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n-LIE STRUCTURES THAT ARE GENERATED BY WRONSKIANS

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Abstract: We study the $(k+1)$-Lie structures, k-left commutative and homotopy $(k+1)$-Lie structures with multiplication generated by Wronskians and prove that the nontrivial structures of n-Lie algebras appear only in the case of small characteristic.

Keywords: n-Lie algebra, homotopy algebra, modular Lie algebras, Wronskian, Jacobian

Given a commutative associative algebra U with commuting derivations $\partial_1, \dots, \partial_n$, the *Jacobian* on U is the determinant

$$\mathrm{Jac}_n(u_1, \dots, u_n) = \begin{vmatrix} \partial_1 u_1 & \cdots & \partial_1 u_n \\ \vdots & \vdots & \vdots \\ \partial_n u_1 & \cdots & \partial_n u_n \end{vmatrix}.$$

It is shown in [1, 2] that (U, Jac_n) as an n-arc algebra is n-Lie, for the Jacobian satisfies the Leibniz rule

$$\mathrm{Jac}_n(u_1, \dots, u_{n-1}, \mathrm{Jac}_n(u_n, \dots, u_{2n-1}))$$
$$= \sum_{i=n}^{2n-1} (-1)^{i+n} \mathrm{Jac}_n(\mathrm{Jac}_n(u_1, \dots, u_{n-1}, u_i), u_n, \dots, \hat{u}_i, \dots, u_{2n-1}),$$

where $\hat{u}_i$ indicates the omission of u_i.

Another remarkable determinant is the Wronskian. Given a commutative associative algebra U with derivation ∂, the Wronskian on it is defined by the rule

$$V^{0,1,\dots,k}(u_0, \dots, u_k) = \begin{vmatrix} u_0 & u_1 & \cdots & u_k \\ \partial u_0 & \partial u_1 & \cdots & \partial u_k \\ \vdots & \vdots & \vdots & \vdots \\ \partial^k u_0 & \partial^k u_1 & \cdots & \partial^k u_k \end{vmatrix}.$$

The goal of this article is to study U as an n-arc algebra with respect to the product generated by the Wronskian.

Take some vector spaces A and M, and the space $T^k(A, M) = \mathrm{Hom}(A^{\otimes k}, M)$ of multilinear maps $A \times \dots \times A \to M$ in k arguments. Put $T^0(A, M) = M$, $T^k(A, M) = 0$ for $k < 0$, and $T^*(A, M) = \bigoplus_k T^k(A, M)$.

Take the kth exterior power $\wedge^k A$ and the subspace $C^k(A, M) = \mathrm{Hom}(\wedge^k A, M)$ of $T^k(A, M)$. Put $C^0(A, M) = M$, $C^k(A, M) = 0$ for $k < 0$, and $C^*(A, M) = \bigoplus_k C^k(A, M)$.

Suppose that A is an algebra with signature Ω. This means [3] that Ω is the set of multilinear maps $A \times \dots \times A \to A$. Call $\omega \in \Omega$ an *n-arc product* if $\omega \in T^n(A, A)$. Put $|\omega| = n$ for $\omega \in T^n(A, A)$. When the signature is important, we will write (A, Ω) instead of A. If Ω consists of a single element ω, put $A = (A, \omega)$.

Let (A, ω) be an n-arc algebra on a vector space A over a field K of characteristic $p \geq 0$, where ω is a multilinear n-arc map $A \times \dots \times A \to A$. Recall that a linear map $D : A \to A$ is called a *derivation* of A if

$$D(\omega(a_1, \dots, a_n)) = \sum_{i=1}^{n} \omega(a_1, \dots, a_{i-1}, D(a_i), a_{i+1}, \dots, a_n)$$

for all $a_1, \dots, a_n \in A$. Take the linear map $L_{a_1 \dots a_{n-1}} : A \to A$ defined by the rule $L_{a_1, \dots, a_{n-1}} a = \omega(a_1, \dots, a_{n-1}, a)$.

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Denote by $\operatorname{Der} A$ the algebra of all derivations of (A, ω). The algebra (A, ω) is called *n-Lie* [1] if ω is skew-symmetric and $L_{a_1,\dots,a_{n-1}} \in \operatorname{Der} A$ for all $a_1, \dots, a_{n-1} \in A$. Sometimes the n-Lie algebras are called *Nambu–Takhtajan algebras*, although V. T. Filippov was the first to introduce them.

Denote by Sym_k the permutation group; and by $\operatorname{sgn}\sigma$, the parity of $\sigma \in \mathrm{Sym}_k$. Denote by $\mathrm{Sym}_{k,l}$ the set of (k,l)-shuffles; these are $\sigma \in \mathrm{Sym}_{k+l}$ such that $\sigma(1) < \dots < \sigma(k)$, and $\sigma(k+1) < \dots < \sigma(k+l)$. Usually the set on which Sym_k acts is taken to be the standard $\{1, \dots, k\}$, but we will use some other cardinality k sets, like $\{2, 3, \dots, k+1\}$ in the definition of Q_l in Section 2. It will be clear from context exactly which set is used.

Refer to an algebra (A, ω) with n-arc product ω as *$(n-1)$-left commutative* if

$$\sum_{\sigma \in \mathrm{Sym}_{2n-2}} \operatorname{sgn}\sigma\, \omega(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, \omega(a_{\sigma(n)}, \dots, a_{\sigma(2n-2)}, a_{2n-1})) = 0$$

for all $a_1, \dots, a_{2n-2}, a_{2n-1} \in A$.

Call (A, ω) *homotopy n-Lie* [4] if $\omega \in C^n(A, A)$ and

$$\sum_{\sigma \in \mathrm{Sym}_{n-1,n}} \operatorname{sgn}\sigma\, \omega(a_{\sigma(1)}, \dots, a_{\sigma(n-1)}, \omega(a_{\sigma(n)}, \dots, a_{\sigma(2n-2)}, a_{\sigma(2n-1)})) = 0$$

for all $a_1, \dots, a_{2n-2}, a_{2n-1} \in A$. In [5] these algebras are called *n-Lie algebras*.

Call an algebra (A, Ω) a *homotopy* Lie algebra [4] if Ω is a sequence $\omega_1, \omega_2, \dots$ with $\omega_k \in C^k(A, A)$ such that

$$\sum_{\sigma \in \mathrm{Sym}_{i-1,j}} \operatorname{sgn}\sigma\, \omega_i(a_{\sigma(1)}, \dots, a_{\sigma(i-1)}, \omega_j(a_{\sigma(i)}, \dots, a_{\sigma(i+j-1)})) = 0$$

for all $a_1, \dots, a_{i+j-1} \in A$ and $i, j = 1, 2, \dots$.

