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ALGEBRAS WITH SKEW-SYMMETRIC IDENTITY OF DEGREE 3

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Dedicated to 70th birthday of E.B. Vinberg

Abstract. Algebras with one of the following identities are considered:

$$[[t_1,t_2],t_3]+[[t_2,t_3],t_1]+[[t_3,t_1],t_2]=0,$$
$$[t_1,t_2]t_3+[t_2,t_3]t_1+[t_3,t_1]t_2=0,$$
$$\{[t_1,t_2],t_3\}+\{[t_2,t_3],t_1\}+\{[t_3,t_1],t_2\}=0,$$

where $[t_1,t_2]=t_1t_2-t_2t_1$ and $\{t_1,t_2\}=t_1t_2+t_2t_1$. We prove that any algebra with a skew-symmetric identity of degree 3 is isomorphic or anti-isomorphic to one of such algebras or can be obtained as their q-commutator algebras.

1. Introduction

Denote by $(A,\circ)$ an algebra with a vector space A over a field $\mathbb{K}$ and a multiplication $\circ$. Let $\circ_q$ be a new multiplication on A defined by

$$a\circ_q b=a\circ b+q\,b\circ a \quad (q\text{-}commutator).$$

Notice that $\circ_{-1}$ coincides with ordinary commutator

$$[a,b]=a\circ b-b\circ a=a\circ_{-1}b$$

and $\circ_1$ coincides with anti-commutator

$$\{a,b\}=a\circ b+b\circ a=a\circ_1 b.$$

Call the algebra $(A,\circ_q)$ as q-*algebra of* $(A,\circ)$.

Let $\mathbb{K}\{t_1,\dots,t_k\}$ be an algebra of non-commutative non-associative polynomials with variables $t_1,t_2,\dots,t_k$. For any algebra $(A,\circ)$ we can consider a homomorphism

$$\mathbb{K}\{t_1,\dots,t_k\}\to A,$$

that corresponds to any $f\in\mathbb{K}\{t_1,\dots,t_k\}$ an element $f(a_1,\dots,a_k)\in A$. This means that in $f(t_1,\dots,t_k)$ we make substitutions $t_1:=a_1,\dots,t_k:=a_k$ by elements of A and calculate $f(a_1,\dots,a_k)$ in terms of multiplication $\circ$.

A polynomial $f\in \mathbb{K}\{t_1,t_2,\dots,t_k\}$ is called *identity on* A, if

$$f(a_1,\dots,a_k)=0,\quad \forall a_1,a_2,\dots,a_k\in A.$$

In such cases we say that $f=0$ is an identity of A.

A polynomial $f\in \mathbb{K}\{t_1,t_2,\dots,t_k\}$ is called *skew-symmetric* if

$$f(t_{\sigma(1)},\dots,t_{\sigma(k)})=sign\,\sigma\, f(t_1,\dots,t_k),$$

for any permutation $\sigma\in Sym_k$. An identity $f=0$ is *skew-symmetric* if f as a non-commutative non-associative polynomial is skew-symmetric.

Define polynomials with 2 variables

$$lie(t_1,t_2)=[t_1,t_2]=t_1t_2-t_2t_1,$$

$$jor(t_1,t_2)=\{t_1,t_2\}=t_1t_2+t_2t_1$$

and polynomials with 3 variables

$$lia(t_1,t_2,t_3)=[[t_1,t_2],t_3]+[[t_2,t_3],t_1]+[[t_3,t_1],t_2],$$

$$alia(t_1,t_2,t_3)=\{[t_1,t_2],t_3\}+\{[t_2,t_3],t_1\}+\{[t_3,t_1],t_2\},$$

$$lalia(t_1,t_2,t_3)=[t_1,t_2]t_3+[t_2,t_3]t_1+[t_3,t_1]t_2,$$

$$ralia(t_1,t_2,t_3)=t_1[t_2,t_3]+t_2[t_3,t_1]+t_3[t_1,t_2],$$

$$alia^{(q)}(t_1,t_2,t_3)=[t_1,t_2]t_3+[t_2,t_3]t_1+[t_3,t_1]t_2+q(t_1[t_2,t_3]+t_2[t_3,t_1]+t_3[t_1,t_2]).$$

Introduce the following names for algebras with identities.

identity	name of algebras
$jor=0$	*Anti-commutative*
$lia=0$	*Lie-admissible*
$alia=0$	*Anti-Lie-admissible* or *Alia*
$lalia=0$	*Left Anti-Lie-admissible* or *Left Alia*
$ralia=0$	*Right Anti-Lie-admissible* or *Right Alia*
$alia^{(q)}=0$	*q-Anti-Lie-admissible* or *q-Alia*
$lalia=0, ralia=0$	*Two-sided Alia*

For anti-commutative algebra $(A,\circ)$ a bilinear map $\psi: A\times A\to A$ is called *commutative cocycle*, if

$$\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b)=0,$$

$$\psi(a,b)=\psi(b,a),$$

for any $a,b,c\in A$.

An algebra $(A,\circ)$ is said *anti-isomorphic* to algebra $(A,\star)$ if there exist one-to-one map $f:A\to A$, such that

$$f(a\circ b)=f(b)\star f(a),$$

for any $a,b\in A$.

The aim of our paper is to describe algebras with skew-symmetric identities of degree 3. We reduce the problem of studying algebras with skew-symmetric identities of degree 3 to the problem of studying q-Allia algebras for $q = 0, \pm 1$, anti-commutative algebras and their commutative cocycles. We give standard constructions of 0-Alia algebras and 1-Alia algebras. We give also examples of simple q-Alia algebras.

2. Space of skew-symmetric and symmetric non-associative polynomials

Let $\mathfrak{P}_k$ be a space of multilinear non-associative polynomials with k variables. Since the number of non-associative non-commutative bracketings on k letters is

$$c_k = \frac{1}{k}\binom{2k-2}{k-1} \quad \text{(Catalan number)},$$

it is clear that $\mathfrak{P}_k$ is $\frac{(2k-2)!}{(k-1)!}$-dimensional. Denote by $\mathfrak{P}_k^-$ a subspace of $\mathfrak{P}_k$ generated by skew-symmetric polynomials.

