Journal of Mathematical Sciences, Vol. 161, No. 1, 2009

EXCEPTIONAL 0-ALIA ALGEBRAS

A. S. Dzhumadil'daev and K. M. Tulenbaev UDC 512.55

Abstract. An algebra $(A, \circ)$ with the identity $[a,b]\circ c+[b,c]\circ a+[c,a]\circ b = 0$, where $[a,b] = a\circ b-b\circ a$, is called 0-Alia. We prove that the algebra $(\mathbb{C}[x], \circ)$ with multiplication $a \circ b = \partial^2(2a\partial(b) + \partial(a)b)$ is a simple, exceptional 0-Alia algebra.

Let $(A, \circ)$ be an algebra with multiplication $\circ$. It is 0-*Alia* (see [1]) if

$$[a,b] \circ c + [b,c] \circ a + [c,a] \circ b = 0$$

for any $a, b, c \in A$. Here $[a,b] = a \circ b - b \circ a$.

Let $\mathbb{C}[x]$ be the polynomial algebra with multiplication $(a,b) \mapsto ab$. We equip $\mathbb{C}[x]$ with a new multiplication $(a,b) \mapsto a \circ b$ defined by

$$a \circ b = \partial^3(a)b + 4\partial^2(a)\partial(b) + 5\partial(a)\partial^2(b) + 2a\partial^3(b)$$

or

$$a \circ b = \partial^2(2a\partial(b) + \partial(a)b).$$

As was shown in [1], the algebra $(\mathbb{C}[x], \circ)$ is simple and 0-Alia.

Let $(U, \cdot)$ be an associative, commutative algebra with multiplication $U \times U \to U$, $(u,v) \mapsto u \cdot v$, and $f, g : U \to U$ be linear maps. An algebra $\mathcal{A}(U, \cdot, f, g)$ is defined on the vector space U by the multiplication

$$(a,b) \mapsto a \cdot f(b) + g(a \cdot b).$$

In [1] it was proved that any algebra of the form $\mathcal{A}(U, \cdot, f, g)$ is 0-Alia. A 0-Alia algebra A is called *special* if it can be obtained as a subalgebra of 0-Alia algebra of a form $\mathcal{A}(U, \cdot, f, g)$ for some associative and commutative algebra $(U, \cdot)$ and its endomorphisms f and g. Otherwise, we say that a 0-Alia algebra A is *exceptional.*

The aim of our paper is to prove the following result.

Theorem 1. *The* 0*-Alia algebra* $(\mathbb{C}[x], \circ)$ *is exceptional.*

We must prove that the algebra $(\mathbb{C}[x], \circ)$ is not isomorphic to any subalgebra of the algebra of the form $\mathcal{A}(U, \cdot, f, g)$, where $U \supseteq \mathbb{C}[x]$ and $\cdot$ is an associative and commutative multiplication on U and $f, g : U \to U$ are linear maps.

Assume that Theorem 1 is not true and such an algebra $A(U, \cdot, f, g)$ exists. Then

$$\partial^2(2a\partial(b) + \partial(a)b) = a \cdot f(b) + g(a \cdot b) \tag{1}$$

for any $a, b \in U$. We can assume that U contains $\mathbb{C}[x]$ as a subspace, in particular, it contains elements $1, x, x^2, x^3, \ldots$.

Lemma 2. *For any* $a \in \mathbb{C}[x] \subseteq U$,

$$2\partial^3(a) = 1 \cdot f(a) + g(1 \cdot a).$$

Proof. Substituting $a = 1$ in (1), we have

$$2\partial^3(b) = 1 \cdot f(b) + g(1 \cdot b).$$

Changing in this relation b to a, we obtain the result. □

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 60, Algebra, 2008.