We prove in Section 2 that every n-Lie algebra is $(n-1)$-left commutative and every $(n-1)$-left commutative algebra is homotopy n-Lie. Some examples of Wronskian algebras show that the converse is false.

1. Statement of the Main Result

Take a commutative associative algebra U with derivation ∂. Suppose that $V^{i_1,\dots,i_k} = \partial^{i_1} \wedge \dots \wedge \partial^{i_k}$ is a *generalized Wronskian*:

$$V^{i_1,\dots,i_k}(u_1, \dots, u_k) = \begin{vmatrix} \partial^{i_1} u_1 & \cdots & \partial^{i_1} u_k \\ \vdots & \vdots & \vdots \\ \partial^{i_k} u_1 & \cdots & \partial^{i_k} u_k \end{vmatrix}.$$

For instance, $V^{0,1,2,\dots,k}$ is the standard Wronskian.

Theorem 1.1. *Given a commutative associative algebra U over a field K of characteristic $p \geq 0$ with derivation ∂, we have:*

(i) *For each $k > 0$ the algebra $(U, V^{0,1,\dots,k})$ is homotopy $(k+1)$-Lie. Moreover, $(U, \{0, \lambda_{i+1} V^{0,1,\dots,i}, i = 1, 2, \dots\})$ is a homotopy Lie algebra for all $\lambda_i \in K, \lambda_1 = 0$.*

(ii) *The algebra $(U, V^{0,1,\dots,k})$, where $k > 0$, is k-left commutative iff $k \neq 2$.*

(iii) *The algebra $(U, V^{0,1,\dots,k})$ is $(k+1)$-Lie iff one of the following conditions is fulfilled:*

$k = 1$ and p is an arbitrary prime or 0;

$k = 2$ and $p = 2$;

$k = 3$ and $p = 3$;

$k = 4$ and $p = 2$.

By Theorem 1.1 Wronskians arise as n-Lie products for $n > 2$ only in the case of small positive characteristic $p = 2, 3$. The following result is established in [1]: if A is an n-Lie algebra with product ω then A is $(n-1)$-Lie with product $i(a)\omega$ for all $a \in A$. Here by $i(a)\omega$ we understand the $(n-1)$-product defined by the rule

$$i(a)\omega(a_1, \dots, a_{n-1}) = \omega(a, a_1, \dots, a_{n-1}).$$

Using this construction, we can obtain from $(k+1)$-Lie algebras $V^{0,1,\dots,k}$ other n-Lie algebras for $n \leq k$.

Theorem 1.2. *For $n \geq 2$ and $p = \operatorname{char} K \geq 0$, the following generalized Wronskians are n-Lie products:*

$n = 2$

$p = 2, \quad V^{2^r - 2^l, 2^r}, \; 0 \leq l \leq r;$

$p = 2, \quad \sum_{i=1}^{2^l} V^{i, 2^l + 1 - i}, \; 0 < l;$

$p = 3, \quad V^{2 \cdot 3^r, 3^{r+1}}, \; 0 \leq r;$

$n = 3$

$p = 2, \quad V^{1,2,4};$

$p = 2, \quad V^{2,3,4};$

$p = 2, \quad \sum_{i=1}^{2^l} V^{0, i, 2^l + 1 - i}, \; 0 < l;$

$p = 3, \quad V^{1,2,3};$

$n = 4$

$p = 2, \quad V^{1,2,3,4};$

$p = 3, \quad V^{0,1,2,3};$

$n = 5$

$p = 2, \quad V^{0,1,2,3,4}.$

In case of a field of characteristic 0, the Wronskian $V^{0,1,\dots,k}$ is an n-Lie product only in the case of Lie algebras: $n = 2$, $k = 1$.

Note that every 3-Lie algebra over a field of characteristic 3 forms a Lie triple system. There is a standard method for associating a Lie algebra to a Lie triple system [6]. Thus, to each 3-Lie algebra of characteristic 3 we can associate a Lie algebra. The simple Lie algebras corresponding to our 3-Lie algebras of characteristic 3 are the Kuznetsov–Ermolaev exceptional simple algebras [7]. The series of simple n-Lie algebras of characteristic p are constructed in [8]. Our n-Lie algebras generated by Wronskians are exceptional in the sense that they cannot be defined in the case of characteristic $p > 3$.

Two ideas underlie our calculations. The first is the following "polynomial trick." Suppose that some statement $\mathscr{X}$ about a unital commutative associative algebra U with commuting derivations $\mathscr{D} = \langle \partial_1, \dots, \partial_n \rangle$ follows from the linear properties of U, its commutativity, its associativity, the properties of its identity element, the Leibniz rule for $\partial_1, \dots, \partial_n$, and the commutation $[\partial_i, \partial_j] = 0$ of the derivations, $i, j = 1, 2, \dots, n$. Then $\mathscr{X}$ holds for every unital commutative associative algebra with commuting derivations.

In particular, we can take as U the polynomial algebra $K[x_1, \dots, x_n]$ with $\partial_i = \partial/\partial x_i$, or the divided powers algebra (in the case $p > 0$)

$$O_n(\mathbf{m}) = \left\{ x^\alpha = \prod_{i=1}^{n} x^{(\alpha_i)} : 0 \leq \alpha_i < p^{m_i}, \mathbf{m} = (m_1, \dots, m_n) \right\},$$

$$x^\alpha x^\beta = \prod_{i=1}^{n} \binom{\alpha_i + \beta_i}{\alpha_i} x^{\alpha + \beta},$$

with the special derivations

$$\partial_i : x^\alpha \mapsto x^{\alpha - \epsilon_i}, \quad \epsilon_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0), \quad i = 1, \dots, n.$$

The second idea has to do with $\mathscr{D}$-invariant polynomials [9]. If $\mathscr{D} = \{\partial_1, \dots, \partial_n\}$ is a system of commuting derivations then $\psi \in T^k(U, U)$ is called *$\mathscr{D}$-invariant* provided that

$$\partial \psi(u_1, \dots, u_k) = \sum_{i=1}^{k} \psi(u_1, \dots, u_{i-1}, \partial u_i, u_{i+1}, \dots, u_k)$$

for all $u_1, \dots, u_k \in U$ and $\partial \in \mathscr{D}$. In other words, ψ is $\mathscr{D}$-invariant if each $\partial \in \mathscr{D}$ is a derivation for ψ. Note that U is $\mathscr{D}$-graded:

$$U = \bigoplus_{s \geq 0} U_s, \quad U_s U_l \subseteq U_{s+l}, \quad U_0 = \langle 1 \rangle,$$
$$\partial_i U_s \subseteq U_{s-1}, \quad U^{\mathscr{D}} = \{u \in U : \partial_i u = 0 \ \forall i = 1, \dots, n\} = U_0.$$