Let

$$\pi^- : \mathfrak{P}_k \to \mathfrak{P}_k^-,$$

be skew-symmetrization map,

$$\pi^- f(t_1, \ldots, t_k) = \frac{1}{k!} \sum_{\sigma \in Sym_k} sign\, \sigma\, f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).$$

Theorem 2.1. *The space $\mathfrak{P}_k^-$ is c_k-dimensional and polynomials of a form $\pi^- f_i$, form base, where $i = 1, 2, \ldots, c_k$, and f_i runs monomials corresponding to different types of bracketings.*

Proof. Let g be a skew-symmetric polynomial. Present it as a sum $\sum_{i=1}^{c_k} g_i$, where g_i is a linear combination of monomials of i-th bracketing type. Since skew-symmetrization map does not change bracketing type, we see that g_i is also skew-symmetric polynomial for any $i = 1, 2, \ldots, c_k$ and is uniquely defined by $g_i(t_1, \ldots, t_k)$. This means that polynomials $\pi^- f_1, \ldots, \pi^- f_{c_k}$ form base of $\mathfrak{P}_k^-$.

Corollary 2.2. $\mathfrak{P}_2^-$ *is* 2*-dimensional and has a base* $\{lalia, ralia\}$.

Remark. Theorem 2.1 is true also for symmetric polynomials. Let $\mathfrak{P}_k^+$ be a subspace of $\mathfrak{P}_k$ generated by symmetric polynomials

$$f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) = f(t_1, \ldots, t_k).$$

and

$$\pi^+ : \mathfrak{P}_k \to \mathfrak{P}_k^+$$

be a symmetrization map,

$$\pi^+ f(t_1,\dots,t_k) = \frac{1}{k!}\sum_{\sigma\in Sym_k} f(t_{\sigma(1)},\dots,t_{\sigma(k)}),$$

Then $\dim \mathfrak{P}_k^+ = c_k$ and polynomials of a form $\pi^+ f_i$, form base, where $i=1,2,\dots,c_k$, and f_i runs monomials corresponding to different types of bracketings.

3. q-Alia algebras constructed by 0-Alia algebras

Denote by $\mathfrak{L}ia$, $\mathfrak{A}lia^{(0)}$, $\mathfrak{A}lia^{(1)}$ and $\mathfrak{A}lia^{(\infty)}$ categories of Lie-admissible, 0-Alia, 1-Alia and two-sided Alia algebras. Notice that

$$\mathfrak{L}ia = \mathfrak{Alia}^{(-1)}$$

and

$$\mathfrak{L}ia \cap \mathfrak{A}lia^{(0)} = \mathfrak{L}ia \cap \mathfrak{A}lia^{(1)} = \mathfrak{A}lia^{(0)} \cap \mathfrak{A}lia^{(1)} = \mathfrak{A}lia^{(\infty)}.$$

Theorem 3.1. *Let $q\in\mathbb{K}$, such that $q^2\neq 1$. Then any algebra of a form $A^{(-q)}$, where A is 0-Alia, satisfies the identity $alia^{(q)}=0$. Inversely, any q-Alia algebra is isomorphic to an algebra $A^{(-q)}$ for some 0-Alia algebra A. In other words, categories of q-Alia algebras $\mathfrak{A}lia^{(q)}$ and 0-Alia algebras $\mathfrak{A}^{(0)}$ are equivalent if $q^2\neq 1$.*

If $q^2=1$ this statement is not true. There exist algebras with identity $alia^{(q)}=0$, that can not be obtained from 0-Alia algebras in a form $A^{(q)}$.

Proof. Let $q^2\neq 1$. Prove that $A^{(q)}$ is 0-Alia if A is q-Alia. Prove also that $(A^{(q)})^{(-q)}$ is once again q-Alia and, moreover, it is isomorphic to A.

Denote by $[a,b]^{(-q)}$ a commutator of the multiplication $\circ_{-q}$. Then

$$[a,b]^{(-q)} = a\circ_{-q} b - b\circ_{-q} a = (1+q)(a\circ b - b\circ a) = (1+q)[a,b].$$

Calculate $lalia(a,b,c)$ and $ralia(a,b,c)$ in terms of multiplication $\circ_{-q}$. We have

$$lalia(a,b,c) = [a,b]^{(-q)}\circ_{-q} c + [b,c]^{(-q)}\circ_{-q} a + [c,a]^{(-q)}\circ_{-q} b$$
$$= (1+q)([a,b]\circ c+[b,c]\circ a+[c,a]\circ b) - (1+q)q(c\circ[a,b]+a\circ[b,c]+b\circ[c,a])$$
$$= (1+q)\ lalia(a,b,c) - (1+q)q\ ralia(a,b,c).$$

Similarly,

$$ralia(a,b,c) = c\circ_{-q}[a,b]^{(-q)} + a\circ_{-q}[b,c]^{(-q)} + b\circ_{-q}[c,a]^{(-q)}$$
$$= (1+q)(c\circ[a,b]+a\circ[b,c]+b\circ[c,a]) - (1+q)q([a,b]\circ c+[b,c]\circ a+[c,a]\circ b)$$
$$= (1+q)ralia(a,b,c) - (1+q)q\ lalia(a,b,c).$$

Therefore,

$$alia^{(q)}(a,b,c) = lalia(a,b,c) + q\ ralia(a,b,c)$$

$$= (1+q)(1-q^2)lalia(a,b,c).$$

This means that $A^{(-q)}$ is q-Alia if A is 0-Alia.

Suppose now $(A,\star)$ is q-Alia. Endow A by a new multiplication

$$a\circ b = (1-q^2)^{-1}(a\star b + q\, b\star a).$$

We see that

$$a\circ_{-q} b = a\circ b - q\, b\circ a = a\star b.$$

Therefore, $(A,\circ_{-q})$ is isomorphic to $(A,\star)$. Check that $(A,\circ)$ is 0-Alia. Let $[a,b]^\star = a\star b - b\star a$. We have

$$[a,b] = (1-q^2)^{-1}(a\star b + q\, b\star a - b\star a - q\, a\star b)$$

$$= (1-q^2)^{-1}(1-q)[a,b]^\star.$$

Thus,

$$lalia(a,b,c)$$

$$= (1-q^2)^{-1}(1-q)([a,b]^\star\circ c + [b,c]^\star\circ a + [c,a]^\star\circ b)$$

$$= (1-q^2)^{-1}(1-q)([a,b]^\star\star c+[b,c]^\star\star a+[c,a]^\star\star b+q\, c\star[a,b]^\star+q\, a\star[b,c]^\star+q\, b\star[c,a]^\star$$

$$= (1-q^2)^{-1}(1-q)\ alia^{(q)}(a,b,c)$$

Therefore $(A,\circ)$ is 0-Alia if $(A,\star)$ is q-Alia and $(A\circ_{-q})$ is isomorphic to $(A,\star)$.

Now consider the case $q^2=1$. Notice that any 0-Alia algebra under q-commutator satisfies identity of degree 2 if $q^2=1$. Namely, any algebra obtained from 0-Alia algebra A in a form $A^{(q)}$ for $q^2=1$ should be anti-commutative (in case $q=-1$) or commutative (in case $q=1$). So, algebras with identities $alia^{(q)}=0, q^2=1,$ without identities of degree 2 gives us counter-examples.