Lemma 3. *For any* $a \in \mathbb{C}[x] \subseteq U$,

$$\partial^3(a) = a \cdot f(1) + g(a \cdot 1).$$

Proof. Substitute $b = 1$ in (1). □

Lemma 4. *For any* $a, b \in \mathbb{C}[x] \subseteq U$,

$$\partial^2(a\partial(b) - \partial(a)b) \cdot c = \partial^2(2(a \cdot c)\partial(b) + \partial(a \cdot c)b - 2(b \cdot c)\partial(a) - \partial(b \cdot c)a).$$

Proof. Multiply both sides of (1) by c (multiplication $\cdot$):

$$\partial^2(2a\partial(b) + \partial(a)b) \cdot c = (a \cdot f(b)) \cdot c + g(a \cdot b) \cdot c. \tag{2}$$

Put in (2) $a := a \cdot c$. We have

$$\partial^2(2(a \cdot c)\partial(b) + \partial(a \cdot c)b) = (a \cdot c) \cdot f(b) + g((a \cdot c) \cdot b). \tag{3}$$

Substract (3) from (2). We obtain

$$g(a \cdot b) \cdot c - g(a \cdot c \cdot b) = \partial^2(2a\partial(b) + \partial(a)b) \cdot c - \partial^2(2(a \cdot c)\partial(b) + \partial(a \cdot c)b). \tag{4}$$

In (4), change a to b and b to a. We have

$$g(a \cdot b) \cdot c - g(a \cdot c \cdot b) = \partial^2(2b\partial(a) + \partial(b)a) \cdot c - \partial^2(2(b \cdot c)\partial(a) + \partial(b \cdot c)a). \tag{5}$$

Subtract from (4) the relation (5). We obtain the required relation. □

Lemma 5. $1 \cdot 1 = 0$.

Proof. Apply Lemma 4 for $b = 1$. We obtain

$$-\partial^3(a) \cdot c = \partial^2(\partial(a \cdot c) - 2(1 \cdot c)\partial(a) - \partial(1 \cdot c)a). \tag{6}$$

Change c in Lemma 4 to b and obtain

$$-\partial^3(a) \cdot b = \partial^2(\partial(a \cdot b) - 2(1 \cdot b)\partial(a) - \partial(1 \cdot b)a). \tag{7}$$

Change in (7) a to b and b to a:

$$-\partial^3(b) \cdot a = \partial^2(\partial(b \cdot a) - 2(1 \cdot a)\partial(b) - \partial(1 \cdot a)b). \tag{8}$$

Subtract (8) from (7). The commutativity property of the multiplication $\cdot$ gives us the following relation:

$$\partial^3(b) \cdot a - \partial^3(a) \cdot b = \partial^2(-2(1 \cdot b)\partial(a) - \partial(1 \cdot b)a + 2(1 \cdot a)\partial(b) + \partial(1 \cdot a)b). \tag{9}$$

Put $c = 1$ in Lemma 4 and use (9). We have

$$\partial^2(a\partial(b) - \partial(a)b) \cdot 1 = \partial^3(b) \cdot a - \partial^3(a) \cdot b. \tag{10}$$

Put in (10) $a = x^2$ and $b = x$. We have

$$\partial^2(x^2 - 2x^2) \cdot 1 = 0.$$

Thus, $1 \cdot 1 = 0$. □

Lemma 6. $1 \cdot f(1) = 0$.

Proof. Apply Lemma 3 for $a = 1$ and use Lemma 5. □

Lemma 7. *For any* $a \in \mathbb{C}[x] \subseteq U$,

$$(a \cdot 1) \cdot 1 = 0.$$

Proof. By the associativity of the multiplication $\cdot$ and by Lemma 5, we have

$$(a \cdot 1) \cdot 1 = a \cdot (1 \cdot 1) = a \cdot 0 = 0.$$

□

Lemma 8. *For any $a \in \mathbb{C}[x] \subseteq U$,*

$$\partial^3(1 \cdot a) = 0.$$

Proof. By Lemma 3,

$$\partial^3(1 \cdot a) = (1 \cdot a) \cdot f(1) + g(1 \cdot (1 \cdot a)) = (1 \cdot f(1)) \cdot a + g((a \cdot 1) \cdot 1).$$

