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ENGEL THEOREM FOR NOVIKOV ALGEBRAS

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We prove that, if A is left-nil Novikov algebra, then A^2 is nilpotent.

Key Words: Engel theorem; Nilpotency; Novikov algebra.

Mathematics Subject Classification: 17C; 17B.

Let $A = (A, \circ)$ be an algebra, with A a vector space over a field K of characteristic $p \geq 0$ and $A \times A \to A$, $(a, b) \mapsto a \circ b$, a multiplication. An algebra A is called *Novikov* (Balinskii and Novikov, 1985; Gelfand and Dorfman, 1979; Osborn, 1992), if

$$a_1 \circ (a_2 \circ a_3) - (a_1 \circ a_2) \circ a_3 = a_1 \circ (a_3 \circ a_2) - (a_1 \circ a_3) \circ a_2,$$
$$a_1 \circ (a_2 \circ a_3) = a_2 \circ (a_1 \circ a_3),$$

for any $a_1, a_2, a_3 \in A$.

Example. $(K[x], \circ)$, where $(a \circ b)(x) = \big(\frac{\partial}{\partial x} a(x)\big) b(x)$, is Novikov.

Denote by A^k a subspace of A generated by products of any k elements of A in any type of bracketings. Then

$$A = A^1 \supseteq A^2 \supseteq \cdots \supseteq A^k \supseteq A^{(k+1)} \supseteq \cdots,$$

and

$$A^k \circ A^s \subseteq A^{k+s}, \qquad k, s \geq 1.$$

In particular,

$$A^k \circ A \subseteq A^{k+1} \subseteq A^k,$$
$$A \circ A^k \subseteq A^{k+1} \subseteq A^k.$$

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Therefore, for any algebra A and k, a subspace A^k forms an ideal in A. In particular, A^2 is an ideal generated by products $a \circ b$, where $a, b \in A$.

An algebra A is called *nilpotent* if $A^n = 0$ for some n. Minimal n with a such property is called *index of nilpotency*.

Let $A^{\cdot k}$ be a subspace of A generated by right-normed products $a_1 \circ (a_2 \circ \cdots (a_{k-2} \circ (a_{k-1} \circ a_k)) \cdots)$. Then

$$A = A^{\cdot 1} \supseteq A^2 = A^{\cdot 2} \supseteq A^{\cdot 3} \supseteq \cdots \supseteq A^{\cdot k} \supseteq A^{\cdot (k+1)} \supseteq \cdots .$$

Call A *left-nilpotent* if $A^{\cdot n} = 0$ for some n.

For $a \in A$ set

$$a^{\cdot n} = \underbrace{a \circ (a \circ (\cdots (a \circ a)))}_{n \text{ times}}.$$

Call A *left-nill* if $a^{\cdot n} = 0, \forall a \in A$, for some n.

Zelmanov has proven that, if A is left-nilpotent finite-dimensional Novikov algebra over a field of characteristic zero, then A^2 is nilpotent (Zelmanov, 1987). In our article we establish the following result.

Theorem 1. *Let A be Novikov algebra over a field of characteristic p such that $a^{\cdot n} = 0$, for any $a \in A$. Suppose that $p = 0$ or $p > n$. Then A^2 is nilpotent with index of nilpotency no more than n.*

Let $l_a : A \to A$ be a left multiplication operator

$$l_a(b) = a \circ b.$$

If A is a finite-dimensional Lie algebra, then by the Engel theorem, A is nilpotent if l_a is nil for any $a \in A$. The following analog of Engel theorem for Novikov algebras takes place.

Corollary 2. *Let A be Novikov algebra such that $l_a^n = 0$ for any $a \in A$. If $p = 0$ or $p > n + 1$, then A^2 is nilpotent and nilpotency index is no more than $n + 1$.*

Proof. If $l_a^n = 0$, then $l_a^n(a) = a^{\cdot (n+1)} = 0$. It remains to use Theorem 1.

Corollary 3. *Let $p = 0$ and A be finite-dimensional Novikov algebra with base $\{a_1, \dots, a_m\}$. If l_{a_i} is nill for all $i = 1, \dots, m$, then A^2 is nilpotent.*

Proof. Since $l_a l_b = l_b l_a$, for any $a, b \in A$, the conditions $l_a^s = 0, l_b^q = 0$ imply that $l_{a+b}^{s+q} = 0$. Therefore, l_a is nil for any $a \in A$. Thus by Corollary 2 A^2 is nilpotent.

Remark 1. Zelmanov's example of two-dimensional algebra with base $\{a, b\}$ and multiplication table $a \circ a = b, a \circ b = b, b \circ a = b, b \circ b = 0$ (Zelmanov, 1987) shows that in Corollaries 2 and 3, one cannot change nilpotency of A^2 to nilpotency of A.

