ALGEBRAS WITH SKEW-SYMMETRIC IDENTITY OF DEGREE 3

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Dedicated to 70th birthday of E.B. Vinberg

Abstract. Algebras with one of the following identities are considered:
\[
\begin{align*}
&[[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2] = 0, \\
&[t_1, t_2]t_3 + [t_2, t_3]t_1 + [t_3, t_1]t_2 = 0, \\
&\{[t_1, t_2], t_3\} + \{[t_2, t_3], t_1\} + \{[t_3, t_1], t_2\} = 0,
\end{align*}
\]
where \([t_1, t_2] = t_1t_2 - t_2t_1\) and \(\{t_1, t_2\} = t_1t_2 + t_2t_1\). We prove that any algebra with a skew-symmetric identity of degree 3 is isomorphic or anti-isomorphic to one of such algebras or can be obtained as their \(q\)-commutator algebras.

1. Introduction

Denote by \((A, \circ)\) an algebra with a vector space \(A\) over a field \(K\) and a multiplication \(\circ\). Let \(\circ_q\) be a new multiplication on \(A\) defined by
\[
a \circ_q b = a \circ b + qb \circ a \quad (q\text{-commutator}).
\]
Notice that \(\circ_{-1}\) coincides with ordinary commutator
\[
[a, b] = a \circ b - b \circ a = a \circ_{-1} b
\]
and \(\circ_1\) coincides with anti-commutator
\[
\{a, b\} = a \circ b + b \circ a = a \circ_1 b.
\]
Call the algebra \((A, \circ_q)\) as \(q\)-algebra of \((A, \circ)\).

Let \(K\{t_1, \ldots, t_k\}\) be an algebra of non-commutative non-associative polynomials with variables \(t_1, t_2, \ldots, t_k\). For any algebra \((A, \circ)\) we can consider a homomorphism
\[
K\{t_1, \ldots, t_k\} \to A,
\]
that corresponds to any \(f \in K\{t_1, \ldots, t_k\}\) an element \(f(a_1, \ldots, a_k) \in A\). This means that in \(f(t_1, \ldots, t_k)\) we make substitutions \(t_1 := a_1, \ldots, t_k := a_k\) by elements of \(A\) and calculate \(f(a_1, \ldots, a_k)\) in terms of multiplication \(\circ\).
A polynomial \( f \in \mathbb{K}\{t_1, t_2, \ldots, t_k\} \) is called identity on \( A \), if
\[
f(a_1, \ldots, a_k) = 0, \quad \forall a_1, a_2, \ldots, a_k \in A.
\]
In such cases we say that \( f = 0 \) is an identity of \( A \).

A polynomial \( f \in \mathbb{K}\{t_1, t_2, \ldots, t_k\} \) is called skew-symmetric if
\[
f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) = \text{sign}\, \sigma \, f(t_1, \ldots, t_k),
\]
for any permutation \( \sigma \in \text{Sym}_k \). An identity \( f = 0 \) is skew-symmetric if \( f \) as a non-commutative non-associative polynomial is skew-symmetric.

Define polynomials with 2 variables
\[
\text{lie}(t_1, t_2) = [t_1, t_2] = t_1t_2 - t_2t_1,
\]
\[
\text{jor}(t_1, t_2) = \{t_1, t_2\} = t_1t_2 + t_2t_1
\]
and polynomials with 3 variables
\[
\text{lia}(t_1, t_2, t_3) = [[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2],
\]
\[
\text{alia}(t_1, t_2, t_3) = \{t_1, [t_2, t_3], t_1\} + \{[t_2, t_3], t_1\} + \{[t_3, t_1], t_2\},
\]
\[
\text{lalia}(t_1, t_2, t_3) = [t_1, t_2]t_3 + [t_2, t_3]t_1 + [t_3, t_1]t_2,
\]
\[
\text{ralia}(t_1, t_2, t_3) = t_1[t_2, t_3] + t_2[t_3, t_1] + t_3[t_1, t_2],
\]
\[
\text{alia}^{(q)}(t_1, t_2, t_3) = [t_1, t_2]t_3 + [t_2, t_3]t_1 + [t_3, t_1]t_2 + q(t_1[t_2, t_3] + t_2[t_3, t_1] + t_3[t_1, t_2]).
\]

Introduce the following names for algebras with identities.

<table>
<thead>
<tr>
<th>identity</th>
<th>name of algebras</th>
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</thead>
<tbody>
<tr>
<td>jor = 0</td>
<td>Anti-commutative</td>
</tr>
<tr>
<td>lia = 0</td>
<td>Lie-admissible</td>
</tr>
<tr>
<td>alia = 0</td>
<td>Anti-Lie-admissible or Alia</td>
</tr>
<tr>
<td>lalia = 0</td>
<td>Left Anti-Lie-admissible or Left Alia</td>
</tr>
<tr>
<td>ralia = 0</td>
<td>Right Anti-Lie-admissible or Right Alia</td>
</tr>
<tr>
<td>alia^{(q)} = 0</td>
<td>q-Anti-Lie-admissible or q-Alia</td>
</tr>
<tr>
<td>lalia = 0, ralia = 0</td>
<td>Two-sided Alia</td>
</tr>
</tbody>
</table>

For anti-commutative algebra \( (A, \circ) \) a bilinear map \( \psi : A \times A \rightarrow A \) is called commutative cocycle, if
\[
\psi(a \circ b, c) + \psi(b \circ c, a) + \psi(c \circ a, b) = 0,
\]
\[
\psi(a, b) = \psi(b, a),
\]
for any \( a, b, c \in A \).

An algebra \( (A, \circ) \) is said anti-isomorphic to algebra \( (A, \star) \) if there exist one-to-one map \( f : A \rightarrow A \), such that
\[
f(a \circ b) = f(b) \star f(a),
\]
for any \( a, b \in A \).
The aim of our paper is to describe algebras with skew-symmetric identities of degree 3. We reduce the problem of studying algebras with skew-symmetric identities of degree 3 to the problem of studying $q$-Allia algebras for $q = 0, \pm 1$, anti-commutative algebras and their commutative cocycles. We give standard constructions of 0-Alia algebras and 1-Alia algebras. We give also examples of simple $q$-Alia algebras.

2. Space of skew-symmetric and symmetric non-associative polynomials

Let $\Psi_k$ be a space of multilinear non-associative polynomials with $k$ variables. Since the number of non-associative non-commutative bracketings on $k$ letters is

$$c_k = \frac{1}{k} \binom{2k - 2}{k - 1}$$

(Catalan number),

it is clear that $\Psi_k$ is $\frac{(2k-2)!}{(k-1)!k!}$-dimensional. Denote by $\Psi_k^-$ a subspace of $\Psi_k$ generated by skew-symmetric polynomials.

Let

$$\pi^- : \Psi_k \to \Psi_k^-,$$

be skew-symmetrization map,

$$\pi^- f(t_1, \ldots, t_k) = \frac{1}{k!} \sum_{\sigma \in Sym_k} \text{sign} \sigma f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).$$

Theorem 2.1. The space $\Psi_k^-$ is $c_k$-dimensional and polynomials of a form $\pi^- f_i$, form base, where $i = 1, 2, \ldots, c_k$, and $f_i$ runs monomials corresponding to different types of bracketings.

Proof. Let $g$ be a skew-symmetric polynomial. Present it as a sum $\sum_{i=1}^{c_k} g_i$, where $g_i$ is a linear combination of monomials of $i$-th bracketing type. Since skew-symmetrization map does not change bracketing type, we see that $g_i$ is also skew-symmetric polynomial for any $i = 1, 2, \ldots, c_k$ and is uniquely defined by $g_i(t_1, \ldots, t_k)$. This means that polynomials $\pi^- f_1, \ldots, \pi^- f_{c_k}$ form base of $\Psi_k^-$. 

Corollary 2.2. $\Psi_2^-$ is 2-dimensional and has a base $\{\text{ralia, ralia}\}$.

Remark. Theorem 2.1 is true also for symmetric polynomials. Let $\Psi_k^+$ be a subspace of $\Psi_k$ generated by symmetric polynomials

$$f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) = f(t_1, \ldots, t_k).$$

and

$$\pi^+ : \Psi_k \to \Psi_k^+$$
be a symmetrization map,
\[ \pi^+ f(t_1, \ldots, t_k) = \frac{1}{k!} \sum_{\sigma \in \text{Sym}_k} f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}), \]
Then \( \dim \mathfrak{B}_k^+ = c_k \) and polynomials of a form \( \pi^+ f_i \), form base, where \( i = 1, 2, \ldots, c_k \), and \( f_i \) runs monomials corresponding to different types of bracketings.

