\textbf{n-LIE STRUCTURES THAT ARE GENERATED BY WRONSKIANS}

\textbf{A. S. Dzhumadil’daev}

\textbf{Abstract:} We study the \((k+1)\)-Lie structures, \(k\)-left commutative and homotopy \((k+1)\)-Lie structures with multiplication generated by Wronskians and prove that the nontrivial structures of \(n\)-Lie algebras appear only in the case of small characteristic.

\textbf{Keywords:} \(n\)-Lie algebra, homotopy algebra, modular Lie algebras, Wronskian, Jacobian

Given a commutative associative algebra \(U\) with commuting derivations \(\partial_1, \ldots, \partial_n\), the Jacobian on \(U\) is the determinant

\[
\text{Jac}_n(u_1, \ldots, u_n) = \begin{vmatrix}
\partial_1 u_1 & \cdots & \partial_1 u_n \\
\vdots & \ddots & \vdots \\
\partial_n u_1 & \cdots & \partial_n u_n
\end{vmatrix}
\]

It is shown in [1, 2] that \((U, \text{Jac}_n)\) as an \(n\)-arc algebra is \(n\)-Lie, for the Jacobian satisfies the Leibniz rule

\[
\text{Jac}_n(u_1, \ldots, u_{n-1}, \text{Jac}_n(u_n, \ldots, u_{2n-1})) = \sum_{i=n}^{2n-1} (-1)^{i+n} \text{Jac}_n(u_1, \ldots, u_{n-1}, u_i, u_n, \ldots, \hat{u}_i, \ldots, u_{2n-1}),
\]

where \(\hat{u}_i\) indicates the omission of \(u_i\).

Another remarkable determinant is the Wronskian. Given a commutative associative algebra \(U\) with derivation \(\partial\), the Wronskian on it is defined by the rule

\[
W^{0,1,\ldots,k}(u_0, \ldots, u_k) = \begin{vmatrix}
u_0 & u_1 & \cdots & u_k \\
\partial u_0 & \partial u_1 & \cdots & \partial u_k \\
\vdots & \vdots & \ddots & \vdots \\
\partial^k u_0 & \partial^k u_1 & \cdots & \partial^k u_k
\end{vmatrix}
\]

The goal of this article is to study \(U\) as an \(n\)-arc algebra with respect to the product generated by the Wronskian.

Take some vector spaces \(A\) and \(M\), and the space \(T^k(A, M) = \text{Hom}(A^\otimes k, M)\) of multilinear maps \(A \times \cdots \times A \to M\) in \(k\) arguments. Put \(T^0(A, M) = M\), \(T^k(A, M) = 0\) for \(k < 0\), and \(T^*(A, M) = \bigoplus_k T^k(A, M)\).

Take the \(k\)th exterior power \(\wedge^k A\) and the subspace \(C^k(A, M) = \text{Hom}(\wedge^k A, M)\) of \(T^k(A, M)\). Put \(C^0(A, M) = M\), \(C^k(A, M) = 0\) for \(k < 0\), and \(C^*(A, M) = \bigoplus_k C^k(A, M)\).

Suppose that \(A\) is an algebra with signature \(\Omega\). This means [3] that \(\Omega\) is the set of multilinear maps \(A \times \cdots \times A \to A\). Call \(\omega \in \Omega\) an \(n\)-arc product if \(\omega \in T^n(A, A)\). Put \(|\omega| = n\) for \(\omega \in T^n(A, A)\). When the signature is important, we will write \((A, \Omega)\) instead of \(A\). If \(\Omega\) consists of a single element \(\omega\), put \(A = (A, \omega)\).

Let \((A, \omega)\) be an \(n\)-arc algebra on a vector space \(A\) over a field \(K\) of characteristic \(p \geq 0\), where \(\omega\) is a multilinear \(n\)-arc map \(A \times \cdots \times A \to A\). Recall that a linear map \(D : A \to A\) is called a derivation of \(A\) if

\[
D(\omega(a_1, \ldots, a_n)) = \sum_{i=1}^n \omega(a_1, \ldots, a_{i-1}, D(a_i), a_{i+1}, \ldots, a_n)
\]

for all \(a_1, \ldots, a_n \in A\). Take the linear map \(L_{a_1,\ldots,a_{n-1}} : A \to A\) defined by the rule \(L_{a_1,\ldots,a_{n-1}} a = \omega(a_1, \ldots, a_{n-1}, a)\).
Denote by $\text{Der} A$ the algebra of all derivations of $(A,\omega)$. The algebra $(A,\omega)$ is called $n$-Lie [1] if $\omega$ is skew-symmetric and $L_{a_1,\ldots,a_{n-1}} \in \text{Der} A$ for all $a_1,\ldots,a_{n-1} \in A$. Sometimes the $n$-Lie algebras are called Nambu–Takhtajan algebras, although V. T. Filippov was the first to introduce them.

Denote by $\text{Sym}_k$ the permutation group; and by $\text{sgn}\sigma$, the parity of $\sigma \in \text{Sym}_k$. Denote by $\text{Sym}_{k,l}$ the set of $(k,l)$-shuffles; these are $\sigma \in \text{Sym}_{k+l}$ such that $\sigma(1) < \cdots < \sigma(k)$, and $\sigma(k+1) < \cdots < \sigma(k+l)$. Usually the set on which $\text{Sym}_k$ acts is taken to be the standard $(1,\ldots,k)$, but we will use some other cardinality $k$ sets, like $\{2,3,\ldots,k+1\}$ in the definition of $Q_t$ in Section 2. It will be clear from context exactly which set is used.

Refer to an algebra $(A,\omega)$ with $n$-arc product $\omega$ as $(n-1)$-left commutative if
\[
\sum_{\sigma \in \text{Sym}_{2n-2}} \text{sgn} \sigma \omega(a_{\sigma(1)},\ldots,a_{\sigma(n-1)},\omega(a_{\sigma(n)},\ldots,a_{\sigma(2n-2)},a_{2n-1})) = 0
\]
for all $a_1,\ldots,a_{2n-2},a_{2n-1} \in A$.

Call $(A,\omega)$ homotopy $n$-Lie [4] if $\omega \in C^n(A, A)$ and
\[
\sum_{\sigma \in \text{Sym}_{n-1,n}} \text{sgn} \omega(a_{\sigma(1)},\ldots,a_{\sigma(n-1)},\omega(a_{\sigma(n)},\ldots,a_{\sigma(2n-2)},a_{2n-1})) = 0
\]
for all $a_1,\ldots,a_{2n-2},a_{2n-1} \in A$. In [5] these algebras are called $n$-Lie algebras.