Given a graded $\mathscr{D}$-invariant multilinear map $\psi \in T^k(U, U)$, denote by $\pi\psi$ the multilinear form $\pi\psi \in T^k(U, U_0)$ defined on the homogeneous basis elements $e_1, \dots, e_k \in U$ by $\pi\psi(e_1, \dots, e_k) = \psi(e_1, \dots, e_k)$ if $\psi(e_1, \dots, e_k) \in U_0$, and $\pi\psi(e_1, \dots, e_k) = 0$ if $\psi(e_1, \dots, e_k) \in \bigoplus_{s>0} U_s$. Call $\pi\psi$ the *support* of ψ, and call a k-tuple $(e_1, \dots, e_k)$ of homogeneous basis elements such that $\pi\psi(e_1, \dots, e_k) \neq 0$ a *supporting chain*. Denote by Γ the set of supporting chains. We can [9] reconstruct ψ from $\pi\psi$ uniquely:

$$\psi(u_1, \dots, u_k) = \sum_{\{\alpha_1, \alpha_2, \dots, \alpha_k\} \in \Gamma} \frac{\partial^{\alpha_1}(u_1)}{\alpha_1!} \dots \frac{\partial^{\alpha_k}(u_k)}{\alpha_k!} \pi\psi(x^{\alpha_1}, \dots, x^{\alpha_k}).$$

Therefore, to find a $\mathscr{D}$-multilinear form, it suffices to compute its support. We use this argument below in calculating $Q\psi$, $Q_{\text{short}}\psi$, $Q_{\text{long}}\psi$, and $Q_{\text{alt}}\psi$. In our case, $n = 1$, and the reconstruction formula reduces to

$$\psi = \sum_{i_1, \dots, i_k \in \Gamma} \lambda_{i_1, \dots, i_k} \frac{\partial^{i_1} u_1}{i_1!} \dots \frac{\partial^{i_k} u_k}{i_k!},$$

where $\lambda_{i_1, \dots, i_k} = \pi\psi(x^{i_1}, \dots, x^{i_k}) \in K$. In the divided powers case x^i and $\frac{\partial^i u}{i!}$ should be replaced by $x^{(i)}$ and $\partial^i u$.

2. Connections Between n-Lie, $(n-1)$-Left Commutative, and Homotopy n-Lie Structures

Define the quadratic maps

$$Q, Q_{\text{short}}, Q_{\text{long}}, Q_{\text{alt}} : C^k(A, A) \to T^{2k-1}(A, A)$$

as follows:

$$Q\psi(a_1, \dots, a_{2k-1}) = \psi(a_1, \dots, a_{k-1}, \psi(a_k, \dots, a_{2k-1})) - \sum_{i=0}^{k-1} \psi(a_k, \dots, a_{k+i-1}, \psi(a_1, \dots, a_{k-1}, a_{k+i}), a_{k+i+1}, \dots, a_{2k-1}),$$

$$Q_{\text{long}}\psi(a_1, \dots, a_{2k-1}) = \sum_{\sigma \in \mathrm{Sym}_{k-1,k}, \sigma(k-1) = 2k-1} \operatorname{sgn} \sigma \psi(a_{\sigma(1)}, \dots, a_{\sigma(k-2)}, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \dots, a_{\sigma(2k-1)})),$$

$$Q_{\text{short}}\psi(a_1, \dots, a_{2k-1}) = \sum_{\sigma \in \mathrm{Sym}_{k-1,k}, \sigma(2k-1) = 2k-1} \operatorname{sgn} \sigma \psi(a_{\sigma(1)}, \dots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \dots, a_{\sigma(2k-2)}, a_{2k-1})),$$

$$Q_{\text{alt}}\psi(a_1, \dots, a_{2k-1}) = \sum_{\sigma \in \mathrm{Sym}_{k-1,k}} \operatorname{sgn} \sigma \psi(a_{\sigma(1)}, \dots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \dots, a_{\sigma(2k-2)}, a_{\sigma(2k-1)})).$$

These definitions split $\{a_1, \ldots, a_{2k-1}\}$ into two types. Call the elements of the $(k-1)$-element subset $\{a_1, \ldots, a_{k-1}\}$ *short*, and those of the k-element subset $\{a_k, \ldots, a_{2k-1}\}$ *long*.

If $\psi \in C^k(A, A)$ then $Q\psi \in T^{2k-1}(A, A)$ is skew-symmetric in all short and all long arguments separately; i.e., in the first $k-1$ and last k arguments. It is easy to see that $Q_{\mathrm{short}}\psi \in T^{2k-1}(A, A)$ is skew-symmetric in all short arguments and in all but one long arguments, for $\psi \in C^k(A, A)$. Similarly $Q_{\mathrm{long}}\psi \in T^{2k-1}(A, A)$ is skew-symmetric in all short arguments and in all but one long arguments, for $\psi \in C^k(A, A)$. Note that $Q_{\mathrm{alt}}\psi \in C^{2k-1}(A, A)$.

Proposition 2.1. *Suppose that $\omega \in C^k(A, A)$ and one of the following conditions is fulfilled:*

$p = 0$, $k > 2$;

$p > 0$, $k \not\equiv 0, 1 (\mathrm{mod}\, p)$, $k \equiv 1 (\mathrm{mod}\, 2)$;

$p > 0$, $k \not\equiv -1, 2 (\mathrm{mod}\, p)$, $k \equiv 0 (\mathrm{mod}\, 2)$.

Then

(i) *if $Q\omega = 0$ then $Q_{\mathrm{short}}\omega = 0$ and $Q_{\mathrm{long}}\omega = 0$;*

(ii) *if $Q_{\mathrm{short}}\omega = 0$ or $Q_{\mathrm{long}}\omega = 0$ then $Q_{\mathrm{alt}}\omega = 0$.*

Proof. Call $\sigma \in \mathrm{Sym}_{k-1,k}$ a *short permutation* if $\sigma(k-1) = 2k-1$. To each short permutation σ there correspond a $(k-2)$-tuple $r'(\sigma)$ and a k-tuple $r''(\sigma)$ defined by

$$r'(\sigma) = \{\sigma(1), \ldots, \sigma(k-2)\}, \quad r''(\sigma) = \{\sigma(k), \ldots, \sigma(2k-1)\}.$$

Call $\sigma \in \mathrm{Sym}_{k-1,k}$ a *long permutation* if $\sigma(2k-1) = 2k-1$. To each long permutation σ there correspond a $(k-1)$-tuple $r'(\sigma)$ and a $(k-1)$-tuple $r''(\sigma)$ defined by

$$r'(\sigma) = \{\sigma(1), \ldots, \sigma(k-1)\}, \quad r''(\sigma) = \{\sigma(k), \ldots, \sigma(2k-2)\}.$$

Note that

$$r'(\sigma) \cup r''(\sigma) = \{1, \ldots, 2k-2\}$$

for all $\sigma \in \mathrm{Sym}_{k-1,k}$. In other words, every $r'(\sigma)$ is uniquely determined by $r''(\sigma)$.