3. \( q \)-Alia algebras constructed by \( 0 \)-Alia algebras

Denote by \( \mathfrak{L}ia, \mathfrak{A}lia^{(0)}, \mathfrak{A}lia^{(1)} \) and \( \mathfrak{A}lia^{(\infty)} \) categories of Lie-admissible, \( 0 \)-Alia, \( 1 \)-Alia and two-sided Alia algebras. Notice that
\[ \mathfrak{L}ia = \mathfrak{A}lia^{(-1)} \]
and
\[ \mathfrak{L}ia \cap \mathfrak{A}lia^{(0)} = \mathfrak{L}ia \cap \mathfrak{A}lia^{(1)} = \mathfrak{A}lia^{(0)} \cap \mathfrak{A}lia^{(1)} = \mathfrak{A}lia^{(\infty)}. \]

**Theorem 3.1.** Let \( q \in \mathbb{K} \), such that \( q^2 \neq 1 \). Then any algebra of a form \( A^{(-q)} \), where \( A \) is \( 0 \)-Alia, satisfies the identity \( \text{alia}^{(q)} = 0 \). Inversely, any \( q \)-Alia algebra is isomorphic to an algebra \( A^{(-q)} \) for some \( 0 \)-Alia algebra \( A \). In other words, categories of \( q \)-Alia algebras \( \mathfrak{A}lia^{(q)} \) and \( 0 \)-Alia algebras \( \mathfrak{A}lia^{(0)} \) are equivalent if \( q^2 \neq 1 \).

If \( q^2 = 1 \) this statement is not true. There exist algebras with identity \( \text{alia}^{(q)} = 0 \), that can not be obtained from \( 0 \)-Alia algebras in a form \( A^{(q)} \).

**Proof.** Let \( q^2 \neq 1 \). Prove that \( A^{(q)} \) is \( 0 \)-Alia if \( A \) is \( q \)-Alia. Prove also that \( (A^{(q)})^{(-q)} \) is once again \( q \)-Alia and, moreover, it is isomorphic to \( A \).

Denote by \( [a, b]^{(-q)} \) a commutator of the multiplication \( \circ_{-q} \). Then
\[ [a, b]^{(-q)} = a \circ_{-q} b - b \circ_{-q} a = (1 + q)(a \circ b - b \circ a) = (1 + q)[a, b]. \]
Calculate \( \text{lalia}(a, b, c) \) and \( \text{ralia}(a, b, c) \) in terms of multiplication \( \circ_{-q} \). We have
\[ \text{lalia}(a, b, c) = [a, b]^{(-q)} \circ_{-q} c + [b, c]^{(-q)} \circ_{-q} a + [c, a]^{(-q)} \circ_{-q} b \]
\[ = (1 + q)([a, b] \circ c + [b, c] \circ a + [c, a] \circ b) - (1 + q)q(c \circ [a, b] + a \circ [b, c] + b \circ [c, a]) \]
\[ = (1 + q) \text{lalia}(a, b, c) - (1 + q)q \text{ralia}(a, b, c). \]
Similarly,
\[ \text{ralia}(a, b, c) = c \circ_{-q} [a, b]^{(-q)} + a \circ_{-q} [b, c]^{(-q)} + b \circ_{-q} [c, a]^{(-q)} \]
\[ = (1 + q)(c \circ [a, b] + a \circ [b, c] + b \circ [c, a]) - (1 + q)q([a, b] \circ c + [b, c] \circ a + [c, a] \circ b) \]
\[ = (1 + q)\text{ralia}(a, b, c) - (1 + q)q \text{lalia}(a, b, c). \]
Therefore,

\[ \text{alia}^{(q)}(a, b, c) = \text{lalia}(a, b, c) + q \text{ralia}(a, b, c) \]

\[ = (1 + q)(1 - q^2)\text{lalia}(a, b, c). \]

This means that \( A^{(-q)} \) is \( q \)-Alia if \( A \) is 0-Alia.

Suppose now \((A, \ast)\) is \( q \)-Alia. Endow \( A \) by a new multiplication

\[ a \ast b = (1 - q^2)^{-1}(a \ast b + q b \ast a). \]

We see that

\[ a \circ_q b = a \circ b - q b \circ a = a \ast b. \]

Therefore, \((A, \circ_q)\) is isomorphic to \((A, \ast)\). Check that \((A, \circ)\) is 0-Alia. Let \([a, b]^* = a \ast b - b \ast a\). We have

\[ [a, b] = (1 - q^2)^{-1}(a \ast b + q b \ast a - b \ast a - q a \ast b) \]

\[ = (1 - q^2)^{-1}(1 - q)[a, b]^*. \]

Thus,

\[ \text{lalia}(a, b, c) \]

\[ = (1 - q^2)^{-1}(1 - q)([a, b]^* \circ c + [b, c]^* \circ a + [c, a]^* \circ b) \]

\[ = (1 - q^2)^{-1}(1 - q)\text{alia}^{(q)}(a, b, c) \]

Therefore \((A, \circ)\) is 0-Alia if \((A, \ast)\) is \( q \)-Alia and \((A \circ_q)\) is isomorphic to \((A, \ast)\).

Now consider the case \( q^2 = 1 \). Notice that any 0-Alia algebra under \( q \)-commutator satisfies identity of degree 2 if \( q^2 = 1 \). Namely, any algebra obtained from 0-Alia algebra \( A \) in a form \( A^{(q)} \) for \( q^2 = 1 \) should be anti-commutative (in case \( q = -1 \)) or commutative (in case \( q = 1 \)). So, algebras with identities \( \text{alia}^{(q)} = 0, q^2 = 1 \), without identities of degree 2 gives us counter-examples.

In the case \( q = -1 \) as a such counter-example one gets free left-symmetric algebras, i.e.,algebras with identity

\[ (a, b, c) = (b, a, c). \]

In the case \( q = 1 \) as a counter-example one takes the algebra \((\mathbb{K}[x], \ast)\), where

\[ a \ast b = \partial(\partial(a)b). \]

It is 1-Alia and has no any identity of degree 2.

Thus categories \( \text{Alia}^{(q)} \) and \( \text{Alia} \) are not equivalent if \( q^2 = 1 \).
4. Commutative cocycles

To describe two-sided Alia algebras and 1-Alia algebras we need a new notion. Let $A = (A, ◦)$ be an algebra and $M$ be a vector space. Call a bilinear map $ψ : A × A → M$ \textit{commutative cocycle} with coefficients in $M$, if

1. $ψ(a, b) = ψ(b, a),$

2. $ψ(a ◦ b, c) + ψ(b ◦ c, a) + ψ(c ◦ a, b) = 0$

for any $a, b, c ∈ A$.

If $A$ is a Lie algebra and the condition is changed to anti-commutative condition, then we will obtain well known notion of 2-cocyclicity of $ψ$.

If $M = K$ is the main field, then call commutative 2-cocycle as a \textit{commutative central extension}. In our paper we mainly consider the case $M = A$ and in such cases we call $ψ$ shortly as a commutative cocycle.

Let $Z^2_{\text{com}}(A, M)$ be a space of commutative cocycles with coefficients in $M$. Then

$Z^2_{\text{com}}(A, M) \cong Z^2_{\text{com}}(A, K) ⊗ M.$

For any two-sided Alia algebra $A = (A, ◦)$ one can correspond Lie algebra $L = A^{(-1)} = (A, ◦_{-1})$. We establish that all two-sided Alia algebras with given Lie part $L$ can be characterized by $Z^2_{\text{com}}(L, A)$. Similar situation appears also for 1-Alia algebras. In this case $L$ is just anti-commutative algebra, not necessary Lie.

Let $A = (A, ◦)$ be anti-commutative algebra with commutative cocycle $ψ$. Let $(A, ◦_ψ)$ be an algebra with vector space $A$ and multiplication $◦_ψ$ given by

$a ◦_ψ b = a ◦ b + ψ(a, b)$

\textbf{Theorem 4.1.} (char$K \neq 2$) If $A = (A, ◦)$ is anti-commutative algebra and $ψ$ is commutative cocycle, then algebra $(A, ◦_ψ)$ is 1-Alia. Inversely, any 1-Alia algebra $A = (A, □)$ such that $A^{(-1)} ≅ (A, ◦)$ is isomorphic to algebra of a form $(A, ◦_ψ)$ for some cocycle $ψ$ of the anti-commutative algebra $(A, ◦)$.

Any two-sided Alia algebra is Lie-admissible. If $A = (A, ◦)$ is a Lie algebra and $ψ$ is its commutative cocycle, then the algebra $(A, ◦_ψ)$ is two-sided Alia. Inversely, any two-sided Alia algebra $A = (A, □)$, such that $A^{(-1)} ≅ L$ is isomorphic to algebra of the form $(A, ◦_ψ)$ for some commutative cocycle $ψ$ of the Lie algebra $L$. 