In the case $q=-1$ as a such counter-example one gets free left-symmetric algebras, i.e.,algebras with identity

$$(a,b,c) = (b,a,c).$$

In the case $q=1$ as a counter-example one takes the algebra $(\mathbb{K}[x],\star)$, where

$$a\star b = \partial(\partial(a)b).$$

It is 1-Alia and has no any identity of degree 2.

Thus categories $\mathfrak{A}lia^{(q)}$ and $\mathfrak{A}lia$ are not equivalent if $q^2=1$.

4. Commutative cocycles

To describe two-sided Alia algebras and 1-Alia algebras we need a new notion. Let $A = (A, \circ)$ be an algebra and M be a vector space. Call a bilinear map $\psi : A \times A \to M$ *commutative cocycle* with coefficients in M, if

$$\psi(a,b) = \psi(b,a), \tag{1}$$

$$\psi(a \circ b, c) + \psi(b \circ c, a) + \psi(c \circ a, b) = 0 \tag{2}$$

for any $a, b, c \in A$.

If A is a Lie algebra and the condition is changed to anti-commutative condition, then we will obtain well known notion of 2-cocyclicity of ψ.

If $M = \mathbb{K}$ is the main field, then call commutative 2-cocycle as a *commutative central extension.* In our paper we mainly consider the case $M = A$ and in such cases we call ψ shortly as a commutative cocycle.

Let $Z^2_{com}(A, M)$ be a space of commutative cocycles with coefficients in M. Then

$$Z^2_{com}(A, M) \cong Z^2_{com}(A, \mathbb{K}) \otimes M.$$

For any two-sided Alia algebra $A = (A, \star)$ one can correspond Lie algebra $L = A^{(-1)} = (A, \star_{-1})$ We establish that all two-sided Alia algebras with given Lie part L can be characterized by $Z^2_{com}(L, A)$. Similar situation appears also for 1-Alia algebras. In this case L is just anti-commutative algebra, not necessary Lie.

Let $A = (A, \circ)$ be anti-commutative algebra with commutative cocycle ψ. Let $(A, \circ_\psi)$ be an algebra with vector space A and multiplication $\circ_\psi$ given by

$$a \circ_\psi b = a \circ b + \psi(a, b)$$

Theorem 4.1. ($char \mathbb{K} \neq 2$) *If $A = (A, \circ)$ is anti-commutative algebra and ψ is commutative cocycle, then algebra $(A, \circ_\psi)$ is 1-Alia. Inversely, any 1-Alia algebra $A = (A, \star)$ such that $A^{(-1)} \cong (A, \circ)$ is isomorphic to algebra of a form $(A, \circ_\psi)$ for some cocycle ψ of the anti-commutative algebra $(A, \circ)$.*

Any two-sided Alia algebra is Lie-admissible. If $A = (A, \circ)$ is a Lie algebra and ψ is its commutative cocycle, then the algebra $(A, \circ_\psi)$ is two-sided Alia. Inversely, any two-sided Alia algebra $A = (A, \star)$, such that $A^{(-1)} \cong L$ is isomorphic to algebra of the form $(A, \circ_\psi)$ for some commutative cocycle ψ of the Lie algebra L.

Proof. Let $A=(A,\circ)$ be anti-commutative algebra with multiplication $\circ$ and ψ be commutative bilinear map

$$\psi(a,b)=\psi(b,a), \qquad \forall a,b\in A$$

Let $\star=\circ_\psi$ be multiplication of the algebra $(A,\circ_\psi)$. Let

$$[a,b]^\star = a\star b - b\star a,$$
$$\{a,b\}^\star = a\star b + b\star a$$

be Lie and Jordan commutators for the multiplication $\star$. Then

$$[a,b]^\star = a\star b - b\star a = 2(a\circ b - b\circ a) = 4(a\circ b),$$

and

$$[a,b]^\star\star c = 4((a\circ b)\circ c + \psi(a\circ b, c)),$$
$$c\star[a,b]^\star = 4(c\circ(a\circ b) + \psi(c, a\circ b)).$$

Therefore,

$$\{[a,b]^\star, c\}^\star = 8\psi(a\circ b, c)$$

and

$$\{[a,b]^\star,c\}^\star + \{[b,c]^\star,a\}^\star + \{[c,a]^\star,b\}^\star =$$
$$8(\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b)).$$

Thus, the algebra $(A,\circ_\psi)$ is 1-Alia if and only if ψ is commutative cocycle of the algebra $(A,\circ)$.

Let now $A=(A,\star)$ be 1-Alia. Let $L=(A,\circ)$ be an algebra with a vector space A and a multiplication

$$a\circ b = (a\star b - b\star a)/2.$$

Let $\psi: A\times A\to A$ be a commutative bilinear map given by

$$\psi(a,b) = (a\star b + b\star a)/2.$$

Then the multiplication $\circ$ as a commutator of the multiplication $\star$ is anti-commutative. Further,

$$\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b)$$
$$=(\{a\circ b,c\}+\{b\circ c,a\}+\{c\circ a,b\})/2$$
$$=(\{[a,b]^\star,c\}^\star+\{[b,c]^\star,a\}^\star+\{[c,a]^\star,b\}^\star)/4$$
$$=alia^{(1),\star}(a,b,c)/4=0$$

This means that ψ is commutative cocycle for anti-commutative algebra L. Notice that

$$a\star b = a\circ b + \psi(a,b)$$

So, $(A,\star)\cong(A,\circ_\psi)$.

Now suppose that $A=(A,\star)$ is two-sided Alia. Then as we have noticed above

$$a\star b = a\circ b+\psi(a,b),$$

where

$$a\circ b=[a,b]^\star/2, \qquad \psi(a,b)=\{a,b\}^\star.$$

We know that A is -1-Alia. This means that

$$[[a,b]^\star,c]^\star+[[b,c]^\star,a]^\star+[[c,a]^\star,b]^\star=0.$$

In other words, $(A,\circ)$ is Lie algebra. We also know that A is 1-Alia. This condition is equivalent to the commutative cocyclicity condition of ψ. Thus, A is isomorphic to the algebra $(A,\circ_\psi)$, where $\circ$ is Lie multiplication on A.