Remark 2. Let $p = 0$ or $p \geq n$. From Lemma 6 it follows that $l_{a_1} \ldots l_{a_n} = 0$ for any $a_1, \ldots, a_n \in A$, if $a^{\cdot n} = 0$, for any $a \in A$. So, for Novikov Algebra A the following conditions are equivalent:

(i) A is left-nill;
(ii) The subalgebra of $End\,A$ generated by left-multiplication operators l_a is nilpotent;
(iii) A is left-nilpotent.

Remark 3. It seems that in Theorem 1 instead of $(A^2)^n = 0$ one can write $(A^2)^{n-1} = 0$, and this estimate cannot be improved if $n > 2$. We have checked it for $n = 3, 4$. For the case $n = 3$, see Lemma 8. The case $n = 4$ needs tedious calculations and we omit them.

Lemma 4. *Let A be a right-symmetric algebra. For any k a subspace $A^{\cdot k}$ forms an ideal in A.*

Proof. It is evident that

$$A \circ A^{\cdot k} \subseteq A^{\cdot (k+1)}.$$

Let us prove

$$A^{\cdot k} \circ A \subseteq A^{\cdot k}.$$

We use induction on k to establish that

$$(a_1 \circ (a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots)) \circ b \in A^{\cdot k},$$

for any $a_1, \ldots, a_k, b \in A$.

For $k = 1$ our statement is evident:

$$a \circ b \in A^2 \subseteq A.$$

Suppose that for $k - 1$ our statement is true. Then by right-symmetric identity

$$\begin{aligned} a_1 \circ (a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots) \circ b = a_1 \circ \big((a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots) \circ b) \\ +(a_1 \circ b) \circ (a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots) \\ -a_1 \circ (b \circ (a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots)\big). \end{aligned}$$

By inductive suggestion

$$(a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots) \circ b \in A^{\cdot (k-1)}.$$

Since $a_1 \circ b \in A$, it is clear that

$$(a_1 \circ b) \circ (a_2 \circ (\cdots (a_{k-1} \circ a_k) \cdots)) \in A^{\cdot k}.$$

Similarly,

$$a_1 \circ (b \circ (a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots)) \in A^{\cdot(k+1)} \subseteq A^{\cdot k}.$$

So,

$$(a_1 \circ (a_2 \circ \cdots (a_{k-1} \circ a_k) \cdots)) \circ b \in A^{\cdot k}.$$

Lemma 5. *Let A be Novikov algebra. Then*

$$(A^2)^k \subseteq A^{\cdot(k+1)}.$$

Proof. We need to prove that product of any k elements $c_1, \ldots, c_k \in A^2$ in any type of bracketings can be presented as a linear combination of elements of the form $a_1 \circ (a_2 \circ (\cdots (a_k \circ a_{k+1}) \cdots))$.

We use induction on k. If $k = 1$, our statement is trivial.

Recall that there are $\frac{1}{k}\binom{2k-2}{k-1}$ types of bracketings in k elements. For example, if $k = 4$, we have 5 bracketing types:

$$((a_1 \circ a_1) \circ a_3) \circ a_4, \quad (a_1 \circ a_2) \circ (a_3 \circ a_4), \quad (a_1 \circ (a_2 \circ a_3)) \circ a_4,$$
$$a_1 \circ ((a_2 \circ a_3) \circ a_4), \quad a_1 \circ (a_2 \circ (a_3 \circ a_4)).$$

Let x be some bracketing type in k elements. Denote by $x(a_1, \ldots, a_k)$ an element obtained by elements $a_1, \ldots, a_k$ applying the bracketing x. It is known that any element $x(a_1, \ldots, a_k)$ can be presented as a product

$$x(a_1, \ldots, a_k) = y(a_1, \ldots, a_s) \circ z(a_{s+1}, \ldots, a_{n+m}),$$

for some bracketing types y and z in s and m elements, where $k = s + m$, $s > 0$, $m > 0$.

Suppose that for $k - 1 \geq 1$ our statement is established. As we mentioned above, any product of k elements $c_1, \ldots, c_k$ in any type of bracketings (denoted as C) can be presented as a product of some elements C_1 and C_2. Here C_1 is obtained by elements $c_1, \ldots, c_s$ applying some bracketing type in s elements and C_2 is obtained by elements $c_{s+1}, \ldots, c_{s+m}$, applying some bracketing type in m elements, where $k = s + m$. By inductive suggestion $C_1 \in A^{\cdot(s+1)}$ and $C_2 \in A^{\cdot(m+1)}$.

So, C is a linear combination of elements of a form $Y \circ Z$, where

$$Y = (a_1 \circ (\cdots (a_s \circ a_{s+1}) \cdots)), \qquad a_1, \ldots, a_{s+1} \in A,$$
$$Z = (b_1 \circ (\cdots (b_m \circ b_{m+1}) \cdots)), \qquad b_1, \ldots, b_{m+1} \in A.$$

By left-commutative identity

$$Y \circ Z = b_1 \circ (\cdots (b_m \circ (Y \circ b_{m+1})) \cdots).$$

By Lemma 4

$$Y \circ b_{m+1} \in A^{\cdot(s+1)}.$$

Thus,

$$Y \circ Z \in A^{\cdot(s+m+1)} = A^{\cdot(k+1)}.$$

So, our statement is true for k. Lemma is proven completely.