Proof. Let $A = (A, \circ)$ be anti-commutative algebra with multiplication $\circ$ and $\psi$ be commutative bilinear map

$$\psi(a, b) = \psi(b, a), \quad \forall a, b \in A$$

Let $\star = \circ \psi$ be multiplication of the algebra $(A, \circ \psi)$. Let

$$[a, b]^{\star} = a \star b - b \star a,$$

$$\{a, b\}^{\star} = a \star b + b \star a$$

be Lie and Jordan commutators for the multiplication $\star$. Then

$$[a, b]^{\star} = a \star b - b \star a = 2(a \circ b - b \circ a) = 4(a \circ b),$$

and

$$[a, b]^{\star} \star c = 4((a \circ b) \circ c + \psi(a \circ b, c)),$$

$$c \star [a, b]^{\star} = 4(c \circ (a \circ b) + \psi(c, a \circ b)).$$

Therefore,

$$\{[a, b]^{\star}, c]\}^{\star} = 8\psi(a \circ b, c)$$

and

$$\{[a, b]^{\star}, c}\}^{\star} + \{[b, c]^{\star}, a\}^{\star} + \{[c, a]^{\star}, b\}^{\star} =$$

$$8(\psi(a \circ b, c) + \psi(b \circ c, a) + \psi(c \circ a, b)).$$

Thus, the algebra $(A, \circ \psi)$ is 1-Alia if and only if $\psi$ is commutative cocycle of the algebra $(A, \circ)$.

Let now $A = (A, \star)$ be 1-Alia. Let $L = (A, \circ)$ be an algebra with a vector space $A$ and a multiplication

$$a \circ b = (a \star b - b \star a) / 2.$$ 

Let $\psi : A \times A \rightarrow A$ be a commutative bilinear map given by

$$\psi(a, b) = (a \star b + b \star a) / 2.$$ 

Then the multiplication $\circ$ as a commutator of the multiplication $\star$ is anti-commutative. Further,

$$\psi(a \circ b, c) + \psi(b \circ c, a) + \psi(c \circ a, b)$$

$$= (\{a \circ b, c\} + \{b \circ c, a\} + \{c \circ a, b\}) / 2$$

$$= (\{[a, b]^{\star}, c\}^{\star} + \{[b, c]^{\star}, a\}^{\star} + \{[c, a]^{\star}, b\}^{\star}) / 4$$

$$= \text{alia}^{(1)} \star (a, b, c) / 4 = 0$$

This means that $\psi$ is commutative cocycle for anti-commutative algebra $L$. Notice that

$$a \star b = a \circ b + \psi(a, b)$$

So, $(A, \star) \cong (A, \circ \psi)$.
Now suppose that \( A = (A, \star) \) is two-sided Alia. Then as we have noticed above
\[
a \star b = a \circ b + \psi(a, b),
\]
where
\[
a \circ b = [a, b]^*/2, \quad \psi(a, b) = \{a, b\}^*.
\]
We know that \( A \) is \(-1\)-Alia. This means that
\[
\]
In other words, \( (A, \circ) \) is Lie algebra. We also know that \( A \) is \(1\)-Alia. This condition is equivalent to the commutative cocyclicity condition of \( \psi \). Thus, \( A \) is isomorphic to the algebra \( (A, \circ_\psi) \), where \( \circ_\psi \) is Lie multiplication on \( A \).

Inversely, let \( (A, \circ) \) be Lie algebra and \( \psi \) be commutative cocycle. Then the algebra \( (A, \star) \), where \( \star = \circ_\psi \), has the following properties,
\[
\text{lalia}^\star(a, b, c) = [a, b]^* \star c + [b, c]^* \star a + [c, a]^* \star b
\]
\[
= 2([a, b]^{\circ} \star c + [b, c]^{\circ} \star a + [c, a]^{\circ} \star b)
\]
\[
= 2([a, b]^{\circ} \circ [b, c]^{\circ} \circ a + [c, a]^{\circ} \circ b + \psi(a \circ b, c) + \psi(b \circ c, a) + \psi(c \circ a, b)) = 0,
\]
and similarly,
\[
\text{ralia}^\star(a, b, c) = a \star [b, c]^* + b \star [c, a]^* + c \star [a, b]^*
\]
\[
= 2(a \circ [b, c]^{\circ} + b \circ [c, a]^{\circ} + c \circ [a, b]^{\circ} + \psi(a \circ b, c) + \psi(b \circ c, a) + \psi(c \circ a, b)) = 0.
\]
In other words, \( (A, \circ_\psi) \) is two-sided Alia.

5. **Algebras with skew-symmetric identity of degree 3**

**Theorem 5.1.** Any algebra with a skew-symmetric identity of degree 3 over a field \( K \) of characteristic \( p \neq 2 \) is isomorphic to one of the following algebras:

- Lie-admissible algebra
- left Alia algebra (or \(0\)-Alia algebra)
- right Alia algebra
- \(1\)-Alia algebra
- algebra of a form \( A^{(q)} \) for some \(0\)-Alia algebra \( A \) and \( q \in K \), such that \( q^2 \neq 0, 1 \).
Characterization of two-sided Alia algebras and 1-Alia algebras in terms of anti-commutative algebras and their commutative cocycles is given in Theorem 4.1. Let \((A, \circ)\) be \(q\)-Alia algebra. Then an opposite algebra \((A, \circ_{\text{op}})\) with multiplication \(a \circ_{\text{op}} b = b \circ a\) is \(1/q\)-Alia if \(q \neq 0\). If \(q = 0\) then 0-Alia algebra is left-Alia and its opposite algebra is right-Alia.

Proof of Theorem 5.1. By Corollary 2.2 a space of skew-symmetric polynomials of degree 3 is 2-dimensional and is generated by the left-Alia and right-Alia polynomials \(\text{lalia}\) and \(\text{ralia}\). Therefore any skew-symmetric non-commutative non-associative polynomial of degree 3 has a form \(f = f^{\alpha, \beta} = \alpha \text{lalia} + \beta \text{ralia}\), where \(\alpha, \beta \in \mathbb{K}\). For example,

\[
\text{lia}^{(q)} = \text{lalia} + q \text{ralia}.
\]

In other words, any non-commutative non-associative skew-symmetric polynomial up to scalar is equal to \(\text{lia}^{(q)}\) for some \(q \in \mathbb{K}\) or equal to \(\text{ralia}\). It remains to use Theorems 3.1.

6. 0-Alia algebras


Proposition 6.1. Let \((A, \cdot)\) be right-commutative algebra,

\[
(a \cdot b) \cdot c = (a \cdot c) \cdot b, \quad \forall a, b, c \in A.
\]

Then \((A, \cdot)\) is 0-Alia.

Proof.

\[
[a, b] \cdot c + [b, c] \cdot c + [c, a] \cdot b
\]

\[
= (a \cdot b) \cdot c - (b \cdot a) \cdot c + (b \cdot c) \cdot a - (c \cdot b) \cdot a + (c \cdot a) \cdot b - (a \cdot c) \cdot b
\]

\[
= (a \cdot b) \cdot c - (a \cdot c) \cdot b + (b \cdot c) \cdot a - (b \cdot a) \cdot c + (c \cdot a) \cdot b - (c \cdot b) \cdot a
\]

\[
= 0.
\]

Theorem 6.2. Let \((U, \cdot)\) be an associative commutative algebra and \(f, g : U \to U\) be linear maps. Define on \(U\) a multiplication \(\circ\) by

\[
a \circ b = a \cdot f(b) + g(a \cdot b).
\]

Then \((U, \circ)\) is 0-Alia.

Denote obtained algebra as \(\mathcal{A}_0(U, \cdot, f, g)\). For a 0-Alia algebra \(A\) say that it is special if \(A\) is isomorphic to a subalgebra of some algebra of a form \(\mathcal{A}_0(U, \cdot, f, g)\), where \((U, \cdot)\) is associative commutative algebra and \(f, g : U \to U\) are linear maps. Otherwise say that \(A\) is exceptional.
Proof. We have
\[ [a, b] \circ c = (a \cdot f(b)) \cdot f(c) - (b \cdot f(a)) \cdot f(c) + g((a \cdot f(b)) \cdot c - (b \cdot f(a)) \cdot c). \]

Therefore by commutativity and associativity properties of the multiplication \( \cdot \),
\[ [a, b] \circ c + [b, c] \circ a + [c, a] \circ b = (a \cdot f(b)) \cdot f(c) - (b \cdot f(a)) \cdot f(c) + (b \cdot f(c)) \cdot a - (c \cdot f(a)) \cdot b - (a \cdot f(c)) \cdot b \]
\[ + g((a \cdot f(b)) \cdot c - (b \cdot f(a)) \cdot c + (b \cdot f(c)) \cdot a - (c \cdot f(a)) \cdot b - (a \cdot f(c)) \cdot b) = 0. \]

Let \((A, \circ)\) be any algebra over a field of characteristic 3 with multiplication \( \circ \) and commutator \([a, b] = a \circ b - b \circ a\). A commutative bilinear map \( A \times A \to M \) is called invariant if
\[ \psi([a, b], c) = \psi(a, [b, c]), \]
for any \( a, b, c \in A \).