Inversely, let $(A,\circ)$ be Lie algebra and ψ be commutative cocycle. Then the algebra $(A,\star)$, where $\star=\circ_\psi$, has the following properties,

$$lalia^\star(a,b,c)=[a,b]^\star\star c+[b,c]^\star\star a+[c,a]^\star\star b$$

$$=2([a,b]^\circ\star c+[b,c]^\circ\star a+[c,a]^\circ\star b)$$

$$=2([a,b]^\circ\circ c+[b,c]^\circ\circ a+[c,a]^\circ\circ b+\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b))$$

$$=0,$$

and similarly,

$$ralia^\star(a,b,c)=a\star[b,c]^\star+b\star[c,a]^\star+c\star[a,b]^\star$$

$$=2(a\circ[b,c]^\circ+b\circ[c,a]^\circ+c\circ[a,b]^\circ+\psi(a\circ b,c)+\psi(b\circ c,a)+\psi(c\circ a,b))$$

$$=0.$$

In other words, $(A,\circ_\psi)$ is two-sided Alia.

5. Algebras with skew-symmetric identity of degree 3

Theorem 5.1. *Any algebra with a skew-symmetric identity of degree 3 over a field $\mathbb{K}$ of characteristic $p\neq 2$ is isomorphic to one of the following algebras:*

- *Lie-admissible algebra*
- *left Alia algebra (or 0-Alia algebra)*
- *right Alia algebra*
- *1-Alia algebra*
- *algebra of a form $A^{(q)}$ for some 0-Alia algebra A and $q\in\mathbb{K}$, such that $q^2\neq 0,1$.*

Characterization of two-sided Alia algebras and 1-Alia algebras in terms of anti-commutative algebras and their commutative cocycles is given in Theorem 4.1. Let $(A,\circ)$ be q-Alia algebra. Then an opposite algebra $(A,\circ_{op})$ with multiplication $a\circ_{op}b=b\circ a$, is $1/q$-Alia if $q\neq 0$. If $q=0$ then 0-Alia algebra is left-Alia and its opposite algebra is right-Alia.

Proof of Theorem 5.1. By Corollary 2.2 a space of skew-symmetric polynomials of degree 3 is 2-dimensional and is generated by the left-Alia and right-Alia polynomials $lalia$ and $ralia$. Therefore any skew-symmetric non-commutative non-associative polynomial of degree 3 has a form $f=f^{\alpha,\beta}=\alpha\ lalia+\beta\ ralia$, where $\alpha,\beta\in\mathbb{K}$. For example,

$$lia=f^{1,-1}$$

$$alia^{(q)}=lalia+q\,ralia.$$

In other words, any non-commutative non-associative skew-symmetric polynomial up to scalar is equal to $alia^{(q)}$ for some $q\in\mathbb{K}$ or equal to $ralt$. It remains to use Theorems 3.1.

6. 0-Alia algebras

6.1. General constructions of 0-Alia algebras.

Proposition 6.1. *Let $(A,\cdot)$ be right-commutative algebra,*

$$(a\cdot b)\cdot c=(a\cdot c)\cdot b,\qquad \forall a,b,c\in A.$$

Then $(A,\cdot)$ is 0-Alia.

Proof.

$$\begin{aligned}&[a,b]\cdot c+[b,c]\cdot c+[c,a]\cdot b\\ &=(a\cdot b)\cdot c-(b\cdot a)\cdot c+(b\cdot c)\cdot a-(c\cdot b)\cdot a+(c\cdot a)\cdot b-(a\cdot c)\cdot b\\ &=(a\cdot b)\cdot c-(a\cdot c)\cdot b+(b\cdot c)\cdot a-(b\cdot a)\cdot c+(c\cdot a)\cdot b-(c\cdot b)\cdot a\\ &=0.\end{aligned}$$

Theorem 6.2. *Let $(U,\cdot)$ be an associative commutative algebra and $f,g:U\to U$ be linear maps. Define on U a multiplication $\circ$ by*

$$a\circ b=a\cdot f(b)+g(a\cdot b).$$

Then $(U,\circ)$ is 0-Alia.

Denote obtained algebra as $\mathcal{A}_0(U,\cdot,f,g)$. For a 0-Alia algebra A say that it is *special* if A is isomorphic to a subalgebra of some algebra of a form $\mathcal{A}_0(U,\cdot,f,g)$, where $(U,\cdot)$ is associative commutative algebra and $f,g:U\to U$ are linear maps. Otherwise say that A is *exceptional.*

Proof. We have

$$[a,b]\circ c$$
$$= (a\cdot f(b))\cdot f(c) - (b\cdot f(a))\cdot f(c) + g((a\cdot f(b))\cdot c - (b\cdot f(a))\cdot c).$$

Therefore by commutativity and associativity properties of the multiplication $\cdot$,

$$[a,b]\circ c + [b,c]\circ a + [c,a]\circ b$$
$$= (a\cdot f(b))\cdot f(c) - (b\cdot f(a))\cdot f(c) + (b\cdot f(c))\cdot f(a) - (c\cdot f(b))\cdot f(a) + (c\cdot f(a))\cdot f(b) - (a\cdot f(c))\cdot f(b)$$
$$+ g\left((a\cdot f(b))\cdot c - (b\cdot f(a))\cdot c + (b\cdot f(c))\cdot a - (c\cdot f(b))\cdot a + (c\cdot f(a))\cdot b - (a\cdot f(c))\cdot b\right)$$
$$= 0.$$

6.2. **Killing form and two-sided Alia algebras in characteristic 3.** Let $(A,\circ)$ be any algebra over a field of characteristic 3 with multiplication $\circ$ and commutator $[a,b] = a\circ b - b\circ a$. A commutative bilinear map $A\times A\to M$ is called *invariant* if

$$\psi([a,b],c) = \psi(a,[b,c]),$$

for any $a,b,c\in A$.

Theorem 6.3. *Let A be any algebra over a field of characteristic $p=3$. Then any commutative invariant form $\psi : A\times A\to M$ is a commutative cocycle.*

Proof. We have

$$\psi([a,b],c) = \psi(a,[b,c]),$$
$$\psi([b,c],a) = \psi(a,[b,c]),$$
$$\psi([c,a],b) = -\psi([a,c],b]) = -\psi(a,[c,b]) = \psi(a,[b,c]).$$

Thus,

$$\psi([a,b],c) + \psi([b,c],a) + \psi([c,a],b) = 3\psi(a,[b,c]) = 0,$$

for any $a,b,c\in A$. Proof is completed.