Call an algebra $(A,\Omega)$ a homotopy Lie algebra [4] if $\Omega$ is a sequence $\omega_1,\omega_2,\ldots$ with $\omega_k \in C^k(A, A)$ such that
\[
\sum_{\sigma \in \text{Sym}_{-1,j}} \text{sgn} \sigma \omega_i(a_{\sigma(1)},\ldots,a_{\sigma(i-1)},\omega_j(a_{\sigma(i)},\ldots,a_{\sigma(i+j-1)})) = 0
\]
for all $a_1,\ldots,a_{i+j-1} \in A$ and $i,j = 1,2,\ldots$.

We prove in Section 2 that every $n$-Lie algebra is $(n-1)$-left commutative and every $(n-1)$-left commutative algebra is homotopy $n$-Lie. Some examples of Wronskian algebras show that the converse is false.

### 1. Statement of the Main Result

Take a commutative associative algebra $U$ with derivation $\partial$. Suppose that $V^{i_1,\ldots,i_k}$ is a generalized Wronskian:
\[
V^{i_1,\ldots,i_k}(u_1,\ldots,u_k) = \begin{vmatrix}
\partial^{i_1} u_1 & \cdots & \partial^{i_1} u_k \\
\vdots & \ddots & \vdots \\
\partial^{i_k} u_1 & \cdots & \partial^{i_k} u_k
\end{vmatrix}.
\]

For instance, $V^{0,1,2,\ldots,k}$ is the standard Wronskian.

**Theorem 1.1.** Given a commutative associative algebra $U$ over a field $K$ of characteristic $p \geq 0$ with derivation $\partial$, we have:

(i) For each $k > 0$ the algebra $(U, V^{0,1,\ldots,k})$ is homotopy $(k+1)$-Lie. Moreover, $(U, \{0,\lambda_{i+1}V^{0,1,\ldots,i}, i = 1,2,\ldots\})$ is a homotopy Lie algebra for all $\lambda_i \in K, \lambda_1 = 0$.

(ii) The algebra $(U, V^{0,1,\ldots,k})$, where $k > 0$, is $k$-left commutative iff $k \neq 2$.

(iii) The algebra $(U, V^{0,1,\ldots,k})$ is $(k+1)$-Lie iff one of the following conditions is fulfilled:
- $k = 1$ and $p$ is an arbitrary prime or $0$;
- $k = 2$ and $p = 2$;
- $k = 3$ and $p = 3$;
- $k = 4$ and $p = 2$.

By Theorem 1.1 Wronskians arise as $n$-Lie products for $n > 2$ only in the case of small positive characteristic $p = 2,3$. The following result is established in [1]: if $A$ is an $n$-Lie algebra with product $\omega$ then $A$ is $(n-1)$-Lie with product $i(a)\omega$ for all $a \in A$. Here by $i(a)\omega$ we understand the $(n-1)$-product defined by the rule
\[
i(a)\omega(a_1,\ldots,a_{n-1}) = \omega(a,a_1,\ldots,a_{n-1}).
\]
Using this construction, we can obtain from $(k+1)$-Lie algebras $V^{0,1,\ldots,k}$ other $n$-Lie algebras for $n \leq k$. 602
Theorem 1.2. For \( n \geq 2 \) and \( p = \text{char} K \geq 0 \), the following generalized Wronskians are \( n \)-Lie products:

- \( n = 2 \)
  - \( p = 2 \), \( V^{2^r - 2^l, 2^r}, 0 \leq l \leq r; \)
  - \( p = 3 \), \( V^{2^{3r}, 3^r + 1}, 0 \leq r; \)
- \( n = 3 \)
  - \( p = 2 \), \( V^{1,2,4}; \)
  - \( p = 2 \), \( V^{2,3,4}; \)
  - \( p = 2 \), \( \sum_{i=1}^{3^l} V^{0,2^l+1-i}, 0 < l; \)
  - \( p = 3 \), \( V^{1,2,3}; \)
- \( n = 4 \)
  - \( p = 2 \), \( V^{1,2,3,4}; \)
  - \( p = 3 \), \( V^{0,1,2,3}; \)
- \( n = 5 \)
  - \( p = 2 \), \( V^{0,1,2,3,4}; \)

In case of a field of characteristic 0, the Wronskian \( V^{0,1,\ldots,k} \) is an \( n \)-Lie product only in the case of Lie algebras: \( n = 2, k = 1 \).

Note that every 3-Lie algebra over a field of characteristic 3 forms a Lie triple system. There is a standard method for associating a Lie algebra to a Lie triple system [6]. Thus, to each 3-Lie algebra of characteristic 3 we can associate a Lie algebra. The simple Lie algebras corresponding to our 3-Lie algebras of characteristic 3 are the Kuznetsov–Ermolaev exceptional simple algebras [7]. The series of simple \( n \)-Lie algebras of characteristic \( p \) are constructed in [8]. Our \( n \)-Lie algebras generated by Wronskians are exceptional in the sense that they cannot be defined in the case of characteristic \( p > 3 \).

Two ideas underlie our calculations. The first is the following ”polynomial trick.” Suppose that some statement \( \mathcal{X} \) about a unital commutative associative algebra \( U \) with commuting derivations \( \mathcal{D} = \langle \partial_1, \ldots, \partial_n \rangle \) follows from the linear properties of \( U \), its commutativity, its associativity, the properties of its identity element, the Leibniz rule for \( \partial_1, \ldots, \partial_n \), and the commutation \([\partial_i, \partial_j] = 0\) of the derivations, \( i, j = 1, 2, \ldots, n \). Then \( \mathcal{X} \) holds for every unital commutative associative algebra with commuting derivations.