Theorem 6.3. Let \( A \) be any algebra over a field of characteristic \( p = 3 \). Then any commutative invariant form \( \psi : A \times A \to M \) is a commutative cocycle.

Proof. We have
\[ \psi([a, b], c) = \psi(a, [b, c]), \]
\[ \psi([b, c], a) = \psi(a, [b, c]), \]
\[ \psi([c, a], b) = -\psi([a, c], b) = -\psi(a, [c, b]) = \psi(a, [b, c]). \]

Thus,
\[ \psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 3\psi(a, [b, c]) = 0, \]
for any \( a, b, c \in A \). Proof is completed.

Recall that, for any semi-simple Lie algebra a Killing form
\[(a, b) = \text{tr ad} a \ ad b\]
is invariant and non-degenerate. Let \( A = (A, \circ) \) be Lie algebra and \( \tilde{A} = A + K \) be commutative central extension defined by a commutative cocycle \( \psi \in Z^2_{\text{com}}(A, K) \). The multiplication on \( \tilde{A} \) is defined by
\[ a \star b = a \circ b + \psi(a, b). \]

Then \((\tilde{A}, \star)\) is two-sided Alia. So,

Corollary 6.4. Any semi-simple Lie algebra in characteristic 3 with a nontrivial invariant form has nontrivial structures of two-sided Alia algebras.
6.3. Simple two-sided Alia algebra with Lie part $sl_2$.

**Theorem 6.5.** Let $L = \langle -1, e, e_1 | [e, e_1] = e_0, [e_1, e] = e_0, [e, e] = e_1 \rangle$ be a 3-dimensional simple Lie algebra. Then $Z^2_{com}(L, \mathbb{K})$ is 5-dimensional and is generated by commutative cocycles $\eta_i, i = 1, \ldots, 5$ defined by

$$
\begin{align*}
\eta_1(e, e) &= 1, \\
\eta_2(e, e_0) &= \eta_2(e_0, e) = 1, \\
\eta_3(e, e_1) &= 1, \\
\eta_4(e_0, e_1) &= \eta_4(e_1, e_0) = 1, \\
\eta_5(e_1, e_1) &= 1
\end{align*}
$$

(non-written components are 0).

**Proof.** There is only one nontrivial cocyclicity condition $d\psi(e, e_0, e_1) = 0$. More exactly,

$$2\psi(e, e_1) = \psi(e_0, [e, e]) = \psi(e_0, e_0).$$

Other statements are evident.

Another formulation of Theorem 6.5.

**Theorem 6.6.** Let $(sl_2, \star)$ be an algebra with multiplication table

$$
e_{-1} \star e_{-1} = \alpha_{1,1}e_{-1} + \alpha_{1,2}e_0 + \alpha_{1,3}e_1,
$$

$$
e_{-1} \star e_0 = e_{-1} + \alpha_{2,1}e_{-1} + \alpha_{2,2}e_0 + \alpha_{2,3}e_1, \\
e_{-1} \star e_1 = e_0 + \alpha_{3,1}e_{-1} + \alpha_{3,2}e_0 + \alpha_{3,3}e_1, \\
e_0 \star e_{-1} = -e_{-1} + \alpha_{2,1}e_{-1} + \alpha_{2,2}e_0 + \alpha_{2,3}e_1, \\
e_0 \star e_0 = 2(\alpha_{3,1}e_{-1} + \alpha_{3,2}e_0 + \alpha_{3,3}e_1), \\
e_0 \star e_1 = e_1 + \alpha_{4,1}e_0 + \alpha_{4,2}e_0 + \alpha_{4,3}e_1, \\
e_1 \star e_{-1} = -e_{-1} + \alpha_{4,1}e_{-1} + \alpha_{4,2}e_0 + \alpha_{4,3}e_1, \\
e_1 \star e_0 = \alpha_{5,1}e_{-1} + \alpha_{5,2}e_0 + \alpha_{5,3}e_1,
$$

where $\alpha_{i,j} \in \mathbb{K}, i = 1, 2, 3, 4, 5, j = 1, 2, 3$. Then $(sl_2, \star)$ is two-sided Alia algebra. It is simple for any 5 $\times$ 3 matrix $(\alpha_{i,j})$. Any two-sided Alia algebra connected with $sl_2$ is isomorphic to a such algebra for some 5 $\times$ 3 matrix $(\alpha_{i,j})$.

**Proof.** Follows from Theorems 6.5 and 4.1.

**Remark.** If $p \neq 2, 3$, then the algebra $(sl_2, \star)$ gives us a unique nontrivial example of two-sided algebras connected with classical simple Lie algebras [4].
6.4. **Simple two-sided Alia algebras with Lie part** $W_1$. Let $L = W_1$ be one-sided or two-sided Witt algebra of rank 1 over a field $\mathbb{K}$ of characteristic 0. Recall that, one-sided Witt algebra of rank 1 is generated by vectors $e_i, i \in \mathbb{Z}$ such that $i \geq -1$, and two-sided Witt algebra of rank 1 is generated by elements $e_i, i \in \mathbb{Z}$. In both cases the multiplication is given by

$$[e_i, e_j] = (j - i)e_{i+j}.$$

**Theorem 6.7.** Let $L$ be one-sided or two-sided Witt algebra of rank 1. Then $Z^2_{\text{com}}(L, \mathbb{K})$ is infinite-dimensional and is generated by commutative cocycles $\eta_i, i \in \mathbb{Z}$, defined by

$$\eta_i(u, v) = \text{coefficient of } uv \text{ at } x^{i+2}.$$  

Here $i \geq -2$ if $L$ is one-sided Witt algebra.

**Proof.** Let $\psi \in Z^2_{\text{com}}(L, \mathbb{K})$ be commutative cocycle. Notice that $Z^2_{\text{com}}(L, \mathbb{K})$ is a direct sum of homogeneous subspaces, $Z^2_{\text{com}}(L, \mathbb{K}) = \bigoplus_s Z^2_{\text{com},s}(L, \mathbb{K})$,

$$Z^2_{\text{com},s}(L, \mathbb{K}) = \{ \psi \in Z^2_{\text{com}}(L, \mathbb{K}) | \psi(e_i, e_j) = 0, i + j \neq s \}.$$

We can assume that $\psi$ is a homogeneous.

Commutative cocyclicity conditions on $e_0, e_i, e_j, i + j = s$, gives us the following relations

$$\psi([e_0, e_i], e_j) + \psi([e_i, e_j], e_0) + \psi([e_j, e_0], e_i) = 0 \Rightarrow$$

$$i \psi(e_i, e_j) + (j - i)\psi(e_{i+j}, e_0) - j \psi(e_j, e_i) = 0 \Rightarrow$$

$$(j - i)\psi(e_0, e_{i+j}) = (j - i)\psi(e_i, e_j).$$

Thus, if $i \neq j$,

$$\psi(e_i, e_j) = \psi(e_0, e_{i+j}).$$

Therefore,

$$\psi = \psi(e_0, e_s)\eta_{s-2}.$$  

The proof is finished.

Another formulation of Theorem 6.7

**Theorem 6.8.** Let $f$ be an endomorphism of polynomial space $U = \mathbb{K}[x]$ or Laurent polynomial space $U = \mathbb{K}[x, x^{-1}]$. Then the algebra $(U, \star_f)$, where

$$a \star_f b = \partial(a)b - a\partial(b) + f(ab),$$

is two-sided Alia algebra and simple. Any two-sided Alia algebra connected with (one-sided or two-sided) Witt algebra of rank 1 is isomorphic to $(U, \star_f)$ for some endomorphism $f \in \text{End}U$.

