) = 0,$$

where $[a, b] = a \star b - b \star a$ *and* $(a, b, c) = a \star (b \star c) - (a \star b) \star c$.

In other words, the algebra $(\mathbb{C}[x], \star)$ satisfies identities (1) and (2).

Proof. The right-commutativity identity is evident:

$$(a \star b) \star c = \left(a \iint b\right) \iint c = \left(a \iint c\right) \iint b = (a \star c) \star b.$$

The second part follows from Lemma 8.2.

Lemma 8.4. *Suppose that* $(A, \star)$ *satisfies the right-commutativity identity* (1) *and identity* (3). *Then the algebra* $(A, [\,,\,])$ *under the commutator* $[a, b] = a \star b - b \star a$ *satisfies the identity* $\mathsf{tortkara} = 0$.

Proof. By the right-commutativity identity,

$$\begin{aligned}\mathsf{jac}(a,b,c) &= (a \star b - b \star a) \star c - c \star (a \star b - b \star a) + (b \star c - c \star b) \star a \\ &\quad - a \star (b \star c - c \star b) + (c \star a - a \star c) \star b - b \star (c \star a - a \star c) \\ &= -a \star (b \star c - c \star b) - b \star (c \star a - a \star c) - c \star (a \star b - b \star a)\end{aligned}$$

and

$$\begin{aligned}[\mathsf{jac}(a,b,c), d] &= (-a \star (b \star c - c \star b) - b \star (c \star a - a \star c) - c \star (a \star b - b \star a)) \star d \\ &\quad + d \star (a \star (b \star c - c \star b) + b \star (c \star a - a \star c) + c \star (a \star b - b \star a)) \\ &= -(a \star [b,c] + b \star [c,a] + c \star [a,b]) \star d + d \star (a \star [b,c]) + b \star [c,a] + c \star [a,b].\end{aligned}$$

Similarly, one calculates $[\mathsf{jac}(a,d,c), b]$. We see that

$$\begin{aligned}&[\mathsf{jac}(a,b,c), d] + [\mathsf{jac}(a,d,c), b] \\ &\quad = -(a \star [b,c] + b \star [c,a] + c \star [a,b]) \star d + d \star (a \star [b,c] + b \star [c,a] + c \star [a,b]) \\ &\quad - (a \star [d,c] + d \star [c,a] + c \star [a,d]) \star b + b \star (a \star [d,c] + d \star [c,a] + c \star [a,d]) \\ &\quad = -(a \star d) \star [b,c] - (b \star d) \star [c,a] - (c \star d) \star [a,b] + d \star (a \star [b,c] + b \star [c,a] + c \star [a,b]) \\ &\quad - (a \star b) \star [d,c] - (d \star b) \star [c,a] - (c \star b) \star [a,d] + b \star (a \star [d,c] + d \star [c,a] + c \star [a,d]).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathsf{tortkara}(a,b,c,d) + [c,d] \star [a,b] + [c,b] \star [a,d] \\ &\quad = (a \star b) \star [c,d] - (b \star a) \star [c,d] + (a \star d) \star [c,b] - (d \star a) \star [c,b] \\ &\quad + (a \star d) \star [b,c] + (b \star d) \star [c,a] + (c \star d) \star [a,b] - d \star (a \star [b,c] + b \star [c,a] + c \star [a,b]) \\ &\quad + (a \star b) \star [d,c] + (d \star b) \star [c,a] + (c \star b) \star [a,d] - b \star (a \star [d,c] + d \star [c,a] + c \star [a,d]) \\ &\quad = -(b \star a) \star [c,d] - (d \star a) \star [c,b] + (b \star d) \star [c,a] + (c \star d) \star [a,b] \\ &\quad - d \star (a \star [b,c]) - d \star (b \star [c,a]) - d \star (c \star [a,b]) \\ &\quad + (d \star b) \star [c,a] + (c \star b) \star [a,d] - b \star (a \star [d,c]) - b \star (d \star [c,a]) - b \star (c \star [a,d]) \\ &\quad = T(a,b,c,d) + [c,d] \star [a,b] + [c,b] \star [a,d],\end{aligned}$$

where

$$T(a,b,c,d) = (b,a,[c,d]) + (b,d,[a,c]) + (b,c,[d,a]) + (d,a,[c,b]) + (d,c,[b,a]) + (d,b,[a,c]).$$

By (3), $T(a,b,c,d) = 0$ for any $a,b,c,d \in A$.

9. Proof of Theorem 2.1

The proof follows from Lemmas 3.1, 3.2, 3.3, 4.1, and 5.1.

10. Proof of Theorem 2.2

The proof follows from Lemmas 6.1 and 6.2.

11. Proof of Theorem 2.3

The proof follows from Lemmas 8.3 and 8.4.

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