In particular, we can take as \( U \) the polynomial algebra \( K[x_1, \ldots, x_n] \) with \( \partial_i = \partial/\partial x_i \), or the divided powers algebra (in the case \( p > 0 \))

\[
O_n(m) = \left\{ x^{\alpha} = \prod_{i=1}^{n} x^{(\alpha_i)} : 0 \leq \alpha_i < p^{m_i}, m = (m_1, \ldots, m_n) \right\},
\]

\[
x^{\alpha} x^{\beta} = \prod_{i=1}^{n} \left( \frac{\alpha_i + \beta_i}{\alpha_i} \right) x^{\alpha + \beta},
\]

with the special derivations

\[
\partial_i : x^{\alpha} \mapsto x^{\alpha - \epsilon_i}, \quad \epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad i = 1, \ldots, n.
\]

The second idea has to do with \( \mathcal{D} \)-invariant polynomials [9]. If \( \mathcal{D} = \{\partial_1, \ldots, \partial_n\} \) is a system of commuting derivations then \( \psi \in T^k(U, U) \) is called \( \mathcal{D} \)-invariant provided that

\[
\partial \psi(u_1, \ldots, u_k) = \sum_{i=1}^{k} \psi(u_1, \ldots, u_{i-1}, \partial u_i, u_{i+1}, \ldots, u_k)
\]
for all \( u_1, \ldots, u_k \in U \) and \( \partial \in \mathcal{D} \). In other words, \( \psi \) is \( \mathcal{D} \)-invariant if each \( \partial \in \mathcal{D} \) is a derivation for \( \psi \). Note that \( U \) is \( \mathcal{D} \)-graded:

\[
U = \bigoplus_{s \geq 0} U_s, \quad U_s U_l \subseteq U_{s+l}, \quad U_0 = \langle 1 \rangle,
\]

\[
\partial_i U_s \subseteq U_{s-1}, \quad \mathcal{D} = \{ u \in U : \partial_i u = 0 \ \forall i = 1, \ldots, n \} = U_0.
\]

Given a graded \( \mathcal{D} \)-invariant multilinear map \( \psi \in T^k(U, U) \), denote by \( \pi \psi \) the multilinear form \( \pi \psi \in T^k(U, U_0) \) defined on the homogeneous basis elements \( e_1, \ldots, e_k \in U \) by \( \pi \psi(e_1, \ldots, e_k) = \psi(e_1, \ldots, e_k) \) if \( \psi(e_1, \ldots, e_k) \in U_0 \), and \( \pi \psi(e_1, \ldots, e_k) = 0 \) if \( \psi(e_1, \ldots, e_k) \in \bigoplus_{s>0} U_s \). Call \( \pi \psi \) the support of \( \psi \), and call a \( k \)-tuple \( (e_1, \ldots, e_k) \) of homogeneous basis elements such that \( \pi \psi(e_1, \ldots, e_k) \neq 0 \) a supporting chain. Denote by \( \Gamma \) the set of supporting chains. We can [9] reconstruct \( \psi \) from \( \pi \psi \) uniquely:

\[
\psi(u_1, \ldots, u_k) = \sum_{\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \in \Gamma} \frac{\partial^{\alpha_1}(u_1)}{\alpha_1!}\cdots\frac{\partial^{\alpha_k}(u_k)}{\alpha_k!} \pi \psi(x^{\alpha_1}, \ldots, x^{\alpha_k}).
\]

Therefore, to find a \( \mathcal{D} \)-multilinear form, it suffices to compute its support. We use this argument below in calculating \( Q \psi \), \( Q_{\text{short}} \psi \), \( Q_{\text{long}} \psi \), and \( Q_{\text{alt}} \psi \). In our case, \( n = 1 \), and the reconstruction formula reduces to

\[
\psi = \sum_{i_1, \ldots, i_k \in \Gamma} \lambda_{i_1, \ldots, i_k} \frac{\partial^{i_1} u_1}{i_1!} \cdots \frac{\partial^{i_k} u_k}{i_k!},
\]

where \( \lambda_{i_1, \ldots, i_k} = \pi \psi(x^{i_1}, \ldots, x^{i_k}) \in K \). In the divided powers case \( x^i \) and \( \frac{\partial^i u}{i!} \) should be replaced by \( x^{(i)} \) and \( \partial^i u \).

\section{Connections Between \( n \)-Lie, \((n-1)\)-Left Commutative, and Homotopy \( n \)-Lie Structures}

Define the quadratic maps

\[
Q, Q_{\text{short}}, Q_{\text{long}}, Q_{\text{alt}} : C^k(A, A) \to T^{2k-1}(A, A)
\]

as follows:

\[
Q\psi(a_1, \ldots, a_{2k-1}) = \psi(a_1, \ldots, a_{k-1}, \psi(a_k, \ldots, a_{2k-1}))
\]

\[
- \sum_{i=0}^{k-1} \psi(a_k, \ldots, a_{k+i-1}, \psi(a_1, \ldots, a_{k-1}, a_{k+i}), a_{k+i+1}, \ldots, a_{2k-1}),
\]

\[
Q_{\text{long}}\psi(a_1, \ldots, a_{2k-1}) = \sum_{\sigma \in \text{Sym}_{k-1, k}, \sigma(k-1) = 2k-1} \text{sgn} \sigma \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-2)}, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)})),
\]

\[
Q_{\text{short}}\psi(a_1, \ldots, a_{2k-1}) = \sum_{\sigma \in \text{Sym}_{k-1, k}, \sigma(2k-1) = 2k-1} \text{sgn} \sigma \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-2), a_{\sigma(2k-1)}}),
\]

\[
Q_{\text{alt}}\psi(a_1, \ldots, a_{2k-1}) = \sum_{\sigma \in \text{Sym}_{k-1, k}} \text{sgn} \sigma \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-2), a_{\sigma(2k-1)}))),
\]

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These definitions split \( \{a_1, \ldots, a_{2k-1}\} \) into two types. Call the elements of the \((k-1)\)-element subset \( \{a_1, \ldots, a_{k-1}\} \) short, and those of the \(k\)-element subset \( \{a_k, \ldots, a_{2k-1}\} \) long.

If \( \psi \in C^k(A, A) \) then \( Q \psi \in T^{2k-1}(A, A) \) is skew-symmetric in all short and all long arguments separately; i.e., in the first \( k - 1 \) and last \( k \) arguments. It is easy to see that \( Q_{\text{short}} \psi \in T^{2k-1}(A, A) \) is skew-symmetric in all short arguments and in all but one long argument, for \( \psi \in C^k(A, A) \). Similarly \( Q_{\text{long}} \psi \in T^{2k-1}(A, A) \) is skew-symmetric in all short arguments and in all but one long argument, for \( \psi \in C^k(A, A) \). Note that \( Q_{\text{alt}} \psi \in C^{2k-1}(A, A) \).

**Proposition 2.1.** Suppose that \( \omega \in C^k(A, A) \) and one of the following conditions is fulfilled:

\[
p = 0, \ k > 2;
\]

\[
p > 0, \ k \neq 0, 1(\text{mod } p), \ k \equiv 1(\text{mod } 2);
\]

\[
p > 0, \ k \neq -1, 2(\text{mod } p), \ k \equiv 0(\text{mod } 2).
\]

Then

(i) if \( Q \omega = 0 \) then \( Q_{\text{short}} \omega = 0 \) and \( Q_{\text{long}} \omega = 0 \);

(ii) if \( Q_{\text{short}} \omega = 0 \) or \( Q_{\text{long}} \omega = 0 \) then \( Q_{\text{alt}} \omega = 0 \).