**Proof.** Follows from Theorems 6.7 and 4.1.
6.5. **Simple 0-Alia algebras defined by symmetric matrix.** Let \( \lambda = (\lambda_{i,j}) \) be a symmetric matrix. Endow space of polynomials \( U = \mathbb{K}[x_1, \ldots, x_n] \), by a multiplication

\[
    a \star b = \sum_{i,j} \lambda_{i,j} (\partial_i(a)\partial_j(b) + \frac{1}{2} \partial_i \partial_j(a)b).
\]

In other words,

\[
    a \star b = \sum_{i<j} \lambda_{i,j} (\partial_i(a)\partial_j(b)+\partial_j(a)\partial_i(b)+\sum_i \lambda_{i,i} (\partial_i(a)\partial_i(b)+\frac{1}{2} \partial_i^2(a)b))
\]

Let \( a \cdot b \) be a usual multiplication of polynomials and

\[
    f(a) = -\frac{1}{2} \sum_{i,j} \lambda_{i,j} \partial_i \partial_j(a),
\]

\[
    g(a) = \frac{1}{2} \sum_{i,j} \lambda_{i,j} \partial_i \partial_j(b).
\]

Then

\[
    a \star b = a \cdot f(b) + g(a \cdot b).
\]

So, \( (U, \star) \) is a standard algebra \( A(U, \cdot, f, g) \). Hence by Theorem 7.1 \((U, \star)\) is 0-Alia.

**Theorem 6.9.** The 0-Alia algebra \((U, \star)\) is simple if and only if the matrix \((\lambda_{i,j})\) is non-degenerate.

For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), set

\[
    |\alpha| = \sum_{i=1}^n \alpha_i.
\]

Endow \((U, \star)\) by grading. If

\[
    |x^\alpha| = |\alpha| - 2, \quad \alpha \in \mathbb{Z}_+^n,
\]

then

\[
    U = \bigoplus_{k \geq -2} U_k,
\]

\[
    U_k \star U_s \subseteq U_{k+s}.
\]

For example,

\[
    U_{-2} = \langle 1 \rangle,
\]

\[
    U_{-1} = \langle x_1 | i = 1, \ldots, n \rangle,
\]

\[
    U_0 = \langle x_i x_j | i, j = 1, \ldots, n \rangle.
\]
Notice that 
\[ u \star 1 = \sum_{i<j} \lambda_{i,j} \partial_i \partial_j (u) + \frac{1}{2} \sum_i \lambda_{i,i} \partial_i^2 (u), \quad \forall u \in U, \]
\[ 1 \star u = 0, \quad \forall u \in U, \]
\[ x_i \star x_j = x_j \star x_i = \lambda_{i,j} 1, \]
\[ x_i \star u = \sum_j \lambda_{i,j} \partial_j (u), \]
\[ u \star x_i = \frac{1}{2} \lambda_{i,i} x_i \partial_i^2 (u) + \sum_{j \neq i} \lambda_{i,j} x_i \partial_i \partial_j (u) + \sum_j \lambda_{i,j} \partial_j (u). \]
In particular,
\[ [u,x_i] = u \star x_i - x_i \star u = \frac{1}{2} \lambda_{i,i} x_i \partial_i^2 (u) + \sum_{j \neq i} \lambda_{i,j} x_i \partial_i \partial_j (u). \]
The following Lemma states that the algebra \((U, \star)\) is transitive.

**Lemma 6.10.** If \( x_i \star u = 0, u \star x_i = 0, u \star 1 = 0 \), then \( u \in U \_2 = <1> \).

**Proof.** From the condition \( u \star 1 = 0 \) it follows that 
\[ u = \theta_0 1 + \sum_i \theta_i x_i + \sum_{i \leq j} \theta_{i,j} x_i x_j, \]
for some \( \theta_0, \theta_i, \theta_{i,j} = \theta_{j,i} \in \mathbb{K}, i \leq j \), with property 
\[ \sum_{i \leq j} \lambda_{i,j} \theta_{i,j} = 0. \]
Further, for any \( i = 1, \ldots, n \),
\[ x_i \star u = 0 \Rightarrow \sum_j \lambda_{i,j} \partial_j (u) = 0 \Rightarrow \sum_j \lambda_{i,j} \theta_j + \sum_j \lambda_{i,j} (\sum_{i' < j} \theta_{i',j} x_{i'} + \sum_{j' > j} \theta_{j,j'} x_{j'}) = 0 \]
\[ \Rightarrow \sum_s \lambda_{i,s} \theta_s + \sum_s \lambda_{i,s} \sum_{j < s} \theta_{j,s} x_j + \sum_s \lambda_{i,s} \sum_{j > s} \theta_{s,j} x_j + 2 \sum_j \lambda_{i,j} \theta_{j,j} x_j = 0 \]
\[ \Rightarrow \sum_s \lambda_{i,s} \theta_s + \sum_{j < s} \lambda_{i,s} \theta_{j,s} x_j + \sum_{j > s} \lambda_{i,s} \theta_{s,j} x_j + 2 \sum_j \lambda_{i,j} \theta_{j,j} x_j = 0 \]
\[ \Rightarrow \sum_j \lambda_{i,j} \theta_j = 0, \]
\[ 2 \lambda_{i,j} \theta_{j,j} + \sum_{j < s} \lambda_{i,s} \theta_{j,s} + \sum_{j > s} \lambda_{i,s} \theta_{s,j} = 0, \quad \forall j = 1, \ldots, n. \]
\[ \Rightarrow \sum_j \lambda_{i,j} \theta_j = 0, \]
\[
\sum_{s=1}^{j-1} \lambda_{i,s} \theta_{s,j} + 2 \lambda_{i,j} \theta_{j,j} + \sum_{s=j+1}^{n} \lambda_{i,s} \theta_{s,j} = 0, \quad \forall j = 1, \ldots, n.
\]

In other words,

\[
\lambda T = 0, \quad \lambda \theta = 0,
\]

where \( \lambda \) is \( n \times n \)-matrix \((\lambda_{i,j})\), \( T \) is a column with coordinates \((\theta_1, \ldots, \theta_n)\), and \( \theta \) is a matrix of a form

\[
\theta = \begin{pmatrix}
2\theta_{1,1} & \theta_{1,2} & \theta_{1,3} & \cdots & \theta_{1,n} \\
\theta_{1,2} & 2\theta_{2,2} & \theta_{2,3} & \cdots & \theta_{2,n} \\
\theta_{1,3} & \theta_{2,3} & 2\theta_{3,3} & \cdots & \theta_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta_{1,n} & \theta_{2,n} & \theta_{3,n} & \cdots & 2\theta_{n,n}
\end{pmatrix}
\]

Since, \( \det (\lambda_{i,j}) \neq 0 \), this means that \( T = 0, \theta = 0 \). Lemma is proved.

**Lemma 6.11.** Suppose that \( \lambda_{i_0,j_0} \neq 0 \), for some \( 1 \leq i_0, j_0 \leq n \). Then for any \( v \in U \), there exists \( u \in U \), such that

\[
v = \sum_{i,j} \lambda_{i,j} \partial_i \partial_j (u).
\]

**Proof.** Endow \( \mathbb{Z}_n^+ \) by lexicographical ordering. For \( \alpha, \beta \in \mathbb{Z}_n^+ \) say that \( \alpha < \beta \), in the following situations:

- \( |\alpha| < |\beta| \) or
- \( |\alpha| = |\beta| \) and \( \alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}, \alpha_k < \beta_k \) for some \( k \leq n \).

Suppose that \( \lambda_{i_0,j_0} \neq 0, i_0 \leq j_0 \), and \((i_0, j_0)\) is maximal with such property. In other words, \( \lambda_{i,j} = 0, i \leq j \), if \( i > i_0 \) or \( i = i_0, j > j_0 \).

Show that

\[
x^\alpha \in < \sum_{i<j} \lambda_{i,j} \partial_i \partial_j (u) | u \in \mathbb{K}[x_1, \ldots, x_n] >
\]

for any \( \alpha \in \mathbb{Z}_n^+ \). Use induction by \( s = |\alpha| \) and in any fixed \( s \) use induction by ordered set of \( \alpha \)’s with \( |\alpha| = s \).

If \( s = 0 \), then \( \alpha = (0, \ldots, 0) \) and

\[
1 = \sum_{i<j} \lambda_{i,j} \partial_i \partial_j (\lambda_{i_0,j_0}^{-1} x_{i_0} x_{j_0}), \quad \text{if } i_0 < j_0,
\]

\[
1 = \sum_{i<j} \lambda_{i,j} \partial_i \partial_j ((2\lambda_{i_0,j_0})^{-1} x_{i_0}^2), \quad \text{if } i_0 < j_0.
\]

Therefore base of induction is established.
Suppose that for \( s - 1 \) our statement is true. Suppose that for any \( \beta \in \mathbb{Z}_1^n \), such that \(|\beta| = s\) and \( \beta < \alpha \) this statement is also true. Set

\[
u = \begin{cases} x_{i_0}^{\alpha_{i_0} + 1} x_{j_0}^{\alpha_{j_0} + 1} \prod_{i \neq i_0, j_0} x_i^{\alpha_i}, & \text{if } i_0 < j_0, \\ x_{i_0}^{\alpha_{i_0} + 2} \prod_{i \neq i_0} x_i^{\alpha_i}, & \text{if } i_0 = j_0. \end{cases}
\]

Then

\[
\sum_{i \leq j} \lambda_{i,j} \partial_i \partial_j(u) = \lambda_{i_0,j_0} (\alpha_{i_0} + 1)(\alpha_{j_0} + 1)x^\alpha + u',
\]

if \( i_0 < j_0 \) or

\[
\sum_{i \leq j} \lambda_{i,j} \partial_i \partial_j(u) = \lambda_{i_0,i_0} (\alpha_{i_0} + 2)(\alpha_{i_0} + 1)x^\alpha + u'',
\]

if \( i_0 = j_0 \). Here \( u', u'' \) are linear combination of monomials of a form \( x^\beta \) with \( \beta < \alpha \). So, by inductive suggestion

\[
x^\alpha \in \langle \sum_{i < j} \lambda_{i,j} \partial_i \partial_j(u) | u \in K[x_1, \ldots, x_n] \rangle.
\]

Lemma is proved.