Call the element

$$A_\sigma := \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)}))$$

short (*long*) *σ-element*, or simply *short* (*long*) *element* if σ is a short (long) permutation. Denote by $\mathrm{Sym}^s_{k-1,k}$ and $\mathrm{Sym}^l_{k-1,k}$ the sets of all short and long permutations. It is obvious that

$$\mathrm{Sym}_{k-1,k} = \mathrm{Sym}^s_{k-1,k} \cup \mathrm{Sym}^l_{k-1,k}$$

and

$$Q_{\mathrm{alt}}\psi = Q_{\mathrm{short}}\psi + Q_{\mathrm{long}}\psi. \tag{1}$$

The identity $Q\psi = 0$ for $\psi \in C^k(A, A)$ yields

$$\begin{aligned} &\psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)})) \\ &= \sum_{i=0}^{k-1} (-1)^{k-i-1} \psi(a_{\sigma(k)}, \ldots, \widehat{a_{\sigma(k+i)}}, \ldots, a_{\sigma(2k-1)}, \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, a_{\sigma(k+i)})). \end{aligned} \tag{2}$$

In particular, (2) means that each short σ-element can be written as a sum of k long elements. More precisely, a short element A_σ is the sum of k long elements A_τ for such long permutations τ that $r'(\tau) \subset r''(\sigma)$. Since $|r''(\sigma) \setminus r'(\tau)| = 1$, there is only one element, say i, such that $r''(\sigma) = r'(\tau) \cup \{i\}$. Hence, $i \le 2k-2$ and i is not equal to $k-1$ elements in $r'(\tau)$. Thus, there exist $k-1$ choices for i.

In other words, in accordance with (2)

$$Q_{\mathrm{long}}\psi(a_1, \ldots, a_{2k-1}) = \sum_{\sigma \in \mathrm{Sym}^s_{k-1,k}} \mathrm{sgn}\, \sigma \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)}))$$

can be written as

$$(k-1)\sum_{\sigma\in \mathrm{Sym}^l_{k-1,k}} \psi(a_{\sigma(1)},\dots,a_{\sigma(k-1)},\psi(a_{\sigma(k)},\dots,a_{\sigma(2k-1)})).$$

Therefore, the conditions $\psi \in C^k(A,A)$ and $Q\psi = 0$ yield

$$Q_{\mathrm{long}}\psi = (k-1)Q_{\mathrm{short}}\psi. \tag{3}$$

Let us reinterpret (2). If σ is a long permutation then A_σ is a sum of $k-1$ short elements and one long element $A_{\tilde\sigma}$, where

$$\tilde\sigma = \begin{pmatrix} 1 & \cdots & k-1 & k & \cdots & 2k-2 & 2k-1 \\ \sigma(k) & \cdots & \sigma(2k-2) & \sigma(1) & \cdots & \sigma(k-1) & 2k-1 \end{pmatrix}.$$

Thus, A_σ can be written as the sum of short elements A_τ, where $r'(\tau) \subset r''(\sigma)$. More precisely, $r''(\sigma) \setminus r'(\tau) = \{i\}$ for some $i \in \{1,2,\dots,2k-2\}$, and i can be equal to one of $k-2$ elements in $r'(\tau)$. Thus, here there are k possibilities for a long permutation σ such that A_τ can be one of the terms in A_σ. Note that

$$\operatorname{sgn}\sigma = (-1)^{k-1}\operatorname{sgn}\tilde\sigma.$$

Consequently, the sum of $\operatorname{sgn}\sigma A_\sigma$ over all $\sigma \in \mathrm{Sym}^l_{k-1,k}$ in accordance with (2) yields

$$Q_{\mathrm{short}}\psi = kQ_{\mathrm{long}}\psi + (-1)^{k-1}Q_{\mathrm{short}}\psi. \tag{4}$$

The determinant of the linear system (3)–(4) is equal to

$$\begin{vmatrix} 1 & -k+1 \\ k & -1-(-1)^k \end{vmatrix} = -k^2+k+1+(-1)^k;$$

hence, the conditions $Q\psi = 0$ and $\psi \in C^k(A,A)$ imply the identities

$$\begin{gathered} Q_{\mathrm{long}}\psi + 2Q_{\mathrm{short}}\psi = 0, \quad k \equiv -1 (\operatorname{mod} p),\ k \equiv 0 (\operatorname{mod} 2),\ p>0, \\ Q_{\mathrm{long}}\psi - Q_{\mathrm{short}}\psi = 0, \quad k \equiv 2 (\operatorname{mod} p),\ k \equiv 0 (\operatorname{mod} 2),\ p>0, \\ Q_{\mathrm{long}}\psi + Q_{\mathrm{short}}\psi = 0, \quad k \equiv 0 (\operatorname{mod} p),\ k \equiv 1 (\operatorname{mod} 2),\ p>0, \\ Q_{\mathrm{long}}\psi = 0, \quad k \equiv 1 (\operatorname{mod} p),\ k \equiv 1 (\operatorname{mod} 2),\ p>0, \end{gathered}$$

and

$$Q_{\mathrm{long}}\psi = 0, \quad Q_{\mathrm{short}}\psi = 0, \quad p = 0,\ k>2.$$

Thus, by (1)

$$Q_{\mathrm{alt}}\psi = 0$$

if $p=0$, and $k>2$ or $k \not\equiv 0,1 (\operatorname{mod} p)$, $k \equiv 1 (\operatorname{mod} 2)$, or $k \not\equiv -1,2 (\operatorname{mod} p)$, $k \equiv 0 (\operatorname{mod} 2)$.

Corollary 2.2. *If (A,ω) is an n-Lie algebra then it is $(n-1)$-left commutative. If (A,ω) is $(n-1)$-left commutative then it is homotopy n-Lie.*

In particular, every n-Lie algebra is homotopy n-Lie. Theorems 1.1 and 1.2 show that the converse is not true.