**Proof.** Call \( \sigma \in \text{Sym}_{k-1,k} \) a short permutation if \( \sigma(k-1) = 2k - 1 \). To each short permutation \( \sigma \) there correspond a \((k-2)\)-tuple \( r'(\sigma) \) and a \(k\)-tuple \( r''(\sigma) \) defined by

\[
r'(\sigma) = \{\sigma(1), \ldots, \sigma(k-2)\}, \quad r''(\sigma) = \{\sigma(k), \ldots, \sigma(2k-1)\}.
\]

Call \( \sigma \in \text{Sym}_{k-1,k} \) a long permutation if \( \sigma(2k-1) = 2k - 1 \). To each long permutation \( \sigma \) there correspond a \((k-1)\)-tuple \( r'(\sigma) \) and a \((k-1)\)-tuple \( r''(\sigma) \) defined by

\[
r'(\sigma) = \{\sigma(1), \ldots, \sigma(k-1)\}, \quad r''(\sigma) = \{\sigma(k), \ldots, \sigma(2k-2)\}.
\]

Note that

\[
r'(\sigma) \cup r''(\sigma) = \{1, \ldots, 2k-2\}
\]

for all \( \sigma \in \text{Sym}_{k-1,k} \). In other words, every \( r'(\sigma) \) is uniquely determined by \( r''(\sigma) \).

Call the element

\[
A_{\sigma} := \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)}))
\]

short (long) \( \sigma \)-element, or simply short (long) element if \( \sigma \) is a short (long) permutation. Denote by \( \text{Sym}^s_{k-1,k} \) and \( \text{Sym}^l_{k-1,k} \) the sets of all short and long permutations. It is obvious that

\[
\text{Sym}_{k-1,k} = \text{Sym}^s_{k-1,k} \cup \text{Sym}^l_{k-1,k}
\]

and

\[
Q_{\text{alt}} \psi = Q_{\text{short}} \psi + Q_{\text{long}} \psi.
\]

The identity \( Q \psi = 0 \) for \( \psi \in C^k(A, A) \) yields

\[
\psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)}))
\]

\[
= \sum_{i=0}^{k-1} (-1)^{k-i-1} \psi(a_{\sigma(k)}, \ldots, a_{\sigma(k+i)}, \ldots, a_{\sigma(2k-1)}, \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, a_{\sigma(k+i)})).
\]

In particular, (2) means that each short \( \sigma \)-element can be written as a sum of \( k \) long elements. More precisely, a short element \( A_{\sigma} \) is the sum of \( k \) long elements \( A_{\tau} \) for such long permutations \( \tau \) that \( r'(\tau) \subseteq r''(\sigma) \). Since \( |r''(\sigma) \setminus r'(\tau)| = 1 \), there is only one element, say \( i \), such that \( r''(\sigma) = r'(\tau) \cup \{i\} \). Hence, \( i \leq 2k - 2 \) and \( i \) is not equal to \( k - 1 \) elements in \( r'(\tau) \). Thus, there exist \( k - 1 \) choices for \( i \).

In other words, in accordance with (2)

\[
Q_{\text{long}} \psi(a_1, \ldots, a_{2k-1}) = \sum_{\sigma \in \text{Sym}^s_{k-1,k}} \text{sgn} \sigma \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)}))
\]
can be written as
\[(k - 1) \sum_{\sigma \in \text{Sym}_{k-1,k}} \psi(a_{\sigma(1)}, \ldots, a_{\sigma(k-1)}, \psi(a_{\sigma(k)}, \ldots, a_{\sigma(2k-1)})).\]

Therefore, the conditions \(\psi \in C^k(A, A)\) and \(Q\psi = 0\) yield
\[Q_{\text{long}}\psi = (k - 1)Q_{\text{short}}\psi. \tag{3}\]

Let us reinterpret (2). If \(\sigma\) is a long permutation then \(A_{\sigma}\) is a sum of \(k - 1\) short elements and one long element \(A_{\tilde{\sigma}}\), where
\[\tilde{\sigma} = \begin{pmatrix} 1 & \cdots & k - 1 & k & \cdots & 2k - 2 & 2k - 1 \\ \sigma(k) & \cdots & \sigma(2k - 2) & \sigma(1) & \cdots & \sigma(k - 1) & 2k - 1 \end{pmatrix}.\]

Thus, \(A_{\sigma}\) can be written as the sum of short elements \(A_{\tau}\), where \(r'(\tau) \subset r''(\sigma)\). More precisely, \(r''(\sigma) \setminus r'(\tau) = \{i\}\) for some \(i \in \{1, 2, \ldots, 2k - 2\}\), and \(i\) can be equal to one of \(k - 2\) elements in \(r'(\tau)\). Thus, here there are \(k\) possibilities for a long permutation \(\sigma\) such that \(A_{\tau}\) can be one of the terms in \(A_{\sigma}\). Note that
\[\text{sgn} \sigma = (-1)^{k-1} \text{sgn} \, \tilde{\sigma}.\]

Consequently, the sum of \(\text{sgn} \sigma A_{\sigma}\) over all \(\sigma \in \text{Sym}_{k-1,k}\) in accordance with (2) yields
\[Q_{\text{short}}\psi = kQ_{\text{long}}\psi + (-1)^{k-1}Q_{\text{short}}\psi. \tag{4}\]

The determinant of the linear system (3)–(4) is equal to
\[\begin{vmatrix} 1 & \cdots & -k + 1 \\ k & \cdots & -1 \end{vmatrix} = -k^2 + k + 1 + (-1)^k;\]

hence, the conditions \(Q\psi = 0\) and \(\psi \in C^k(A, A)\) imply the identities
\[Q_{\text{long}}\psi + 2Q_{\text{short}}\psi = 0, \quad k \equiv -1(\text{mod } p), \quad k \equiv 0(\text{mod } 2), \quad p > 0,\]
\[Q_{\text{long}}\psi - Q_{\text{short}}\psi = 0, \quad k \equiv 2(\text{mod } p), \quad k \equiv 0(\text{mod } 2), \quad p > 0,\]
\[Q_{\text{long}}\psi + Q_{\text{short}}\psi = 0, \quad k \equiv 0(\text{mod } p), \quad k \equiv 1(\text{mod } 2), \quad p > 0,\]
\[Q_{\text{long}}\psi = 0, \quad k \equiv 1(\text{mod } p), \quad k \equiv 1(\text{mod } 2), \quad p > 0,\]

and
\[Q_{\text{long}}\psi = 0, \quad Q_{\text{short}}\psi = 0, \quad p = 0, \quad k > 2.\]

Thus, by (1)
\[Q_{\text{alt}}\psi = 0\]

if \(p = 0\), and \(k > 2\) or \(k \not\equiv 0, 1(\text{mod } p), k \equiv 1(\text{mod } 2), \) or \(k \not\equiv -1, 2(\text{mod } p), k \equiv 0(\text{mod } 2).\)

**Corollary 2.2.** If \((A, \omega)\) is an \(n\)-Lie algebra then it is \((n - 1)\)-left commutative. If \((A, \omega)\) is \((n - 1)\)-left commutative then it is homotopy \(n\)-Lie.