Let

$$S_k(a_1, \ldots, a_k) = \sum_{\sigma \in Sym_k} a_{\sigma(1)} \circ (\cdots \circ (a_{\sigma(k-2)} \circ (a_{\sigma(k-1)} \circ a_{\sigma(k)})) \cdots).$$

Lemma 6. *Let A be Novikov algebra over a field of characteristic $p > k$. Then for any $a_1, \ldots, a_{k+1} \in A$, the following relation takes place:*

$$a_1 \circ (\cdots \circ (a_k \circ a_{k+1}) \cdots) = \frac{1}{k!} S_{k+1}(a_1, \ldots, a_k, a_{k+1}) - \frac{1}{(k-1)!} a_{k+1} \circ S_k(a_1, \ldots, a_k).$$

Proof. By left-commutative identity

$$\begin{aligned}
&S_{k+1}(a_1, \ldots, a_{k+1}) \\
&\quad = \sum_{\sigma \in Sym_{k+1}} sign\,\sigma\, a_{\sigma(1)} \circ (\cdots \circ (a_{\sigma(k)} \circ a_{\sigma(k+1)})) \\
&\quad = \sum_{i=1}^{k+1} \sum_{\sigma \in Sym_{k+1}, \sigma(i)=i} sign\,\sigma\, a_{\sigma(1)} \circ (\cdots \circ (a_{\sigma(k)} \circ a_{\sigma(k+1)})) \\
&\quad = \sum_{i=1}^{k+1} k!\, a_1 \circ (\cdots a_{i-1} \circ (a_{i+1} \circ (\cdots \circ (a_{k+1} \circ a_i)))) \\
&\quad = k!\, a_1 \circ (\cdots \circ (a_k \circ a_{k+1})) + \sum_{i=1}^{k} k!\, a_{k+1} \circ (a_1 \circ (\cdots a_{i-1} \circ (a_{i+1} \circ (\cdots (a_k \circ a_i))))) \\
&\quad = k!\, a_1 \circ (\cdots \circ (a_k \circ a_{k+1})) \\
&\qquad + k\, a_{k+1} \circ \left(\sum_{i=1}^{k} (k-1)! a_1 \circ (\cdots a_{i-1} \circ (a_{i+1} \circ (\cdots (a_k \circ a_i)))) \right) \\
&\quad = k!\, a_1 \circ (\cdots \circ (a_k \circ a_{k+1})) + k\, a_{k+1} \circ (S_k(a_1, \ldots, a_{k+1}).
\end{aligned}$$

Lemma is proven.

Lemma 7. *Let A be an algebra over a field of characteristic $p = 0$ or $p \geq k$. Let J_k be an ideal of algebra A generated by right-normed elements $a^{\cdot k}, a \in A$. Then J_k is generated by elements of the form $S_k(a_1, \ldots, a_k)$, where $a_1, \ldots, a_k \in A$.*

Proof. We have

$$S_k(a_1, a_2, \ldots, a_k) = \sum (-1)^{k-r} (a_{i_1} + a_{i_2} + \cdots + a_{i_r})^{\cdot k},$$

where the summation is over all nonempty subsets $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$ and all products are right-bracketed. So,

$$S_k(a_1, \ldots, a_k) \in J_k.$$

Conversely, by left-symmetry identity

$$a^{\cdot k} = \frac{1}{(k-1)!} S_k(a, a, \ldots, a).$$

Lemma 8. *Suppose that A is Novikov algebra with the identity* $a^{\cdot 3} = 0$ *for any* $a \in A$. *Then* $(A^2)^2 = 0$.

Proof. Direct calculations show that

$$(a \circ b) \circ (c \circ d) = \frac{1}{4}(S_3(a \circ b, c, d) - S_3(a \circ c, b, d) + S_3(a \circ d, b, c) + b \circ S_3(a, c, d) + c \circ S_3(a, b, d) - d \circ S_3(a, b, c)) - \frac{1}{12} S_4(a, b, c, d).$$

By Lemma 7

$$S_3(a, b, c) = 0 \quad \text{and} \quad S_4(a, b, c, d) = 0$$

for any $a, b, c, d \in A$. Therefore, $(a \circ b) \circ (c \circ d) = 0$ for any $a, b, c, d \in A$. In other words, $(A^2)^2 = 0$.

Proof of Theorem 1. By Lemmas 6 and 7,

$$A^{\cdot(n+1)} = 0,$$

if $a^{\cdot n} = 0$, for any $a \in A$. Then by Lemma 5

$$(A^2)^n = 0.$$

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