3. Proofs of Theorems 1.1 and 1.2

Take $\mathbf{Z}^n = \{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbf{Z}\}$ and $\mathbf{Z}^n_+ = \{\alpha \in \mathbf{Z}^n : \alpha_i \geq 0,\ i = 1, \ldots, n\}$. Given $\alpha \in \mathbf{Z}^n$, put

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Take a commutative associative algebra U with a derivation ∂. Take $\psi \in C^k(U,U)$ and $g \in C^l(U,U)$. Define $f \smile g, f \wedge g \in C^{k+l}(U,U)$ and bilinear maps

$$Q(f,g), Q_{\mathrm{alt}}(f,g) \in C^{k+l-1}(U,U), \quad Q_{\mathrm{short}}(f,g) \in T^{k+l-1}(U,U)$$

as follows:

$$f \smile g(u_1, \ldots, u_{k+l}) = f(u_1, \ldots, u_k) g(u_{k+1}, \ldots, u_{k+l}),$$

$$f \wedge g(u_1, \ldots, u_{k+l}) = \sum_{\sigma \in \mathrm{Sym}_{k+l}} \operatorname{sgn} \sigma (f \smile g)(u_{\sigma(1)}, \ldots, u_{\sigma(k+l)}),$$

$$f \star g(u_1, \ldots, u_{k+l-1}) = f(u_1, \ldots, u_{k-1}, g(u_k, \ldots, u_{k+l-1})),$$

$$Q(f,g) = f(u_1, \ldots, u_{k-1}, g(u_k, \ldots, u_{k+l-1}))$$

$$- \sum_{i=1}^{l} g(u_k, \ldots, u_{k+i-2}, f(u_1, \ldots, u_{k-1}, u_{k+i-1}), u_{k+i}, \ldots, u_{k+l-1}),$$

$$Q_{\mathrm{alt}}(f,g)(u_1, \ldots, u_{k+l-1}) = \sum_{\sigma \in \mathrm{Sym}_{k-1,l}} \operatorname{sgn} \sigma (f \star g)(u_{\sigma(1)}, \ldots, u_{\sigma(k+l-1)}),$$

$$Q_{\mathrm{short}}(f,g)(u_1, \ldots, u_{k+l-1}) = \sum_{\sigma \in \mathrm{Sym}_{k-1,l-1}} \operatorname{sgn} \sigma (f \star g)(u_{\sigma(1)}, \ldots, u_{\sigma(k+l-2)}, u_{k+l-1}).$$

Note that the definitions of Q as bilinear maps agree with the definitions of Q as quadratic maps in the previous section:

$$Q(f,f) = Q(f), \quad Q_{\mathrm{alt}}(f,f) = Q_{\mathrm{alt}}(f), \quad Q_{\mathrm{short}}(f,f) = Q_{\mathrm{short}}(f).$$

Since $(C^*(U,U), \smile)$ is associative, so is $(C^*(U,U), \wedge)$. Put

$$C^k_{\mathrm{loc},s}(U) = \{\partial^{i_1} \wedge \cdots \wedge \partial^{i_k} : 0 \leq i_1 < \cdots < i_k, i_1 + \cdots + i_k = s\}$$

and

$$C^k_{\mathrm{loc}}(U) = \bigoplus_s C^k_{\mathrm{loc},s}(U).$$

Note that $V^\alpha \in C^k_{\mathrm{loc},|\alpha|}(U)$ for all $\alpha \in \mathbf{Z}^n_+$. Put $|\psi| = s$ for $\psi \in C^k_{\mathrm{loc},s}(U)$.

Given $\psi \in T^k(A,A)$, define $i(a)\psi \in T^{k-1}(A,A)$ by

$$i(a)\psi(a_1, \ldots, a_{k-1}) = \psi(a, a_1, \ldots, a_{k-1}).$$

Proposition 3.1. *If ψ is k-Lie then $i(a)\psi$ is $(k-1)$-Lie for each $a \in A$.*

Proof. See [1].

Proposition 3.1 can be modified as follows:

Lemma 3.2. *If ψ is k-Lie then for the $(k-l)$-Lie product $\psi_l := i(a_1)i(a_2)\cdots i(a_l)\psi$ we have $Q(\psi_i, \psi_j) = 0$, $i \leq j$.*

Lemma 3.3. *$C^k_{\mathrm{loc},s}(U) = 0$ for $s < k(k-1)/2$.*

Proof. If $0 \neq \partial^{i_1} \wedge \cdots \wedge \partial^{i_k} \in C^k_{\mathrm{loc},s}(U)$ then $s = i_1 + \cdots + i_k \geq 0+1+2+\cdots+(k-1) = (k-1)k/2$.

Lemma 3.4 (see [10]). *$Q_{\mathrm{alt}}(V^\alpha, V^\beta) \in C^{k+l-1}_{\mathrm{loc},|\alpha|+|\beta|}(U)$ for all $\alpha \in \mathbf{Z}^k_+$ and $\beta \in \mathbf{Z}^l_+$.*

Corollary 3.5. *$Q_{\mathrm{alt}}(V^{0,1,\dots,k}, V^{0,1\dots,l}) = 0$ for all $k, l > 0$.*

Proof. Note that $|V^{0,1,\dots,k}| = k(k+1)/2$; hence,

$$Q_{\mathrm{alt}}(V^{0,1,\dots,k}, V^{0,1,\dots,l}) \in C^{k+l+1}_{\mathrm{loc},(k^2+l^2+k+l)/2}(U).$$

It is obvious that $(k^2+l^2+k+l)/2 < (k+l+1)(k+l)/2$, and so $C^{k+l+1}_{\mathrm{loc},(k^2+l^2+k+l)/2}(U) = 0$ by Lemma 3.3. Thus, $Q_{\mathrm{alt}}(V^{0,1,\dots,k}, V^{0,1,\dots,l}) = 0$.

Lemma 3.6. *If $k > 2$ then $Q_{\mathrm{short}}(V^{0,1,\dots,k}, V^{0,1,\dots,k}) = 0$. If $k = 2$ then $Q_{\mathrm{short}}(V^{0,1,2}, V^{0,1,2}) = 2V^{0,1,2,3} \smile \mathrm{id}$.*

Proof. Note that $Q_{\mathrm{short}}(V^{0,1,\dots,k}, V^{0,1,\dots,k})$ is a linear combination of cochains $(\partial^{i_1} \wedge \cdots \wedge \partial^{i_{2k}}) \smile \partial^{i_{2k+1}}$ such that $i_1 + \cdots + i_{2k} + i_{2k+1} = k^2 + k$ and $0 \leq i_1 < i_2 < \cdots < i_{2k}$. We have $i_1 + \cdots + i_{2k} \geq 0 + 1 + 2 + \cdots + (2k-1) = (2k-1)k$. Consequently,

$$k^2 + k = i_1 + \cdots + i_{2k+1} > (2k-1)k.$$

The inequality is impossible for $k > 2$.