In particular, every \(n\)-Lie algebra is homotopy \(n\)-Lie. Theorems 1.1 and 1.2 show that the converse is not true.
3. Proofs of Theorems 1.1 and 1.2

Take $\mathbb{Z}^n = \{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{Z}\}$ and $\mathbb{Z}_+^n = \{\alpha \in \mathbb{Z}^n : \alpha_i \geq 0, \ i = 1, \ldots, n\}$. Given $\alpha \in \mathbb{Z}^n$, put

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$ 

Take a commutative associative algebra $U$ with a derivation $\partial$. Take $\psi \in C^k(U, U)$ and $g \in C^l(U, U)$. Define $f \sim g, f \wedge g \in C^{k+l}(U, U)$ and bilinear maps

$$Q(f, g), Q_{alt}(f, g) \in C^{k+l-1}(U, U), \quad Q_{short}(f, g) \in T^{k+l-1}(U, U)$$

as follows:

$$f \sim g(u_1, \ldots, u_{k+l}) = f(u_1, \ldots, u_k)g(u_{k+1}, \ldots, u_{k+l}),$$

$$f \wedge g(u_1, \ldots, u_{k+l}) = \sum_{\sigma \in \text{Sym}_{k+l}} \text{sgn} \sigma(f \sim g)(u_{\sigma(1)}, \ldots, u_{\sigma(k+l)}),$$

$$f \ast g(u_1, \ldots, u_{k+l-1}) = f(u_1, \ldots, u_{k-1}, g(u_k, \ldots, u_{k+l-1})),$$

$$Q(f, g) = f(u_1, \ldots, u_{k-1}, g(u_k, \ldots, u_{k+l-1})) - \sum_{i=1}^l g(u_k, \ldots, u_{k+i-2}, f(u_1, \ldots, u_{k-1}, u_{k+i-1}), u_{k+i}, \ldots, u_{k+l-1}),$$

$$Q_{alt}(f, g)(u_1, \ldots, u_{k+l-1}) = \sum_{\sigma \in \text{Sym}_{k-1+l}} \text{sgn} \sigma(f \ast g)(u_{\sigma(1)}, \ldots, u_{\sigma(k+l-1)}),$$

$$Q_{short}(f, g)(u_1, \ldots, u_{k+l-1}) = \sum_{\sigma \in \text{Sym}_{k-1+l-1}} \text{sgn} \sigma(f \ast g)(u_{\sigma(1)}, \ldots, u_{\sigma(k+l-2)}, u_{k+l-1}).$$

Note that the definitions of $Q$ as bilinear maps agree with the definitions of $Q$ as quadratic maps in the previous section:

$$Q(f, f) = Q(f), \quad Q_{alt}(f, f) = Q_{alt}(f), \quad Q_{short}(f, f) = Q_{short}(f).$$

Since $(C^*(U, U), \sim)$ is associative, so is $(C^*(U, U), \wedge)$. Put

$$C^k_{loc,s}(U) = \{\partial^{i_1} \wedge \cdots \wedge \partial^{i_k} : 0 \leq i_1 < \cdots < i_k, i_1 + \cdots + i_k = s\}$$

and

$$C^k_{loc}(U) = \bigoplus_s C^k_{loc,s}(U).$$

Note that $V^s \subset C^k_{loc,\alpha}(U)$ for all $\alpha \in \mathbb{Z}_+^n$. Put $|\psi| = s$ for $\psi \in C^k_{loc,s}(U)$.

Given $\psi \in T^k(A, A)$, define $i(a)\psi \in T^{k-1}(A, A)$ by

$$i(a)\psi(a_1, \ldots, a_{k-1}) = \psi(a, a_1, \ldots, a_{k-1}).$$

**Proposition 3.1.** If $\psi$ is $k$-Lie then $i(a)\psi$ is $(k-1)$-Lie for each $a \in A$.

**Proof.** See [1].

Proposition 3.1 can be modified as follows:

**Lemma 3.2.** If $\psi$ is $k$-Lie then for the $(k-l)$-Lie product $\psi_1 := i(a_1)i(a_2)\cdots i(a_l)\psi$ we have $Q(\psi_i, \psi_j) = 0, \ i \leq j$. 
Lemma 3.3. $C^k_{\text{loc},s}(U) = 0$ for $s < k(k-1)/2$.

Proof. If $0 \neq \partial^1 \wedge \cdots \wedge \partial^k \in C^k_{\text{loc},s}(U)$ then $s = i_1 + \cdots + i_k \geq 0 + 1 + 2 + \cdots + (k-1) = (k-1)k/2$.

Lemma 3.4 (see [10]). $Q_{\text{alt}}(V^{\alpha},V^{\beta}) \in C_{\text{loc},s}[\alpha|+|\beta](U)$ for all $\alpha \in Z^k_+$ and $\beta \in Z^k_+$.

Corollary 3.5. $Q_{\text{alt}}(V^{0,1,\ldots,k},V^{0,1,\ldots,l}) = 0$ for all $k,l > 0$.

Proof. Note that $|V^{0,1,\ldots,k}| = k(k+1)/2$; hence,

$$Q_{\text{alt}}(V^{0,1,\ldots,k},V^{0,1,\ldots,l}) \in C_{\text{loc},s}[k^2+i^2+k+l]/2(U).$$

It is obvious that $(k^2+l^2+k+l)/2 < (k+l+1)(k+l)/2$, and so $C_{\text{loc},s}[k^2+i^2+k+l]/2(U) = 0$ by Lemma 3.3. Thus, $Q_{\text{alt}}(V^{0,1,\ldots,k},V^{0,1,\ldots,l}) = 0$.