By Lemmas 6 and 7 our statement is proved. □

Lemma 9. *For any $a \in \mathbb{C}[x] \subseteq U$,*

$$1 \cdot a = 0.$$

Proof. By Lemmas 2 and 3,

$$2\partial^3(a) = 1 \cdot f(a) + g(1 \cdot a), \quad \partial^3(a) = a \cdot f(1) + g(a \cdot 1).$$

Thus, by the commutativity of the multiplication $\cdot$,

$$\partial^3(a) = 1 \cdot f(a) - a \cdot f(1).$$

Multiply both sides of this relation by 1 under the multiplication $\cdot$ and use the associativity and commutativity of the multiplication $\cdot$. We obtain

$$1 \cdot \partial^3(a) = 1 \cdot (1 \cdot f(a)) - a \cdot (1 \cdot f(1)).$$

Thus, by Lemmas 6 and 7,

$$1 \cdot \partial^3(a) = 0.$$

Note that any element $u \in \mathbb{C}[x]$ can be presented in the form $u = \partial^3(a)$ for some $a \in \mathbb{C}[x]$. Thus,

$$1 \cdot u = 0, \qquad \forall u \in \mathbb{C}[x].$$

□

Lemma 10. *For any $a \in \mathbb{C}[x] \subseteq U$,*

$$2\partial^3(a) = 1 \cdot f(a).$$

Proof. Follows from Lemmas 2 and 9. □

Lemma 11. *For any $a, b \in \mathbb{C}[x] \subseteq U$,*

$$a \cdot b = 0.$$

Proof. For any nonnegative integers $k, l \geq 0$, note that

$$x^k = 2\partial^3(b_1), x^l = 2\partial^3(b_2),$$

where

$$b_1 = x^{k+3}/(2(k+3)(k+2)(k+1), b_2 = (x^{l+3}/(2(l+3)(l+2)(l+1).$$

Therefore, by Lemma 10,

$$x^k \cdot x^l = (1 \cdot f(b_1)) \cdot (1 \cdot f(b_2)).$$

By the associativity and commutativity of the multiplication $\cdot$ and by Lemma 5 we have

$$x^k \cdot x^l = 0.$$

□

Proof of Theorem 1. We set

$$e_k = f(x^k).$$

By Lemma 11

$$a \circ b = a \cdot f(b),$$

and

$$\partial^2(2a\partial(b) + \partial(a)b) = a \cdot f(b). \tag{11}$$

By (11), we have

$$x^k \cdot e_l = (k+2l)\partial^2(x^{k+l-1}), \tag{12}$$

Therefore, by (12),

$$(x^k \cdot e_s) \cdot e_l = (k+2s)\partial^2(x^{k+s-1}) \cdot e_l = (k+2s)(k+s-1)(k+s-2)x^{k+s-3} \cdot e_l$$
$$= (k+2s)(k+s-1)(k+s-2)(k+s+2l-3)\partial^2(x^{k+s+l-4}).$$

On the other hand, for similar reasons,

$$(x^k \cdot e_l) \cdot e_s = (k+2l)(k+l-1)(k+l-2)(k+l+2s-3)\partial^2(x^{k+s+l-4}).$$

By the associativity and commutativity of the multiplication $\cdot$,

$$(x^k \cdot e_l) \cdot e_s = (x^k \cdot e_s) \cdot e_l$$

for any nonnegative integers k, l, s. Therefore, the identity

$$(k+2s)(k+s-1)(k+s-2)(k+s+2l-3) = (k+2l)(k+l-1)(k+l-2)(k+l+2s-3)$$

holds for sufficiently large nonnegative integers k, l, and s (for example, $k+l+s > 6$). It is obvious that this is not true. For example, for $k = 2$, $l = 3$, and $s = 4$, we obtain a counterexample: $1800 = 960$. The theorem is proved completely. ☐

REFERENCES

1. A. S. Dzhumadil'daev, "Algebras with skew-symmetric identity of degree 3," *J. Math. Sci.*, **161**, No. 1, 11–30 (2009).

A. S. Dzhumadil'daev
Kazakh-British Technical University, Almaty, Kazakhstan
E-mail: askar56@homail.com
K. M. Tulenbaev
S. Demirel University, Almaty, Kazakhstan