Consider the case $k = 2$. We have

$$Q_{\mathrm{short}}(V^{0,1,2}, V^{0,1,2}) = \lambda V^{0,1,2,3} \smile \partial^0 \tag{5}$$

for some $\lambda \in K$. We deduced this formula using only the associativity, commutativity, linear properties of U, and the Leibniz rule for derivations. Therefore, (5) holds for every commutative associative algebra U with derivation ∂, and λ is independent of U and ∂. In particular, we can take $U = K[x]$ and $\partial = \partial/\partial x$. We have

$$Q_{\mathrm{short}}(V^{0,1,2}, V^{0,1,2})(1, x, x^2, x^3, 1) = \lambda V^{0,1,2,3} \smile \partial^0(1, x, x^2, x^3, 1).$$

Furthermore,

$$\begin{aligned} &Q_{\mathrm{short}}(V^{0,1,2}, V^{0,1,2})(1, x, x^2, x^3, 1) \\ &= V^{0,1,2}(1, x, V^{0,1,2}(x^2, x^3, 1)) - V^{0,1,2}(1, x^2, V^{0,1,2}(x, x^3, 1)) \\ &+ V^{0,1,2}(1, x^3, V^{0,1,2}(x, x^2, 1)) + V^{0,1,2}(x, x^2, V^{0,1,2}(1, x^3, 1)) \\ &- V^{0,1,2}(x, x^3, V^{0,1,2}(x, x^2, 1)) + V^{0,1,2}(x^2, x^3, V^{0,1,2}(1, x, 1)) \\ &= \partial^2(V^{0,1,2}(1, x^2, x^3)) - V^{0,1,2}(1, x^2, 6x)) + V^{0,1,2}(1, x^3, 2x^0) \\ &+ V^{0,1,2}(x, x^2, 0) - V^{0,1,2}(x, x^3, 2x^0) + V^{0,1,2}(x^2, x^3, 0) = 24 \end{aligned}$$

and

$$\lambda V^{0,1,2,3} \smile \partial^0(1, x, x^2, x^3, 1) = 12\lambda.$$

Thus, $\lambda = 2$.

Lemma 3.7. *Take $\psi \in C^k(A, A)$. Let X and Y be the linear spans of the sets $\{a_1, \dots, a_{k-1}\}$ and $\{b_1, \dots, b_k\}$. If $X \subseteq Y$ then $Q\psi(a_1, \dots, a_{k-1}, b_1, \dots, b_k) = 0$.*

Proof. Suppose that $X \subseteq Y$ and $\dim Y = l \leq k$.

Since ψ is skew-symmetric for $l < k$, we have $\psi(b_1, \dots, b_k) = 0$ for all $b_1, \dots, b_k \in Y$, and $\psi(a_1, \dots, a_{k-1}, b_i) = 0$ for all $a_1, \dots, a_{k-1} \in X \subseteq Y, b_i \in Y$. Thus, in this case

$$Q\psi(a_1, \dots, a_{k-1}, b_1, \dots, b_k) = 0.$$

Suppose that $\dim Y = k$. If $\dim X < k-1$ then similar reasoning shows that

$$\psi(e_{i_1}, \dots, e_{i_{k-1}}, c) = 0$$

for all $c \in A$. Thus, the lemma holds in this case.

It remains to consider the case $\dim X = k-1$, $\dim Y = k$.

Take a basis $\{e_1, \dots, e_k\}$ of Y such that $\{e_1, \dots, e_{k-1}\}$ is a basis of X. Since $Q\psi$ is multilinear in $a_1, \dots, a_{k-1}$ and $b_1, \dots, b_k$; to prove the lemma, it suffices to establish that

$$Q\psi(e_1, \dots, e_{k-1}, e_1, \dots, e_k) = 0.$$

Since ψ is skew-symmetric,

$$\psi(e_1, \dots, e_{k-1}, e_i) = 0$$

for all $i \le k-1$. Thus,

$$\sum_{i=1}^{k} \psi(e_1, \dots, e_{i-1}, \psi(e_1, \dots, e_{k-1}, e_i), e_{i+1}, \dots, e_k) = \psi(e_1, \dots, e_{k-1}, \psi(e_1, \dots, e_{k-1}, e_k)).$$

In other words, $Q\psi(e_1, \dots, e_{k-1}, e_1, \dots, e_k) = 0$. This completes the proof of the lemma.

Lemma 3.8. *If $p = 3$ then $V^{0,1,2,3}$ is a 4-Lie product.*

Proof. We have

$$|V^{0,1,2,3}| = 6 \Rightarrow |QV^{0,1,2,3}| = 12.$$

Here is the list of $(3,4)$-partitions of 12:

$$\Gamma_{3,4}(12) = \big\{(\{0,1,2\},\{0,1,2,6\}),\ (\{0,1,2\},\{0,1,3,5\}),\ (\{0,1,2\},\{0,2,3,4\}),$$
$$(\{0,1,3\},\{0,1,2,5\}),\ (\{0,1,3\},\{0,1,3,4\}),\ (\{0,1,4\},\{0,1,2,4\}),\ (\{0,1,5\},\{0,1,2,3\}),$$
$$(\{0,2,3\},\{0,1,2,4\}),\ (\{0,2,4\},\{0,1,2,3\}),\ (\{1,2,3\},\{0,1,2,3\})\big\}.$$

Thus,

$$QV^{0,1,2,3} = \sum_{(\alpha,\beta)\in\Gamma_{3,4}(12)} \lambda_{(\alpha,\beta)} V^{\alpha} \smile V^{\beta}, \tag{6}$$

where $\alpha = \{i_1, i_2, i_3\}$, $\beta = \{i_4, i_5, i_6, i_7\}$, $0 \le i_1 < i_2 < i_3$, $0 \le i_4 < i_5 < i_6 < i_7$, $i_1 + \dots + i_7 = 12$. Formula (6) is deduced using only the Leibniz rule, and so it is universal: the coefficients $\lambda_{\alpha,\beta}$ are independent of U and ∂. In particular, we can take $U = \mathbf{Q}[x]$ and $\partial = \partial/\partial_x$. To find $\lambda_{\alpha,\beta}$, we can take $a_l = x^{i_l}$, $l = 1, \dots, 7$, and compute $QV^{0,1,2,3}$ in $k[x]$. We have

$$\lambda_{\alpha,\beta} = \frac{1}{i_1! \dots i_7!} QV^{0,1,2,3}.$$

By Lemma 3.7

$$QV^{0,1,2,3}(1, x, x^2, 1, x, x^2, x^6) = 0,$$
$$QV^{0,1,2,3}(1, x, x^3, 1, x, x^3, x^4) = 0,$$
$$QV^{0,1,2,3}(1, x, x^4, 1, x, x^2, x^4) = 0.$$