Lemma 3.6. If $k > 2$ then $Q_{\text{short}}(V^{0,1,\ldots,k},V^{0,1,\ldots,k}) = 0$. If $k = 2$ then $Q_{\text{short}}(V^{0,1,2},V^{0,1,2}) = 2V^{0,1,2,3} \sim \text{id}$.

Proof. Assume $Q_{\text{short}}(V^{0,1,\ldots,k},V^{0,1,\ldots,k})$ is a linear combination of cochains $(\partial i_1 \wedge \cdots \wedge \partial i_{2k}) \sim \partial i_{2k+1}$ such that $i_1 + \cdots + i_{2k} = k^2 + k$ and $0 \leq i_1 < i_2 < \cdots < i_{2k}$. We have $i_1 + \cdots + i_{2k} \geq 0 + 1 + 2 + \cdots + (2k-1) = (2k-1)k$. Consequently,

$$k^2 + k = i_1 + \cdots + i_{2k+1} > (2k-1)k.$$ 

The inequality is impossible for $k > 2$.

Consider the case $k = 2$. We have

$$Q_{\text{short}}(V^{0,1,2},V^{0,1,2}) = \lambda V^{0,1,2,3} \sim \partial^0$$

for some $\lambda \in K$. We deduced this formula using only the associativity, commutativity, linear properties of $U$, and the Leibniz rule for derivations. Therefore, (5) holds for every commutative associative algebra $U$ with derivation $\partial$, and $\lambda$ is independent of $U$ and $\partial$. In particular, we can take $U = K[x]$ and $\partial = \partial/\partial x$. We have

$$Q_{\text{short}}(V^{0,1,2},V^{0,1,2})(1,x,x^2,x^3,1) = \lambda V^{0,1,2,3} \sim \partial^0(1,x,x^2,x^3,1).$$

Furthermore,

$$Q_{\text{short}}(V^{0,1,2},V^{0,1,2})(1,x,x^2,x^3,1) = V^{0,1,2}(1,x,V^{0,1,2}(x^2,x^3,1)) - V^{0,1,2}(1,x^2,V^{0,1,2}(x,x^3,1)) + V^{0,1,2}(1,x^3,V^{0,1,2}(x^2,1)) - V^{0,1,2}(1,x^2,1) + V^{0,1,2}(1,x,1)$$

$$= \partial^2(V^{0,1,2}(1,x^2,x^3)) - V^{0,1,2}(1,x^2,6x) + V^{0,1,2}(1,x^3,2x^0) + V^{0,1,2}(x,x^2,0) - V^{0,1,2}(x^3,2x^0) + V^{0,1,2}(x^2,x^3,0) = 24$$

and

$$\lambda V^{0,1,2,3} \sim \partial^0(1,x,x^2,x^3,1) = 12\lambda.$$

Thus, $\lambda = 2$.

Lemma 3.7. Take $\psi \in C^k(A,A)$. Let $X$ and $Y$ be the linear spans of the sets $\{a_1,\ldots,a_{k-1}\}$ and $\{b_1,\ldots,b_k\}$. If $X \subseteq Y$ then $Q\psi(a_1,\ldots,a_{k-1},b_1,\ldots,b_k) = 0$.

Proof. Suppose that $X \subseteq Y$ and $\dim Y = l \leq k$.

Since $\psi$ is skew-symmetric for $l < k$, we have $\psi(b_1,\ldots,b_k) = 0$ for all $b_1,\ldots,b_k \in Y$, and $\psi(a_1,\ldots,a_{k-1},b_i) = 0$ for all $a_1,\ldots,a_{k-1} \in X \subseteq Y, b_i \in Y$. Thus, in this case

$$Q\psi(a_1,\ldots,a_{k-1},b_1,\ldots,b_k) = 0.$$
Suppose that \( \dim Y = k \). If \( \dim X < k - 1 \) then similar reasoning shows that

\[
\psi(e_{i_1}, \ldots, e_{i_{k-1}}, c) = 0
\]

for all \( c \in A \). Thus, the lemma holds in this case.

It remains to consider the case \( \dim X = k - 1 \), \( \dim Y = k \).

Take a basis \( \{e_1, \ldots, e_k\} \) of \( Y \) such that \( \{e_1, \ldots, e_{k-1}\} \) is a basis of \( X \). Since \( Q\psi \) is multilinear in \( a_1, \ldots, a_{k-1} \) and \( b_1, \ldots, b_k \); to prove the lemma, it suffices to establish that

\[
Q\psi(e_1, \ldots, e_{k-1}, e_1, \ldots, e_k) = 0.
\]

Since \( \psi \) is skew-symmetric,

\[
\psi(e_1, \ldots, e_{k-1}, e_i) = 0
\]

for all \( i \leq k - 1 \). Thus,

\[
\sum_{i=1}^{k} \psi(e_1, \ldots, e_{i-1}, \psi(e_1, \ldots, e_{k-1}, e_i), e_{i+1}, \ldots, e_k) = \psi(e_1, \ldots, e_{k-1}, \psi(e_1, \ldots, e_{k-1}, e_k)).
\]

In other words, \( Q\psi(e_1, \ldots, e_{k-1}, e_1, \ldots, e_k) = 0 \). This completes the proof of the lemma.

**Lemma 3.8.** If \( p = 3 \) then \( V^{0,1,2,3} \) is a 4-Lie product.

**Proof.** We have

\[
|V^{0,1,2,3}| = 6 \Rightarrow |QV^{0,1,2,3}| = 12.
\]

Here is the list of \((3,4)\)-partitions of 12:

\[
\Gamma_{3,4}(12) = \{(0,1,2), (0,1,2,6), (0,1,2,3,4), (0,1,3,4,5), (0,1,2,3,4,6)\}.
\]

Thus,

\[
QV^{0,1,2,3} = \sum_{(\alpha,\beta)\in\Gamma_{3,4}(12)} \lambda_{(\alpha,\beta)} V^\alpha \wedge V^\beta,
\]

where \( \alpha = \{i_1, i_2, i_3\} \), \( \beta = \{i_4, i_5, i_6, i_7\} \), \( 0 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i_7 \), \( i_1 + \cdots + i_7 = 12 \).