Recall that, for any semi-simple Lie algebra a Killing form

$$(a,b) = tr\, ad\, a\ ad\, b$$

is invariant and non-degenerate. Let $A = (A,\circ)$ be Lie algebra and $\tilde{A} = A + \mathbb{K}$ be commutative central extension defined by a commutative cocycle $\psi\in Z^2_{com}(A,\mathbb{K})$. The multiplication on $\tilde{A}$ is defined by

$$a\star b = a\circ b + \psi(a,b).$$

Then $(\tilde{A},\star)$ is two-sided Alia. So,

Corollary 6.4. *Any semi-simple Lie algebra in characteristic 3 with a nontrivial invariant form has nontrivial structures of two-sided Alia algebras.*

6.3. **Simple two-sided Alia algebra with Lie part** sl_2 **.**

Theorem 6.5. *Let* $L =< e_{-1}, e_0, e_1 | [e_{-1}, e_1] = e_0, [e_{-1}, e_1] = e_0, [e_0, e_1] = e_1 >$ *be* 3*-dimensional simple Lie algebra. Then* $Z^2_{com}(L, \mathbb{K})$ *is* 5 *- dimensional and is generated by commutative cocycles* $\eta_i, i = 1, \ldots, 5$ *defined by*

$$\eta_1(e_{-1}, e_{-1}) = 1, \qquad \eta_2(e_{-1}, e_0) = \eta_2(e_0, e_{-1}) = 1,$$

$$\eta_3(e_{-1}, e_1) = 1, \qquad \eta_3(e_0, e_0) = 2, \qquad \eta_3(e_1, e_{-1}) = 1,$$

$$\eta_4(e_0, e_1) = \eta_4(e_1, e_0) = 1, \qquad \eta_5(e_1, e_1) = 1$$

(non-written components are 0 *).*

Proof. There is only one nontrivial cocyclicity condition $d\psi(e_{-1}, e_0, e_1) = 0$. More exactly,

$$2\psi(e_{-1}, e_1) = \psi(e_0, [e_{-1}, e_1]) = \psi(e_0, e_0).$$

Other statements are evident.

Another formulation of Theorem 6.5.

Theorem 6.6. *Let* $(sl_2, \star)$ *be an algebra with multiplication table*

$$e_{-1} \star e_{-1} = \alpha_{1,1}e_{-1} + \alpha_{1,2}e_0 + \alpha_{1,3}e_1,$$

$$e_{-1}\star e_0 = e_{-1}+\alpha_{2,1}e_{-1}+\alpha_{2,2}e_0+\alpha_{2,3}e_1, \quad e_0\star e_{-1} = -e_{-1}+\alpha_{2,1}e_{-1}+\alpha_{2,2}e_0+\alpha_{2,3}e_1,$$

$$e_{-1}\star e_1 = e_0+\alpha_{3,1}e_{-1}+\alpha_{3,2}e_0+\alpha_{3,3}e_1, \quad e_1\star e_{-1} = -e_0+\alpha_{3,1}e_{-1}+\alpha_{3,2}e_0+\alpha_{3,3}e_1,$$

$$e_0 \star e_0 = 2(\alpha_{3,1}e_{-1} + \alpha_{3,2}e_0 + \alpha_{3,3}e_1),$$

$$e_0\star e_1 = e_1+\alpha_{4,1}e_0+\alpha_{4,2}e_0+\alpha_{4,3}e_1, \quad e_1\star e_0 = -e_1+\alpha_{4,1}e_{-1}+\alpha_{4,2}e_0+\alpha_{4,3}e_1,$$

$$e_1 \star e_1 = \alpha_{5,1}e_{-1} + \alpha_{5,2}e_0 + \alpha_{5,3}e_1,$$

where $\alpha_{i,j} \in \mathbb{K}, \; i = 1, 2, 3, 4, 5, j = 1, 2, 3$. *Then* $(sl_2, \star)$ *is two-sided Alia algebra. It is simple for any* 5×3*-matrix* $(\alpha_{i,j})$. *Any two-sided Alia algebra connected with* sl_2 *is isomorphic to a such algebra for some* 5×3*-matrix* $(\alpha_{i,j})$.

Proof. Follows from Theorems 6.5 and 4.1.

Remark. If $p \neq 2, 3,$ then the algebra $(sl_2, \star)$ gives us a unique nontrivial example of two-sided algebras connected with classical simple Lie algebras [4] .

6.4. **Simple two-sided Alia algebras with Lie part W_1.** Let $L = W_1$ be one-sided or two-sided Witt algebra of rank 1 over a field $\mathbb{K}$ of characteristic 0. Recall that, one-sided Witt algebra of rank 1 is generated by vectors $e_i, i \in \mathbb{Z}$ such that $i \geq -1$, and two-sided Witt algebra of rank 1 is generated by elements $e_i, i \in \mathbb{Z}$. In both cases the multiplication is given by

$$[e_i, e_j] = (j - i)e_{i+j}.$$

Theorem 6.7. *Let L be one-sided or two-sided Witt algebra of rank 1. Then $Z^2_{com}(L, \mathbb{K})$ is infinite-dimensional and is generated by commutative cocycles $\eta_i, i \in \mathbb{Z}$, defined by*

$$\eta_i(u, v) = \textit{coefficient of } uv \textit{ at } x^{i+2}.$$

Here $i \geq -2$ if L is one-sided Witt algebra.

Proof. Let $\psi \in Z^2_{com}(L, \mathbb{K})$ be commutative cocycle. Notice that $Z^2_{com}(L, \mathbb{K})$ is a direct sum of homogeneous subspaces,

$$Z^2_{com}(L, \mathbb{K}) = \oplus_s Z^2_{com,s}(L, \mathbb{K}),$$

$$Z^2_{com,s}(L, \mathbb{K}) =< \psi \in Z^2_{com}(L, \mathbb{K}) | \psi(e_i, e_j) = 0, i + j \neq s > .$$

We can assume that ψ is a homogeneous.

Commutative cocyclicity conditions on $e_0, e_i, e_j, i + j = s,$ gives us the following relations

$$\psi([e_0, e_i], e_j) + \psi([e_i, e_j], e_0) + \psi([e_j, e_0], e_i) = 0 \Rightarrow$$

$$i\,\psi(e_i, e_j) + (j - i)\psi(e_{i+j}, e_0) - j\,\psi(e_j, e_i) = 0 \Rightarrow$$

$$(j - i)\psi(e_0, e_{i+j}) = (j - i)\psi(e_i, e_j).$$

Thus, if $i \neq j$,

$$\psi(e_i, e_j) = \psi(e_0, e_{i+j}).$$

Therefore,

$$\psi = \psi(e_0, e_s)\eta_{s-2}.$$

The proof is finished.

Another formulation of Theorem6.7

Theorem 6.8. *Let f be an endomorphism of polynomial space $U = \mathbb{K}[x]$ or Laurent polynomial space $U = \mathbb{K}[x, x^{-1}]$. Then the algebra $(U, \star_f)$, where*

$$a \star_f b = \partial(a)b - a\partial(b) + f(ab),$$

is two-sided Alia algebra and simple. Any two-sided Alia algebra connected with (one-sided or two-sided) Witt algebra of rank 1 is isomorphic to $(U, \star_f)$ for some endomorphism $f \in End\, U$.