**Proof of Theorem 6.9.** Suppose that \( \det (\lambda_{i,j}) = 0 \). Then there exists some \( \eta_i \in K, i = 1, \ldots, n \), such that

\[
\sum_{j=1}^n \lambda_{i,j} \eta_j = 0, \quad i = 1, \ldots, n.
\]

Set

\[
X = \sum_{i=1}^n \eta_i x_i.
\]

Let \( J \) be subspace of \( A \), that consists of elements of a form \( Xu, u \in U = K[x_1, \ldots, x_n] \), where \( Xu \) denotes usual multiplication of polynomials. Prove that \( J \) is ideal of \( U \).

We have

\[
(Xu) \ast a =
\]

\[
\sum_{i,j} \lambda_{i,j} (\partial_i(Xu)\partial_j(a) + \frac{1}{2}\partial_i \partial_j(Xu)a) =
\]

\[
= \sum_{i,j} \lambda_{i,j} \{ \partial_i(X)u\partial_j(a) + Xu\partial_i(a)\partial_j(a) + \frac{1}{2}\partial_i(X)\partial_j(u)a + \frac{1}{2}X\partial_i \partial_j(u)a \}.
\]
\[ X' = X' + X_1 + X_2, \]

where

\[ X' = \sum_{i,j} \lambda_{i,j} \{ \partial_i(X)u\partial_j(a) + \frac{1}{2} \partial_i(X)\partial_j(u)a + \frac{1}{2} \partial_j(X)\partial_i(u)a \}, \]

\[ X_1 = X(\sum_{i,j} \lambda_{i,j} \partial_i(u)\partial_j(a)) \in J, \]

\[ X_2 = X(\sum_{i,j} \frac{1}{2} \partial_i\partial_j(u)a) \in J. \]

By (3)

\[ X' = \sum_{i} (\sum_{j=1}^{n} \lambda_{i,j} \eta_j) u \partial_i(a) + \frac{1}{2} \sum_{j} (\sum_{i=1}^{n} \lambda_{i,j} \eta_i) \partial_j(u)a + \frac{1}{2} \sum_{i} (\sum_{j=1}^{n} \lambda_{i,j} \eta_j) \partial_i(u)a \]

\[ = 0. \]

Hence,

\[ (Xu) \ast a = X_1 + X_2 \in J, \]

for any \( a, u \in U \). Similarly,

\[ a \ast (Xu) = \sum_{i,j} \lambda_{i,j} (\partial_i(a)\partial_j(Xu) + \partial_j(a)\partial_i(Xu) + \frac{1}{2} \partial_i\partial_j(a)Xu) \]

\[ = X'' + X_5 + X_6 + X_7, \]

where

\[ X'' = \sum_{i,j} \lambda_{i,j} (\partial_i(a)\partial_j(X)u + \partial_j(a)\partial_i(X)u) \]

\[ X_5 = X(\sum_{i,j} \lambda_{i,j} (\partial_i(a)\partial_j(u))) \in J, \]

\[ X_6 = X(\sum_{i,j} \lambda_{i,j} \partial_j(a)\partial_i(u)) \in J, \]

\[ X_7 = X(\sum_{i,j} \frac{1}{2} \partial_i\partial_j(a)u) \in J. \]

By (3),

\[ X'' = \sum_{i} (\sum_{j=1}^{n} \lambda_{i,j} \eta_j) \partial_i(a)u \]
\[ + \sum_{j} \left( \sum_{i=1}^{n_1} \lambda_{i,j} \eta_i \right) \partial_j (a) u \]

Therefore,

\[ a \ast (Xu) = X_5 + X_6 + X_7 \in J, \]

for any \( a, u \in U \).

So, we have proved that \( J = \langle Xu : u \in U \rangle \) is ideal of \((U, \ast)\). It remains to note that it is non-trivial ideal. It is evident: \( 1 \notin J \).

Now suppose that \( \det (\lambda_{i,j}) \neq 0 \). Prove that \((U, \ast)\) is simple.

Suppose that it is not true: \( I \) is some non-trivial ideal of \((U, \ast)\). Take some \( 0 \neq R \in I \). Suppose that \( R = \sum_{\alpha \in \mathbb{Z}^n_{+}} \mu_{\alpha} x^\alpha \), for \( \mu_{\alpha} \in \mathbb{K} \), where \( x^\alpha = \prod_i x_i^{\alpha_i}, \alpha = (\alpha_1, \ldots, \alpha_n) \). Assume that \( \mu_{\alpha} = 0 \), for any \( \alpha \), such that \( |\alpha| > k \), but \( \mu_{\beta} \neq 0 \), for some \( \beta \in \mathbb{Z}^n_+ \) with \( |\beta| = k \). Call \( k = \deg R \) degree of \( R \). Take \( R \in I \) with minimal \( \deg R \).

Since

\[ \deg R \ast 1 < \deg R, \quad \deg R \ast x_i < \deg R, \quad \deg x_i \ast R < \deg R, \]

if \( R \ast 1, x_i \ast R, R \ast x_i \neq 0 \), by Lemma 6.10 we obtain that

\[ \deg R = 0. \]

In other words, \( R \in I \). So,

\[ 1 \in U; \]

if \( \det \lambda \neq 0 \).

Then

\[ 1 \in I \Rightarrow u \ast 1 = \frac{1}{2} \sum_{i,j} \lambda_{i,j} \partial_i \partial_j (u) \in J, \]

for any \( u \in U \). By Lemma 6.11, \( I = U \). This means that \((U, \ast)\), is simple, if \( \det (\lambda_{i,j}) \neq 0 \).

6.6. Simple exceptional 0-Alia algebra. All 0-Alia algebras constructed above are special. In other words they can be constructed in a form \( \mathcal{A}_0(U, f, g) \) for some associative commutative algebra \((U, \cdot)\) and endomorphisms \( f, g \). In [3] is proved that the following algebra will be exceptional.

**Theorem 6.12.** The algebra \((\mathbb{K}[x], \ast)\) with multiplication

\[ a \ast b = \partial^3(a) b + 4 \partial^3(a) \partial(b) + 5 \partial(a) \partial^2(b) + 2a \partial^3(b), \]

is 0-Alia and simple.
Proof. Let $U = \mathbb{K}[x]$. Direct calculations show that $(U, \star)$ is 0-Alia.

Let $e_i = x^{i+3}$. Then

$$e_i \star e_j = (4 + i + j)(5 + i + j)(9 + i + 2j)e_{i+j}.$$ 

So, $A$ is graded:

$$A = \bigoplus_{i \geq -3} A_i, \quad A_i = \langle x^{i+3} \rangle,$$

$$A_i \star A_j \subseteq A_{i+j}.$$

Lemma 6.13. If $e_{-1} \star u = 0$, then $u \in A_{-3}$.

Proof. Let

$$u = \sum_{j \leq j_0} \lambda_j e_j, \quad \lambda_{j_0} \neq 0.$$ 

Suppose that $e_{-1} \star u = 0$. We have to prove that $j_0 = -3$. Since $(A, \star)$ is graded,

$$e_{-1} \star u = 0 \Rightarrow \lambda_{j_0} e_i \star e_{j_0-1} = 0$$

$$\Rightarrow (3 + j_0)(4 + j_0)(8 + 2j_0)e_{j_0-1} = 0 \Rightarrow j_0 = -3.$$

Lemma 6.14. For any $u \in A$ there exists $v$ such that $u = e_{-1} \star v$.

Proof. Let $j \geq -3$. Then

$$(4 + j)(5 + j)(10 + 2j) \neq 0.$$ 

Therefore, we can take the element

$$v = e_{j+1}/((4 + j)(5 + j)(10 + 2j)) \in A.$$ 

Then,

$$e_j = e_{-1} \star v.$$ 

This means that any element of $A$ can be presented in a form $e_{-1} \star v$.