Thus, $\lambda_{\alpha,\beta} = 0$ for

$$(\alpha,\beta) \in \big\{(\{0,1,2\},\{0,1,2,6\}),\ (\{0,1,3\},\{0,1,3,4\}),\ (\{0,1,4\},\{0,1,2,4\})\big\}.$$

Further,

$$V^{0,1,2,3}(1,x^2,x^3,x^4)=\begin{vmatrix}1 & x^2 & x^3 & x^4\\ 0 & 2x & 3x^2 & 4x^3\\ 0 & 2 & 6x & 12x^2\\ 0 & 0 & 6 & 24x\end{vmatrix}=48x^3,$$

$$V^{0,1,2,3}(1,x,x^2,x^4)=\begin{vmatrix}1 & x & x^2 & x^4\\ 0 & 1 & 2x & 4x^3\\ 0 & 0 & 2 & 12x^2\\ 0 & 0 & 0 & 24x\end{vmatrix}=48x.$$

Thus,

$$QV^{0,1,2,3}(1,x,x^2,1,x^2,x^3,x^4)=V^{0,1,2,3}(1,x,x^2,V^{0,1,2,3}(1,x^2,x^3,x^4))-0-0$$
$$-V^{0,1,2,3}(1,x^2,x^3,V^{0,1,2,3}(1,x,x^2,x^4))=48V^{0,1,2,3}(1,x,x^2,x^3)-48V^{0,1,2,3}(1,x^2,x^3,x)=0$$

and

$$\lambda_{\{0,1,2\},\{0,2,3,4\}}=0.$$

We have

$$V^{0,1,2,3}(1,x,x^3,x^5)=\begin{vmatrix}1 & x & x^3 & x^5\\ 0 & 1 & 3x^2 & 5x^4\\ 0 & 0 & 6x & 20x^3\\ 0 & 0 & 6 & 60x^2\end{vmatrix}=240x^3,$$

$$V^{0,1,2,3}(1,x,x^2,x^5)=\begin{vmatrix}1 & x & x^2 & x^5\\ 0 & 1 & 2x & 5x^4\\ 0 & 0 & 2 & 20x^3\\ 0 & 0 & 0 & 60x^2\end{vmatrix}=120x^2.$$

Consequently,

$$\begin{gathered}QV^{0,1,2,3}(1,x,x^2,1,x,x^3,x^5)\\ =V^{0,1,2,3}(1,x,x^2,V^{0,1,2,3}(1,x,x^3,x^5))-0-0-V^{0,1,2,3}(1,x,x^3,V^{0,1,2,3}(1,x,x^2,x^5))\\ =240V^{0,1,2,3}(1,x,x^2,x^3)-120V^{0,1,2,3}(1,x,x^3,x^2)\\ =360V^{0,1,2,3}(1,x,x^2,x^3)=4320\end{gathered}$$

and

$$\lambda_{\{0,1,2\},\{0,1,3,5\}}=\frac{4320}{0!1!2!0!1!3!5!}=3.$$

Similar calculations show that

$$\lambda_{\{0,1,3\},\{0,1,2,5\}}=-3,\lambda_{\{0,1,5\},\{0,1,2,3\}}=3,\lambda_{\{0,2,3\},\{0,1,2,4\}}=\lambda_{\{0,2,4\},\{0,1,2,3\}}=0.$$

We have thus established that

$$QV^{0,1,2,3}=3V^{0,1,2}V^{0,1,3,5}-3V^{0,1,3}V^{0,1,2,5}+3V^{0,1,5}V^{0,1,2,3}.$$

In particular, $QV^{0,1,2,3}=0$ if $p=3$.

Corollary 3.9. *If $p=3$ then $V^{1,2,3}$ is a 3-Lie product, and $V^{2,3}$ is a 2-Lie product.*

Proof. This follows from Lemma 3.8, Proposition 3.1, and the formulas

$$i(1)V^{0,1,2,3}=V^{1,2,3},\quad i(x)V^{1,2,3}=V^{2,3}.$$

Lemma 3.10. *If $p = 2$ then $V^{0,1,2,3,4}$ is a 5-Lie product.*

The proof is similar to that of Lemma 3.8, and so we skip details. We have

$$|V^{0,1,2,3,4}| = 10 \Rightarrow |QV^{0,1,2,3}| = 20.$$

Consequently, $QV^{0,1,2,3}$ is a linear combination of $V^\alpha \smile V^\beta$, where $(\alpha, \beta) \in \Gamma_{4,5}(20)$, and

$$\begin{gathered}\Gamma_{4,5}(20) = \big\{(\{0,1,2,3\},\{0,1,2,3,8\}),\ (\{0,1,2,3\},\{0,1,2,4,7\}),\\ (\{0,1,2,3\},\{0,1,2,5,6\}),\ (\{0,1,2,3\},\{0,1,3,4,6\}),\ (\{0,1,2,3\},\{0,2,3,4,5\}),\\ (\{0,1,2,4\},\{0,1,2,3,7\}),\ (\{0,1,2,4\},\{0,1,2,4,6\}),\ (\{0,1,2,4\},\{0,1,3,4,5\}),\\ (\{0,1,2,5\},\{0,1,2,3,6\}),\ (\{0,1,2,5\},\{0,1,2,4,5\}),\ (\{0,1,3,4\},\{0,1,2,3,6\}),\\ (\{0,1,3,4\},\{0,1,2,4,5\}),\ (\{0,1,2,6\},\{0,1,2,3,5\}),\ (\{0,1,3,5\},\{0,1,2,3,5\}),\\ (\{0,2,3,4\},\{0,1,2,3,5\}),\ (\{0,1,2,7\},\{0,1,2,3,4\}),\ (\{0,1,3,6\},\{0,1,2,3,4\}),\\ (\{0,1,4,5\},\{0,1,2,3,4\}),\ (\{0,2,3,5\},\{0,1,2,3,4\}),\ (\{1,2,3,4\},\{0,1,2,3,4\})\big\}.\end{gathered}$$

Thus, there exists $\lambda_{\alpha,\beta} \in \mathbf{Z}$ such that

$$QV^{0,1,2,3,4} = \sum_{(\alpha,\beta)\in\Gamma_{4,5}(20)} \lambda_{(\alpha,\beta)} V^\alpha \smile V^\beta.$$