Proof. Follows from Theorems 6.7 and 4.1.

6.5. **Simple 0-Alia algebras defined by symmetric matrix.** Let $\lambda = (\lambda_{i,j})$ be a symmetric matrix. Endow space of polynomials $U = \mathbb{K}[x_1, \ldots, x_n]$, by a multiplication

$$a \star b = \sum_{i,j} \lambda_{i,j}(\partial_i(a)\partial_j(b) + \frac{1}{2}\partial_i\partial_j(a)b).$$

In other words,

$$a \star b = \sum_{i<j} \lambda_{i,j}(\partial_i(a)\partial_j(b) + \partial_j(a)\partial_i(b) + \partial_i\partial_j(a)b) + \sum_i \lambda_{i,i}(\partial_i(a)\partial_i(b) + \frac{1}{2}\partial_i^2(a)b)$$

Let $a \cdot b$ be a usual multiplication of polynomials and

$$f(a) = -\frac{1}{2}\sum_{i,j} \lambda_{i,j}\partial_i\partial_j(a),$$

$$g(a) = \frac{1}{2}\sum_{i,j} \lambda_{i,j}\partial_i\partial_j(b).$$

Then

$$a \star b = a \cdot f(b) + g(a \cdot b).$$

So, $(U, \star)$ is a standard algebra $\mathcal{A}(U, \cdot, f, g)$. Hence by Theorem 7.1 $(U, \star)$ is 0-Alia.

Theorem 6.9. *The 0-Alia algebra $(U, \star)$ is simple if and only if the matrix $(\lambda_{i,j})$ is non-degenerate.*

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Endow $(U, \star)$ by grading. If

$$|x^\alpha| = |\alpha| - 2, \qquad \alpha \in \mathbb{Z}_+^n,$$

$$U_k = < x^\alpha | \, |\alpha| = k + 2 >,$$

then

$$U = \oplus_{k \geq -2} U_k,$$

$$U_k \star U_s \subseteq U_{k+s}.$$

For example,

$$U_{-2} = < 1 >,$$

$$U_{-1} = < x_i | i = 1, \ldots, n >,$$

$$U_0 = < x_i x_j | i, j = 1, \ldots, n > .$$

Notice that

$$u\star 1=\sum_{i<j}\lambda_{i,j}\partial_i\partial_j(u)+\frac{1}{2}\sum_i\lambda_{i,i}\partial_i^2(u),\qquad \forall u\in U,$$

$$1\star u=0,\qquad \forall u\in U,$$

$$x_i\star x_j=x_j\star x_i=\lambda_{i,j}1,$$

$$x_i\star u=\sum_j\lambda_{i,j}\partial_j(u),$$

$$u\star x_i=\frac{1}{2}\lambda_{i,i}x_i\partial_i^2(u)+\sum_{j\neq i}\lambda_{i,j}x_i\partial_i\partial_j(u)+\sum_j\lambda_{i,j}\partial_j(u).$$

In particular,

$$[u,x_i]=u\star x_i-x_i\star u=\frac{1}{2}\lambda_{i,i}x_i\partial_i^2(u)+\sum_{j\neq i}\lambda_{i,j}x_i\partial_i\partial_j(u).$$

The following Lemma states that the algebra $(U,\star)$ is transitive.

Lemma 6.10. *If* $x_i{\star}u=0, u{\star}x_i=0, u{\star}1=0,$ *then* $u\in U_{-2}=<1>$.

Proof. From the condition $u\star 1=0$ it follows that

$$u=\theta_0 1+\sum_i\theta_i x_i+\sum_{i\le j}\theta_{i,j}x_ix_j,$$

for some $\theta_0,\theta_i,\theta_{i,j}=\theta_{j,i}\in\mathbb{K}, i\le j,$ with property

$$\sum_{i\le j}\lambda_{i,j}\theta_{i,j}=0.$$

Further, for any $i=1,\dots,n,$

$$x_i{\star}u=0\Rightarrow\sum_j\lambda_{i,j}\partial_j(u)=0\Rightarrow\sum_j\lambda_{i,j}\theta_j+\sum_j\lambda_{i,j}(\sum_{i'<j}\theta_{i',j}x_{i'}+\sum_{j'>j}\theta_{j,j'}x_{j'}+2\theta_{j,j}x_j)=0$$

$$\Rightarrow\sum_s\lambda_{i,s}\theta_s+\sum_s\lambda_{i,s}\sum_{j<s}\theta_{j,s}x_j+\sum_s\lambda_{i,s}\sum_{j>s}\theta_{s,j}x_j+2\sum_j\lambda_{i,j}\theta_{j,j}x_j)=0$$

$$\Rightarrow\sum_s\lambda_{i,s}\theta_s+\sum_j\sum_{j<s}\lambda_{i,s}\theta_{j,s}x_j+\sum_j\sum_{j>s}\lambda_{i,s}\theta_{s,j}x_j+2\sum_j\lambda_{i,j}\theta_{j,j}x_j=0$$

$$\Rightarrow\sum_j\lambda_{i,j}\theta_j=0,$$

$$2\lambda_{i,j}\theta_{j,j}+\sum_{j<s}\lambda_{i,s}\theta_{j,s}+\sum_{j>s}\lambda_{i,s}\theta_{s,j}=0,\qquad \forall j=1,\dots,n.$$

$$\Rightarrow\sum_j\lambda_{i,j}\theta_j=0,$$

$$\sum_{s=1}^{j-1} \lambda_{i,s}\theta_{s,j} + 2\lambda_{i,j}\theta_{j,j} + \sum_{s=j+1}^{n} \lambda_{i,s}\theta_{j,s} = 0, \qquad \forall j = 1, \dots, n.$$

In other words,

$$\lambda T = 0,$$

$$\lambda\theta = 0,$$

where λ is $n \times n$-matrix $(\lambda_{i,j})$, T is a column with coordinates $(\theta_1, \dots, \theta_n)$, and θ is a matrix of a form

$$\theta = \begin{pmatrix} 2\theta_{1,1} & \theta_{1,2} & \theta_{1,3} & \cdots & \theta_{1,n} \\ \theta_{1,2} & 2\theta_{2,2} & \theta_{2,3} & \cdots & \theta_{2,n} \\ \theta_{1,3} & \theta_{2,3} & 2\theta_{3,3} & \cdots & \theta_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \theta_{1,n} & \theta_{2,n} & \theta_{3,n} & \cdots & 2\theta_{n,n} \end{pmatrix}$$

Since, $det\,(\lambda_{i,j}) \neq 0$, this means that $T = 0$, $\theta = 0$. Lemma is proved.