Proof of Theorem 6.12. Prove that $0$-Alia algebra $(\mathbb{K}[x], \star)$ is simple. Let $J$ be some nontrivial ideal of $(\mathbb{K}[x], \star)$ and $0 \neq X = \sum_{i \leq i_1} \lambda_i e(i) \in J$ with $\lambda_{i_1} \neq 0$. Call $i_1 = \deg X$ degree of $X$ and take such $X$ with minimal degree. By Lemma 6.13

$$\deg X = -3.$$ 

In other words,

$$1 \in J.$$

So, by Lemma 6.14 $J = \mathbb{K}[x]$. 

7. 1-Alia algebras


**Theorem 7.1.** Let \((U, \cdot)\) be associative commutative algebra and \(f, g : U \to U\) be linear maps. Define on \(U\) a multiplication \(\circ\) by
\[
a \circ b = a \cdot f(b) - b \cdot f(a) + g(a \cdot b).
\]
Then \((U, \circ)\) is 1-Alia.

Denote obtained algebra as \(A_1(U, \cdot, f, g)\).

**Proof.** Follows by Theorem 9.1.

**Corollary 7.2.** Define a multiplication on \(U = \mathbb{K}[x]\) by
\[
a \star b = -a \partial^m(b) + \partial^m(a)b + \partial^m(ab).
\]
Then \((U, \star)\) is 1-Alia for any \(m \geq 1\).

7.2. Identities for 1-Alia algebra. Let \(U\) be differential associative commutative algebra with derivation \(\partial\). Endow \(U\) by multiplication
\[
a \ast_b u = u \partial(a) \partial^2(b).
\]
Denote \(\ast_1\) shortly as \(\ast\).

**Theorem 7.3.** Let
\[
\begin{align*}
f_1 &= \text{alia}^{(1)} = \{[t_1, t_2, t_3] \} + \{[t_2, t_3, t_1] \} + \{[t_3, t_1, t_2] \}, \\
f_2 &= [t_1, t_2]t_3 - t_1(t_2t_3) + t_2(t_1t_3) + 2(t_1t_3)t_2 - 2(t_2t_3)t_1, \\
f_3 &= \text{ass}(t_3t_1, t_4, t_2) - \text{ass}(t_3t_2, t_4, t_1) - \text{ass}(t_4t_1, t_3, t_2) + \text{ass}(t_4t_2, t_3, t_1), \\
f_4 &= \sum_{\sigma \in S_{5m_3}} \text{sign} \sigma ((t_4\sigma(1))t_{\sigma(2))}t_{\sigma(3)}, \\
f_5 &= 2((t_3t_1)_{t_2})t_4)t_5 - 2(((t_3t_1)t_2)t_5) - (((t_3t_1)t_2)t_5) + (((t_3t_1)t_4)t_5)t_2 - ((t_3t_1)t_4)t_5 + ((t_3t_2)t_1)t_4) + (((t_3t_2)t_4)t_5)t_1 - ((t_3t_2)t_5)t_1)t_4 \\
&+ (((t_3t_4)t_1)t_2) - (((t_3t_4)t_1)t_5) - (((t_3t_4)t_2)t_5) + (((t_3t_4)t_5)t_1) + (((t_3t_4)t_5)t_1)t_2 \\
&+ (((t_3t_5)t_1)t_2) - ((t_3t_5)t_1)t_4)_{t_2}
\end{align*}
\]
be non-commutative non-associative polynomials. Then
- \(f_i = 0, 1 \leq i \leq 5\) are identities for \((U, \ast)\)
- Identities \(f_2 = 0, f_3 = 0, f_4 = 0, f_5 = 0\) are independent
- \(f_2 = 0 \Rightarrow f_1 = 0\)
- \(f_1 = 0, f_4 = 0, f_5 = 0\) are identities for \((U, \ast_u)\)
- \(f_2 = 0, f_3 = 0\) are identities of the algebra \((U, \ast_u)\) iff \(u = 1\).

Here \(\text{ass}(t_1, t_2, t_3) = (t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_2)t_s\) is an associator.
We omit proof of this result. It needs long calculations. Just note that the multiplication \((a, b) \mapsto \partial(a)\partial^2(b)\) is opposite to the multiplication \(a \ast b = \partial^2(a)\partial(b)\). For the last multiplication Theorem 7.3 partially is proved above.

8. **Simple 1-Alia algebra** \((\mathbb{K}[x], \circ)\) **with multiplication**

\[a \circ b = \partial(\partial(a)b)\]

Let

\[a \circ b = \partial(\partial(a)b).\]

Note that

\[2\partial(\partial(a)b) = a\partial^2(b) - \partial^2(a)b + \partial^2(ab).\]

Therefore, \((U, \circ)\) can be obtained by standard construction of 1-Alia algebras \(A_1(U, \cdot, f, g)\), if one sets

\[f(a) = \partial^2(a)/2, g(a) = \partial^2(a)/2.\]

Any commutative or anti-commutative algebra is 1-Alia. It will be interesting to describe simple algebras with minimal identity \(alia^{(q)} = 0\) for \(q = 0, \pm 1\). Minimality condition exclude from the consideration standard examples of \(q\)-Alia algebras, like Lie algebras, (anti)commutative algebras, right-commutative algebras, left-symmetric algebras. One of such non-trivial examples of 1-Alia algebras gives us the algebra \((\mathbb{K}[x], \circ)\).

**Theorem 8.1.** The algebra \((\mathbb{K}[x], \circ)\) is simple.

**Proof.** Let

\[e_i = x^{i+2}, \quad i \geq -2.\]

Then

\[e_i \circ e_j = (i + 2)(i + j + 3)e_{i+j}, \quad -2 \leq i, j.\]

For example,

\[e_{-2} \circ e_j = 0,\]
\[e_j \circ e_{-2} = (j + 2)(j + 1)e_{j-2},\]
\[e_{-1} \circ e_j = (j + 2)e_{j-1},\]
\[e_j \circ e_{-1} = (j + 2)e_{j-1},\]
\[e_0 \circ e_j = 2(j + 3)e_j,\]
\[e_j \circ e_0 = (j + 3)(j + 2)e_j.\]

Suppose that non-trivial ideal \(J\) has element \(X = \sum_{i \geq i_0} \lambda_i e_i \in J\), such that \(\lambda_{i_0} \neq 0\) and \(i_0\) is minimal with this property,

\[\sum_j \mu_j e_j \in J \Rightarrow \mu_j = 0, \forall j < i_0.\]
Prove that \( i_0 = -2 \). Suppose that it is not true.

If \( i_0 \geq 0 \), then

\[
X \in J \Rightarrow X \circ e_{-2} = \sum_{i \geq i_0} \lambda_i (i+2)(i+1)e_{i-2} \in J, \quad \lambda_{i_0} (i_0+2)(i_0+1) \neq 0.
\]

This contradicts to minimality \( i_0 \). So, the case \( i_0 \geq 0 \) is not possible.

Let \( i_0 = -1 \). Then

\[
e_{-1} \circ X = \sum_{i \geq i_0} \mu_i e_{i-1} \in J,
\]

where \( \mu_i = \lambda_i (i+2), \quad \mu_{-2} = \lambda_{-1} \neq 0 \).

This contradicts to minimality of \( i_0 \). We proved that the case \( i_0 = -1 \) is also not possible.

So, we have proved that \( i_0 = -2 \). We see that elements \( X \circ e_j \) has a form \( \sum_{i \geq j-2} \gamma_i e_i \) with \( \gamma_{j-2} \neq 0 \) if \( j \) runs elements \( 0, 1, 2, \ldots \). This means that \( J = \mathbb{K}[x] \). So, \((\mathbb{K}[x], \circ)\) is simple, where \( a \circ b = \partial(\partial(a) b) \).

**Remark.** A map \( f : A \to A, \quad f : a \mapsto \partial(a) \), induces a homomorphism of algebras

\[
f : (A, \ast) \to (A, \circ),
\]

where

\[
a \ast b = \partial^2(\partial(a) \partial(b)).
\]

Check it:

\[
f(a \ast b) = \partial(\partial^2(\partial(a) \partial(b))) = \partial(a) \circ \partial(b) = f(a) \circ f(b).
\]

So, we see that \((\mathbb{K}[x], \ast)\) is 1-Alia and there exists exact sequence of 1-Alia algebras

\[
0 \to \mathbb{K} \to (\mathbb{K}[x], \ast) \to (\mathbb{K}[x], \circ) \to 0.
\]

In other words, \((\mathbb{K}[x], \ast)\) is a central extension of \((\mathbb{K}[x], \circ)\).