Calculations like those in the proof of Lemma 3.8 show that

$$\begin{aligned}QV^{0,1,2,3,4} &= 4V^{0,1,2,7}V^{0,1,2,3,4} + 2V^{0,1,3,6}V^{0,1,2,3,4} - 2V^{0,1,4,5}V^{0,1,2,3,4}\\ &+2V^{0,2,3,5}V^{0,1,2,3,4} + 2V^{0,1,2,6}V^{0,1,2,3,5} - 2V^{0,2,3,4}V^{0,1,2,3,5}\\ &-2V^{0,1,2,5}V^{0,1,2,3,6} - 2V^{0,1,3,4}V^{0,1,2,3,6} - 4V^{0,1,2,4}V^{0,1,2,3,7}\\ &+2V^{0,1,3,4}V^{0,1,2,4,5} + 4V^{0,1,2,3}V^{0,1,2,4,7} + 2V^{0,1,2,3}V^{0,1,2,5,6}\\ &-2V^{0,1,2,4}V^{0,1,3,4,5} + 2V^{0,1,2,3}V^{0,1,3,4,6} + 2V^{0,1,2,3}V^{0,2,3,4,5}.\end{aligned}$$

In particular, if $p = 2$ then $QV^{0,1,2,3,4} = 0$.

Corollary 3.11. *If $p = 2$ then $V^{1,2,3,4}$ is a 4-Lie product, $V^{2,3,4}$ is a 3-Lie product, and $V^{3,4}$ is a 2-Lie product.*

Proof. This follows from Lemma 3.10, Proposition 3.1, and the formulas

$$V^{1,2,3,4} = i(1)V^{0,1,2,3,4}, \quad V^{2,3,4} = i(x)V^{1,2,3,4}, \quad V^{3,4} = i(x^2)V^{2,3,4}/2.$$

Remark. The explicit expressions for $QV^{1,2,3,4}, QV^{2,3,4}, QV^{3,4}$, $QV^{1,2,3}, QV^{2,3}$ over $\mathbf{Z}$ follow easily from the calculations of $QV^{0,1,2,3,4}$ and $QV^{0,1,2,3}$. For instance,

$$\begin{aligned}QV^{1,2,3,4} &= 4V^{1,2,7}V^{1,2,3,4} + 2V^{1,3,6}V^{1,2,3,4} - 2V^{1,4,5}V^{1,2,3,4}\\ &+2V^{2,3,5}V^{1,2,3,4} + 2V^{1,2,6}V^{1,2,3,5} - 2V^{2,3,4}V^{1,2,3,5}\\ &-2V^{1,2,5}V^{1,2,3,6} - 2V^{1,3,4}V^{1,2,3,6} - 4V^{1,2,4}V^{1,2,3,7}\\ &+2V^{1,3,4}V^{1,2,4,5} + 4V^{1,2,3}V^{1,2,4,7} + 2V^{1,2,3}V^{1,2,5,6}\\ &-2V^{1,2,4}V^{1,3,4,5} + 2V^{1,2,3}V^{1,3,4,6} + 2V^{1,2,3}V^{2,3,4,5}\end{aligned}$$

and

$$QV^{1,2,3} = 3V^{1,2}V^{1,3,5} - 3V^{1,3}V^{1,2,5} + 3V^{1,5}V^{1,2,3}.$$

Lemma 3.12. *Suppose that $U = O_1(m), p > 0$. We have*

(i) $\partial^q \in \operatorname{Der} U$, *iff $q = p^k$ for some $k > 0$;*

(ii) $\partial^{p^k-1} \wedge \partial^{p^k}$ *is 2-Lie iff either $p = 2$ or $p = 3$ and $k = 1$;*

(iii) $\partial^{p^k-2} \wedge \partial^{p^k-1} \wedge \partial^{p^k}$ *is 3-Lie iff $p = 3$ and $k = 1$, or $p = 2$ and $k = 1$, or $p = 2$ and $k = 2$.*

Proof. The case $k = 1$ is dealt with above. Since ∂^{p^k} is a derivation, the polynomial principle shows that the statement holds in general.

Corollary 3.13. (i) $p = 3$. *For all $k \in \mathbf{Z}_+$ the operation $\partial^{p^k-p^{k-1}} \wedge \partial^{p^k}$ is 2-Lie.*

(ii) $p = 2$. *For all $k, l \in \mathbf{Z}_+$ with $k > l$ the operation $\partial^{p^k-p^l} \wedge \partial^{p^k}$ is 2-Lie.*

Proof. (i) For $p = 3$ we have $p^k - p^{k-1} = 2p^{k-1}$ and $p^k = 3p^{k-1}$. Since $F = \partial^{p^{k-1}} \in \operatorname{Der} U$, the claim follows from Lemma 3.12(ii) applied to F in place of ∂.

(ii) For $p = 2$ we have $p^k - p^l = p^l(p^{k-l} - 1)$ and $p^k = p^{k-l}p^l$. Thus, for $F = \partial^{p^l}$, we have $\partial^{p^k} = F^{p^{k-l}}$ and $\partial^{p^k-p^l} = F^{p^{k-l}-1}$. The claim follows from Lemma 3.12(ii) applied to $F^{p^{k-l}}$ in place of ∂^{p^k}.

Proof of Theorem 1.1.

(i) See Corollary 3.5.

(ii) See Lemma 3.6.

(iii) Suppose that $(U, V^{0,1,\dots,q})$ is $(q+1)$-Lie. If $q = 1$ then it is also 2-Lie for every characteristic p.

Suppose that $q > 1$. By Proposition 3.1 $V^q = i(1)i(x)\cdots i(x^{(q-1)})V^{0,1,\dots,q}$ is 1-Lie; i.e., $\partial^q \in \operatorname{Der} U$. This is impossible for $q > 1$ and $p = 0$.

Thus, $p > 0$. Take $U = O_1(m)$. By Lemma 3.12(i) q must be a power of p. Suppose that $q = p^t$.

By Proposition 3.1 $V^{p^t-1,p^t} = i(1)i(x)\dots i(x^{(p^t-2)})V^{0,1,\dots,p^t}$ is 2-Lie. By Lemma 3.12(ii) this is possible in the cases $p = 2$ or $p = 3$ and $t = 1$.

By Proposition 3.1 $V^{2^t-2,2^t-1,2^t} = i(1)i(x)\cdots i(x^{(2^t-3)})V^{0,1,\dots,2^t}$ is 3-Lie. By Lemma 3.12(iii) this is possible only in the cases $p = 2$, $t = 1$ or $p = 2$, $t = 2$. This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 follows from Corollaries 3.9 and 3.11. Similarly it is possible to prove that $\sum_{i=1}^{2^l} V^{0,i,2^l+1-i}$ is 3-Lie for $p = 2$. By Proposition 3.1 $\sum_{i=1}^{2^l} V^{i,2^l+1-i}$ is 2-Lie for $p = 2$.

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