Lemma 6.11. *Suppose that* $\lambda_{i_0,j_0} \neq 0$, *for some* $1 \leq i_0, j_0 \leq n$. *Then for any* $v \in U$, *there exists* $u \in U$, *such that*

$$v = \sum_{i,j} \lambda_{i,j}\partial_i\partial_j(u).$$

Proof. Endow $\mathbb{Z}^n_+$ by lexicographical ordering. For $\alpha, \beta \in \mathbb{Z}^n_+$ say that $\alpha < \beta$, in the following situations:

- $|\alpha| < |\beta|$ or
- $|\alpha| = |\beta|$ and $\alpha_1 = \beta_1, \dots, \alpha_{k-1} = \beta_{k-1}, \alpha_k < \beta_k$ for some $k \leq n$.

Suppose that $\lambda_{i_0,j_0} \neq 0, i_0 \leq j_0$, and (i_0, j_0) is maximal with such property. In other words, $\lambda_{i,j} = 0, i \leq j$, if $i > i_0$ or $i = i_0, j > j_0$.

Show that

$$x^\alpha \in < \sum_{i<j} \lambda_{i,j}\partial_i\partial_j(u) | u \in \mathbb{K}[x_1, \dots, x_n] >$$

for any $\alpha \in \mathbb{Z}^n_+$. Use induction by $s = |\alpha|$ and in any fixed s use induction by ordered set of α's with $|\alpha| = s$.

If $s = 0$, then $\alpha = (0, \dots, 0)$ and

$$1 = \sum_{i<j} \lambda_{i,j}\partial_i\partial_j(\lambda^{-1}_{i_0,j_0} x_{i_0} x_{j_0}), \text{ if } i_0 < j_0,$$

$$1 = \sum_{i<j} \lambda_{i,j}\partial_i\partial_j((2\lambda_{i_0,i_0})^{-1} x^2_{i_0}), \text{ if } i_0 < j_0, .$$

Therefore base of induction is established.

Suppose that for $s-1$ our statement is true. Suppose that for any $\beta\in\mathbb{Z}^n_+$, such that $|\beta|=s$ and $\beta<\alpha$ this statement is also true. Set

$$u=x_{i_0}^{\alpha_{i_0}+1}x_{j_0}^{\alpha_{j_0}+1}\prod_{i\neq i_0,j_0}x_i^{\alpha_i},\ \text{if}\ i_0<j_0$$

$$u=x_{i_0}^{\alpha_{i_0}+2}\prod_{i\neq i_0}x_i^{\alpha_i},\ \text{if}\ i_0=j_0\,.$$

Then

$$\sum_{i\le j}\lambda_{i,j}\partial_i\partial_j(u)=\lambda_{i_0,j_0}(\alpha_{i_0}+1)(\alpha_{j_0}+1)x^\alpha+u',$$

if $i_0<j_0$ or

$$\sum_{i\le j}\lambda_{i,j}\partial_i\partial_j(u)=\lambda_{i_0,i_0}(\alpha_{i_0}+2)(\alpha_{i_0}+1)x^\alpha+u'',$$

if $i_0=j_0$. Here u',u'' are linear combination of monomials of a form x^β with $\beta<\alpha$. So, by inductive suggestion

$$x^\alpha\in<\sum_{i<j}\lambda_{i,j}\partial_i\partial_j(u)|u\in\mathbb{K}[x_1,\ldots,x_n]>.$$

Lemma is proved.

Proof of Theorem 6.9. Suppose that $det\,(\lambda_{i,j})=0$. Then there exists some $\eta_i\in K, i=1,\ldots,n,$ such that

$$\sum_{j=1}^n\lambda_{i,j}\eta_j=0,\quad i=1,\ldots,n. \tag{3}$$

Set

$$X=\sum_{i=1}^n\eta_ix_i.$$

Let J be subspace of $A,$ that consists of elements of a form $Xu,u\in U=\mathbb{K}[x_1,\ldots,x_n],$ where Xu denotes usual multiplication of polynomials. Prove that J is ideal of $U.$

We have

$$(Xu)\star a=$$

$$\sum_{i,j}\lambda_{i,j}(\partial_i(Xu)\partial_j(a)+\frac{1}{2}\partial_i\partial_j(Xu)a)$$

$$=\sum_{i,j}\lambda_{i,j}\{\partial_i(X)u\partial_j(a)+X\partial_i(u)\partial_j(a)$$

$$+\frac{1}{2}\partial_i(X)\partial_j(u)a+\frac{1}{2}\partial_j(X)\partial_i(u)a+\frac{1}{2}X\partial_i\partial_j(u)a\}$$

$$= X' + X_1 + X_2,$$

where

$$X' =$$

$$\sum_{i,j} \lambda_{i,j}\{\partial_i(X)u\partial_j(a) + \frac{1}{2}\partial_i(X)\partial_j(u)a + \frac{1}{2}\partial_j(X)\partial_i(u)a\},$$

$$X_1 = X(\sum_{i,j} \lambda_{i,j}\partial_i(u)\partial_j(a)) \in J,$$

$$X_2 = X(\sum_{i,j} \frac{1}{2}\partial_i\partial_j(u)a) \in J.$$

By (3)

$$X' =$$

$$\sum_j(\sum_{i=1}^n \lambda_{i,j}\eta_i)u\partial_j(a) + \frac{1}{2}\sum_j(\sum_{i=1}^n \lambda_{i,j}\eta_i)\partial_j(u)a + \frac{1}{2}\sum_i(\sum_{j=1}^n \lambda_{i,j}\eta_j)\partial_i(u)a$$

$$= 0.$$

Hence,

$$(Xu) \star a = X_1 + X_2 \in J,$$

for any $a, u \in U$. Similarly,

$$a \star (Xu) =$$

$$\sum_{i,j} \lambda_{i,j}(\partial_i(a)\partial_j(Xu) + \partial_j(a)\partial_i(Xu) + \frac{1}{2}\partial_i\partial_j(a)Xu)$$

$$= X'' + X_5 + X_6 + X_7,$$

where

$$X'' = \sum_{i,j} \lambda_{i,j}(\partial_i(a)\partial_j(X)u + \partial_j(a)\partial_i(X)u)$$

$$X_5 = X(\sum_{i,j} \lambda_{i,j}(\partial_i(a)\partial_j(u)) \in J,$$

$$X_6 = X(\sum_{i,j} \lambda_{i,j}\partial_j(a)\partial_i(u)) \in J,$$

$$X_7 = X(\sum_{i,j} \frac{1}{2}\partial_i\partial_j(a)u) \in J.$$

By (3),

$$X'' =$$

$$\sum_i(\sum_{j=1}^{n_1} \lambda_{i,j}\eta_j)\partial_i(a)u$$

$$+\sum_j(\sum_{i=1}^{n_1}\lambda_{i,j}\eta_i)\partial_j(a)u)$$
$$=0.$$

Therefore,

$$a\star(Xu)=X_5+X_6+X_7\in J,$$

for any $a,u\in U$.