9. **Standard construction of \( q \)-Alia algebras**

**Theorem 9.1.** Let \( A = (A, \cdot) \) be associative commutative algebra with multiplication \( a \cdot b \) and \( f, g : A \to A \) linear maps. Define a multiplication \( a \circ b \) by

\[
a \circ b = a \cdot f(b) - q b \cdot f(a) + g(a \cdot b).
\]

Then \( (A, \circ) \) is \( q \)-Alia.

**Proof.** Easy calculations. If \( q^2 \neq 1 \), it follows from Theorem 3.1.

**Theorem 9.2.** Let $U = \mathbb{K}[x]$ and
\[ a \star b = a \partial^m(b) - q \partial^m(a)b + q \partial^m(ab). \]
Then $(U, \star)$ are $q$-Alia and simple for $q^2 \neq 1$.

**Proof.** Calculate $q$-commutator of the multiplication $\star$
\[ a \star b = a \partial^m(b) - q \partial^m(a)b + q \partial^m(ab) + q^2 a \partial^m(b) + q^2 \partial^m(ab) \]
This multiplication is standard. In other words, for associative commutative algebra $U$ with usual polynomial multiplication $a \cdot b = ab$ and linear maps
\[ f : U \to U, \quad f(a) = (1 - q^2) \partial^m(a), \]
\[ g : U \to U, \quad g(a) = (q^2 + q) \partial^m(a), \]
the algebra $(U, \star_q)$ has a form $A_0(U, \cdot, f, g)$. So, by Theorem 7.1 $(U, \star_q)$ is $0$-Alia. Then by Theorem 3.1 the algebra $(U, \star)$ is $q$-Alia.

Set
\[ e_i = x^{i+m} / (i+m)!, \quad i = -m, -m+1, \ldots. \]
Then
\[ e_i \star e_j = \binom{i+j+m}{i+m} - q \binom{i+j+m}{j+m} + q \binom{i+j+2m}{i+m} e_{i+j}. \]
So, $(U, \star)$ is graded,
\[ U = \bigoplus_{i \geq -m} U_i, \quad U_i = \ll e_i \gg, \]
\[ U_i \star U_j \subseteq U_{i+j}. \]
Notice that
\[ e_{-m} \star e_j = (q - 1) e_{j-m}, \quad (4) \]
\[ e_i \star e_j = q \binom{m}{-j} e_{-m}, \text{ if } -m < i, j < 0, i + j = -m. \quad (5) \]
Let $J$ is a non-trivial ideal of $(U, \star)$. Take $X = \sum_{-m \leq i \leq i_0} \lambda_i e_i \in J$, such that $\lambda_{i_0} \neq 0$ and $i_0$ is minimal with such property. Since $Y = e_{-m} \star X \in J$ and $i_0$ is minimal, by grading property $Y = 0$. In particular, by (4),
\[ \lambda_{i_0} (q - 1) = 0, \]
and
\[ \lambda_{i_0} = 0. \]
if \( i_0 \geq 0 \). So, we can assume that \( i_0 < 0 \). Similar arguments that uses (5) shows that the case \( i_0 > -m \) is not possible. So, \( i_0 = -m \). In other words
\[
e_{-m} \in J.
\]
Then by (4)
\[
e_j = (q - 1)^{-1}e_{-m} \star e_{j+m} \in J.
\]
This means that
\[
J = U.
\]
Therefore, \((U, \star)\) is simple.

10. **Dual operads to Alia algebras**

**Theorem 10.1.** Koszul dual algebras to left-Alia algebras is defined by identities
\[
[t_1, t_2] t_3 = 0,
\]
\[
(t_1 t_2) t_3 = (t_1 t_3) t_2,
\]
\[
t_1 (t_2 t_3) = 0.
\]
Left-Alia operads are not Koszul. Dimensions of multilinear parts of Koszul dual to Left-Alia algebras are \( d_1 = 1, d_2 = 2, d_3 = 1, d_4 = 1, \ldots \)

Koszul dual to 1-Alia algebras is defined by identities
\[
(t_1 t_2) t_3 = -t_1 (t_2 t_3),
\]
\[
(t_1 t_2) t_3 = (t_2 t_1) t_3,
\]
\[
(t_1 t_2) t_3 = (t_1 t_3) t_2.
\]
Multilinear parts of degree \( n \) of free algebra with these identities has the following dimensions \( d_1 = 1, d_2 = 2, d_3 = 1, d_i = 0, i > 3 \).

**Proof.** According left-Alia identity in degree 3 there is only one non-trivial relation between 6 left-bracketed elements
\[
(c \circ b) \circ a = (a \circ b) \circ c - (b \circ a) \circ c + (b \circ c) \circ a + (c \circ a) \circ b - (a \circ c) \circ b
\]
and no condition between 6 right-bracketed elements. Therefore we can take as a base elements of free left-Alia algebra of degree 3 all 12 elements except \((c \circ b) \circ a\).

We have
\[
[[a \otimes u, b \otimes v], c \otimes w] =
\]
\[
((a \cdot b) \cdot c) \otimes ((uv)w) - ((a \cdot b) \cdot c) \otimes ((vu)w) - (c \cdot (a \cdot b)) \otimes (w(uv)) + (c \cdot (b \cdot a)) \otimes (w(vu)),
\]
\[
[[b \otimes v, c \otimes w], a \otimes u] =
\]
\[
((b \cdot c) \cdot a) \otimes ((vw)u) - ((b \cdot c) \cdot a) \otimes ((uv)w) - (a \cdot (b \cdot c)) \otimes (u(vw)) + (a \cdot (c \cdot b)) \otimes (u(wv)) =
\]
(according to (6))

\[(b\cdot c)\cdot a \otimes (uv\cdot wv) - (a\cdot b)\otimes c\otimes ((uv\cdot wv) + (b\cdot a)\otimes c\otimes ((uv\cdot wv) - (b\cdot c)\otimes a\otimes ((uv\cdot wv) - (c\cdot a)\otimes b\otimes ((uv\cdot wv) - (a\cdot b\cdot c)\otimes (u\cdot wv)\otimes (v\cdot wv)).

Thus, dimensions of multilinear parts are \(1, 2, 11, 100\).

\[
((a \cdot b) \cdot c) \otimes \{(uv)w - (uv)w\} - ((b \cdot a) \cdot c) \otimes \{(uv)w - (uv)w\} - (c \cdot a) \otimes b \otimes \{(uv)w - (uv)w\} + (a \cdot c) \otimes b \otimes \{(uv)w - (uv)w\} - (c \cdot (b \cdot a)) \otimes (w(uv)) + ((c \cdot (b \cdot a)) \otimes (w(uv)) + ((a \cdot (b \cdot c)) \otimes (w(uv)) - (b \cdot (c \cdot a)) \otimes (v(wu)) + ((b \cdot (c \cdot a)) \otimes (v(wu)) = 0.

Therefore Koszul dual operad is generated by relations that follow from identities

\[(7) \quad (t_1 t_2) t_3 = (t_2 t_1) t_2, \quad (t_1 t_2) t_3 = (t_1 t_3) t_1, \quad t_1 t_2 t_3 = 0.

It is easy to see that multilinear part of degree \(n\) of free algebra with identities (7) has the following base

\[n = 1, \quad \{a_1\},\]

\[n = 2, \quad \{a_1 a_2, a_2 a_1\},\]

\[n > 2, \quad \{\cdots ((a_1 a_2) a_3) \cdots a_n\}.

Thus, dimensions of multilinear parts are \(d_2 = 2, d_i = 1, i \neq 2\).

We omit long calculations that shows that first four dimensions of multilinear parts of free left-Alia algebras are \(1, 2, 11, 100\).

So, generating functions are

\[f_{\text{left}}(x) = -x + x^2 - 11x^3/6 + 25x^4/6 + O(x^5),\]

\[f_{\text{dual(left)}}(x) = -x + x^2 - x^3/6 + x^4/24 + O(x^5).\]

We see that

\[f_{\text{left}}(f_{\text{dual(left)}}(x)) = x - x^4/24 + O(x^5) \neq x.

Therefore, necessary condition for Koszulity [1] for left-Alia algebras is not fulfilled.

The case of 1-Alia algebras is considered in a similar ways.
Remark. We do not know whether 1-Alia algebras form Koszul operad. Generating functions look like

\[ f_{1-\text{alia}}(x) = -x + x^2 - 11x^3/3! + 100x^4/4! - 1270x^5/5! + O(x^6), \]

\[ f_{\text{dual}(1-\text{alia})}(x) = -x + x^2 - x^3/3!. \]

No contradiction for Koszulity condition until degree 5:

\[ f_{1-\text{alia}}(f_{\text{dual}(1-\text{alia})}(x)) = x + O(x^6). \]

References


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