Formula (6) is deduced using only the Leibniz rule, and so it is universal: the coefficients \( \lambda_{\alpha,\beta} \) are independent of \( U \) and \( \partial \). In particular, we can take \( U = Q[x] \) and \( \partial = \partial/\partial x \). To find \( \lambda_{\alpha,\beta} \), we can take \( a_l = x^{i_l}, l = 1, \ldots, 7 \), and compute \( QV^{0,1,2,3} \) in \( k[x] \). We have

\[
\lambda_{\alpha,\beta} = \frac{1}{i_1! \cdots i_7!} QV^{0,1,2,3}.
\]

By Lemma 3.7

\[
QV^{0,1,2,3}(1, x, x^2, 1, x, x^2, x^6) = 0,
\]

\[
QV^{0,1,2,3}(1, x, x^3, 1, x, x^3, x^4) = 0,
\]

\[
QV^{0,1,2,3}(1, x, x^4, 1, x, x^2, x^4) = 0.
\]

Thus, \( \lambda_{\alpha,\beta} = 0 \) for

\[
(\alpha, \beta) \in \{(0,1,2), (0,1,2,6), (0,1,3), (0,1,3,4), (0,1,4), (0,1,2,4)\}.
\]
Further,

\[ V^{0,1,2,3}(1, x^2, x^3, x^4) = \begin{vmatrix} 1 & x^2 & x^3 & x^4 \\ 0 & 2x & 3x^2 & 4x^3 \\ 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 6 & 24x \end{vmatrix} = 48x^3, \]

\[ V^{0,1,2,3}(1, x, x^2, x^4) = \begin{vmatrix} 1 & x & x^2 & x^4 \\ 0 & 1 & 2x & 4x^3 \\ 0 & 0 & 2 & 12x^2 \\ 0 & 0 & 0 & 24x \end{vmatrix} = 48x. \]

Thus,

\[ QV^{0,1,2,3}(1, x, x^2, 1, x^2, x^3, x^4) = V^{0,1,2,3}(1, x, x^2, V^{0,1,2,3}(1, x^2, x^3, x^4)) - 0 - 0 \]

\[ -V^{0,1,2,3}(1, x^2, x^3, V^{0,1,2,3}(1, x, x^2, x^4)) = 48V^{0,1,2,3}(1, x, x^2, x^3) - 48V^{0,1,2,3}(1, x^2, x^3, x) = 0 \]

and

\[ \lambda_{\{0,1,2\},\{0,2,3,4\}} = 0. \]

We have

\[ V^{0,1,2,3}(1, x, x^3, x^5) = \begin{vmatrix} 1 & x & x^3 & x^5 \\ 0 & 1 & 3x^2 & 5x^4 \\ 0 & 0 & 6x & 20x^3 \\ 0 & 0 & 6 & 60x^2 \end{vmatrix} = 240x^3, \]

\[ V^{0,1,2,3}(1, x, x^2, x^5) = \begin{vmatrix} 1 & x & x^2 & x^5 \\ 0 & 1 & 2x & 5x^4 \\ 0 & 0 & 2 & 20x^3 \\ 0 & 0 & 0 & 60x^2 \end{vmatrix} = 120x^2. \]

Consequently,

\[ QV^{0,1,2,3}(1, x, x^2, 1, x, x^3, x^5) = V^{0,1,2,3}(1, x, x^2, V^{0,1,2,3}(1, x, x^3, x^5)) - 0 - 0 - V^{0,1,2,3}(1, x, x^3, V^{0,1,2,3}(1, x, x^2, x^5)) \]

\[ = 240V^{0,1,2,3}(1, x, x^2, x^3) - 120V^{0,1,2,3}(1, x, x^3, x^2) \]

\[ = 360V^{0,1,2,3}(1, x, x^2, x^3) = 4320 \]

and

\[ \lambda_{\{0,1,2\},\{0,1,3,5\}} = \frac{4320}{0!1!2!0!1!3!5!} = 3. \]

Similar calculations show that

\[ \lambda_{\{0,1,3\},\{0,1,2,5\}} = -3, \lambda_{\{0,1,5\},\{0,1,2,3\}} = 3, \lambda_{\{0,2,3\},\{0,1,2,4\}} = \lambda_{\{0,2,4\},\{0,1,2,3\}} = 0. \]

We have thus established that

\[ QV^{0,1,2,3} = 3V^{0,1,2}V^{0,1,3,5} - 3V^{0,1,3}V^{0,1,2,5} + 3V^{0,1,5}V^{0,1,2,3}. \]

In particular, \( QV^{0,1,2,3} = 0 \) if \( p = 3 \).

**Corollary 3.9.** If \( p = 3 \) then \( V^{1,2,3} \) is a 3-Lie product, and \( V^{2,3} \) is a 2-Lie product.

**Proof.** This follows from Lemma 3.8, Proposition 3.1, and the formulas

\[ i(1)V^{0,1,2,3} = V^{1,2,3}, \quad i(x)V^{1,2,3} = V^{2,3}. \]
Lemma 3.10. If $p = 2$ then $V^{0,1,2,3,4}$ is a 5-Lie product.

The proof is similar to that of Lemma 3.8, and so we skip details. We have

$$|V^{0,1,2,3,4}| = 10 \Rightarrow |QV^{0,1,2,3}| = 20.$$ 

Consequently, $QV^{0,1,2,3}$ is a linear combination of $V^\alpha \sim V^\beta$, where $(\alpha, \beta) \in \Gamma_{4,5}(20)$, and

$$\Gamma_{4,5}(20) = \{(\{0,1,2,3\},\{0,1,2,3,8\}), (\{0,1,2,3,7\},\{0,1,2,4,7\}), (\{0,1,2,4,7\},\{0,1,2,3,8\}), (\{0,1,2,3\},\{0,1,2,4,6\}), (\{0,1,2,4\},\{0,1,2,3,7\}), (\{0,1,2,3,7\},\{0,1,2,4,6\}), (\{0,1,2,4,6\},\{0,1,2,3,7\})\}.$$ 

Thus, there exists $\lambda_{\alpha,\beta} \in \mathbb{Z}$ such that

$$QV^{0,1,2,3,4} = \sum_{(\alpha,\beta) \in \Gamma_{4,5}(20)} \lambda_{\alpha,\beta} V^\alpha \sim V^\beta.$$ 

Calculations like those in the proof of Lemma 3.8 show that

$$QV^{0,1,2,3,4} = 4V^{0,1,2,7}V^{0,1,2,3,4} + 2V^{0,1,3,6}V^{0,1,2,3,4} - 2V^{0,1,4,5}V^{0,1,2,3,4} + 2V^{0,2,3,5}V^{0,1,2,3,4} + 2V^{0,1,2,6}V^{0,1,2,3,5} - 2V^{0,2,3,4}V^{0,1,2,3,5} - 2V^{0,1,2,5}V^{0,1,2,3,6} - 2V^{0,1,3,4}V^{0,1,2,3,6} - 4V^{0,1,2,4}V^{0,1,2,3,7} + 2V^{0,1,2,4}V^{0,1,2,4,5} + 4V^{0,1,2,3}V^{0,1,2,4,7} + 2V^{0,1,2,3}V^{0,1,2,5,6} - 2V^{0,1,2,4}V^{0,1,3,4,5} + 2V^{0,1,2,5}V^{0,1,3,4,6} + 2V^{0,1,2,3}V^{0,2,3,4,5}.$$ 

In particular, if $p = 2$ then $QV^{0,1,2,3,4} = 0$.

Corollary 3.11. If $p = 2$ then $V^{1,2,3,4}$ is a 4-Lie product, $V^{2,3,4}$ is a 3-Lie product, and $V^{3,4}$ is a 2-Lie product.

Proof. This follows from Lemma 3.10, Proposition 3.1, and the formulas

$$V^{1,2,3,4} = i(1)V^{1,2,3,4}, \quad V^{2,3,4} = i(x)V^{1,2,3,4}, \quad V^{3,4} = i(x^2)V^{2,3,4}/2.$$ 

Remark. The explicit expressions for $QV^{1,2,3,4}, QV^{2,3,4}, QV^{3,4}, QV^{1,2,3}, QV^{2,3}$ over $\mathbb{Z}$ follow easily from the calculations of $QV^{0,1,2,3,4}$ and $QV^{0,1,2,3}$. For instance,

$$QV^{1,2,3,4} = 4V^{1,2,7}V^{1,2,3,4} + 2V^{1,3,6}V^{1,2,3,4} - 2V^{1,4,5}V^{1,2,3,4} + 2V^{2,3,5}V^{1,2,3,4} + 2V^{1,2,6}V^{1,2,3,5} - 2V^{2,3,4}V^{1,2,3,5} - 2V^{1,2,5}V^{1,2,3,6} - 2V^{1,3,4}V^{1,2,3,6} - 4V^{1,2,4}V^{1,2,3,7} + 2V^{1,3,4}V^{1,2,4,5} + 4V^{1,2,3}V^{1,2,4,7} + 2V^{1,2,3}V^{1,2,5,6} - 2V^{1,2,4}V^{1,3,4,5} + 2V^{1,2,3}V^{1,3,4,6} + 2V^{1,2,3}V^{2,3,4,5}$$

and

$$QV^{1,2,3} = 3V^{1,2}V^{1,3,5} - 3V^{1,3}V^{1,2,5} + 3V^{1,5}V^{1,2,3}.$$
**Lemma 3.12.** Suppose that \( U = O_1(m), p > 0 \). We have

(i) \( \partial^t \in \text{Der} \, U \), iff \( q = p^k \) for some \( k > 0 \);

(ii) \( \partial^{p^k-1} \wedge \partial^{p^k} \) is 2-Lie iff either \( p = 2 \) or \( p = 3 \) and \( k = 1 \);

(iii) \( \partial^{p^k-2} \wedge \partial^{p^k-1} \wedge \partial^{p^k} \) is 3-Lie iff \( p = 3 \) and \( k = 1 \), or \( p = 2 \) and \( k = 1 \), or \( p = 2 \) and \( k = 2 \).

**Proof.** The case \( k = 1 \) is dealt with above. Since \( \partial^{p^k} \) is a derivation, the polynomial principle shows that the statement holds in general.

**Corollary 3.13.** (i) \( p = 3 \). For all \( k \in \mathbb{Z}_+ \) the operation \( \partial^{p^k-p^k-1} \wedge \partial^{p^k} \) is 2-Lie.

(ii) \( p = 2 \). For all \( k, l \in \mathbb{Z}_+ \) with \( k > l \) the operation \( \partial^{p^k-p^l} \wedge \partial^{p^k} \) is 2-Lie.

**Proof.** (i) For \( p = 3 \) we have \( p^k - p^k-1 = 2p^k-1 \) and \( p^k = 3p^k-1 \). Since \( F = \partial^{p^k-1} \in \text{Der} \, U \), the claim follows from Lemma 3.12(ii) applied to \( F \) in place of \( \partial \).

(ii) For \( p = 2 \) we have \( p^k - p^l = p'(p^k-l-1) \) and \( p^k = p^k-lp' \). Thus, for \( F = \partial^l \), we have \( \partial^{p^k} = F^{p^k-l} \) and \( \partial^{p^k-p^l} = F^{p^k-l-1} \). The claim follows from Lemma 3.12(ii) applied to \( F^{p^k-l} \) in place of \( \partial^{p^k} \).

**Proof of Theorem 1.1.**

(i) See Corollary 3.5.

(ii) See Lemma 3.6.

(iii) Suppose that \( (U, V^{0,1,\ldots-q}) \) is \((q+1)\)-Lie. If \( q = 1 \) then it is also 2-Lie for every characteristic \( p \).

Suppose that \( q > 1 \). By Proposition 3.1 \( V^{q,i} = i(1) i(x) \cdots i(x^{(q-1)}) V^{0,1,\ldots,q} \) is 1-Lie; i.e., \( \partial^t \in \text{Der} \, U \).

This is impossible for \( q > 1 \) and \( p = 0 \).

Thus, \( p > 0 \). Take \( U = O_1(m) \). By Lemma 3.12(i) \( q \) must be a power of \( p \). Suppose that \( q = p^t \).

By Proposition 3.1 \( V^{p^t-1,p^t} = i(1) i(x) \cdots i(x^{(p^t-2)}) V^{0,1,\ldots,p^t} \) is 2-Lie. By Lemma 3.12(ii) this is possible in the cases \( p = 2 \) or \( p = 3 \) and \( t = 1 \).

By Proposition 3.1 \( V^{2^t-2,2^t-1,2^t} = i(1) i(x) \cdots i(x^{(2^t-3)}) V^{0,1,\ldots,2^t} \) is 3-Lie. By Lemma 3.12(iii) this is possible only in the cases \( p = 2 \), \( t = 1 \) or \( p = 2 \), \( t = 2 \). This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 follows from Corollaries 3.9 and 3.11. Similarly it is possible to prove that \( \sum_{i=1}^{2^t} V^{i,2^t+1-i} \) is 3-Lie for \( p = 2 \). By Proposition 3.1 \( \sum_{i=1}^{2^t} V^{i,2^t+1-i} \) is 2-Lie for \( p = 2 \).

**References**