So, we have proved that $J=<Xu:u\in U>$ is ideal of $(U,\star)$. It remains to note that it is non-trivial ideal. It is evident: $1\notin J$.

Now suppose that $\det(\lambda_{i,j})\neq 0$. Prove that $(U,\star)$ is simple.

Suppose that it is not true: I is some non-trivial ideal of $(U,\star)$. Take some $0\neq R\in I$. Suppose that $R=\sum_{\alpha\in\mathbf{Z}_+^n}\mu_\alpha x^\alpha$, for $\mu_\alpha\in\mathbb{K}$, where $x^\alpha=\prod_i x_i^{\alpha_i},\alpha=(\alpha_1,\dots,\alpha_n)$. Assume that $\mu_\alpha=0$, for any α, such that $|\alpha|>k$, but $\mu_\beta\neq 0$, for some $\beta\in\mathbb{Z}_+^n$ with $|\beta|=k$. Call $k=deg\,R$ degree of R. Take $R\in I$ with minimal $deg\,R$.

Since

$$deg\,R\star 1<deg\,R,\quad deg\,R\star x_i<deg\,R,\quad deg\,x_i\star R<deg\,R,$$

if $R\star 1,x_i\star R,R\star x_i\neq 0$, by Lemma 6.10 we obtain that

$$deg\,R=0.$$

In other words, $R\in I$. So,

$$1\in U,$$

if $det\,\lambda\neq 0$.

Then

$$1\in I\Rightarrow u\star 1=\frac{1}{2}\sum_{i,j}\lambda_{i,j}\partial_i\partial_j(u)\in J,$$

for any $u\in U$. By Lemma 6.11, $I=U$. This means that $(U,\star)$, is simple, if $det\,(\lambda_{i,j})\neq 0$.

6.6. **Simple exceptional 0-Alia algebra.** All 0-Alia algebras constructed above are special. In other words they can be constructed in a form $\mathcal{A}_0(U,\cdot,f,g)$ for some associative commutative algebra $(U,\cdot)$ and endomorphisms f,g. In [3] is proved that the following algebra will be exceptional.

Theorem 6.12. *The algebra* $(\mathbb{K}[x],\star)$ *with multiplication*

$$a\star b=\partial^3(a)b+4\partial^2(a)\partial(b)+5\partial(a)\partial^2(b)+2a\,\partial^3(b),$$

is 0*-Alia and simple.*

Proof. Let $U = \mathbb{K}[x]$. Direct calculations show that $(U,\star)$ is 0-Alia.

Let $e_i = x^{i+3}$. Then

$$e_i \star e_j = (4+i+j)(5+i+j)(9+i+2j)e_{i+j}.$$

So, A is graded:

$$A = \oplus_{i\geq -3} A_i, \quad A_i =< x^{i+3} >,$$

$$A_i \star A_j \subseteq A_{i+j}.$$

Lemma 6.13. *If* $e_{-1} \star u = 0$, *then* $u \in A_{-3}$.

Proof. Let

$$u = \sum_{j\leq j_0} \lambda_j e_j, \qquad \lambda_{j_0} \neq 0.$$

Suppose that $e_{-1} \star u = 0$. We have to prove that $j_0 = -3$. Since $(A,\star)$ is graded,

$$e_{-1} \star u = 0 \Rightarrow \lambda_{j_0} e_i \star e_{j_0 - 1} = 0$$

$$\Rightarrow (3+j_0)(4+j_0)(8+2j_0)e_{j_0-1} = 0 \Rightarrow j_0 = -3.$$

Lemma 6.14. *For any* $u \in A$ *there exists* v *such that* $u = e_{-1} \star v$.

Proof. Let $j \geq -3$. Then

$$(4+j)(5+j)(10+2j) \neq 0.$$

Therefore, we can take the element

$$v = e_{j+1}/((4+j)(5+j)(10+2j)) \in A.$$

Then,

$$e_j = e_{-1} \star v.$$

This means that any element of A can be presented in a form $e_{-1} \star v$.

Proof of Theorem 6.12. Prove that 0-Alia algebra $(\mathbb{K}[x],\star)$ is simple. Let J be some nontrivial ideal of $(\mathbb{K}[x],\star)$ and $0 \neq X = \sum_{i\leq i_1} \lambda_i e_{(i)} \in J$ with $\lambda_{i_1} \neq 0$. Call $i_1 = deg\, X$ degree of X and take such X with minimal degree. By Lemma 6.13

$$deg\, X = -3.$$

In other words,

$$1 \in J.$$

So, by Lemma 6.14 $J = \mathbb{K}[x]$.

7. 1-Alia algebras

7.1. **Standard construction of 1-Alia algebras.**

Theorem 7.1. *Let* $(U,\cdot)$ *be associative commutative algebra and* $f,g : U\to U$ *be linear maps. Define on* U *a multiplication* $\circ$ *by*

$$a\circ b = a\cdot f(b) - b\cdot f(a) + g(a\cdot b).$$

Then $(U,\circ)$ *is* 1*-Alia.*

Denote obtained algebra as $\mathcal{A}_1(U,\cdot,f,g)$.
Proof. Follows by Theorem 9.1.

Corollary 7.2. *Define a multiplication on* $U=\mathbb{K}[x]$ *by*

$$a\star b = -a\partial^m(b) + \partial^m(a)b + \partial^m(ab).$$

Then $(U,\star)$ *is* 1*-Alia for any* $m\ge 1$.

7.2. **Identities for 1-Alia algebra.**

Let U be differential associative commutative algebra with derivation ∂. Endow U by multiplication

$$a\star_u b = u\,\partial(a)\partial^2(b).$$

Denote $\star_1$ shortly as $\star$.

Theorem 7.3. *Let*

$$f_1 = alia^{(1)} = \{[t_1,t_2],t_3\} + \{[t_2,t_3],t_1\} + \{[t_3,t_1],t_2\},$$

$$f_2 = [t_1,t_2]t_3 - t_1(t_2t_3) + t_2(t_1t_3) + 2(t_1t_3)t_2 - 2(t_2t_3)t_1,$$

$$f_3 = ass(t_3t_1,t_4,t_2) - ass(t_3t_2,t_4,t_1) - ass(t_4t_1,t_3,t_2) + ass(t_4t_2,t_3,t_1),$$

$$f_4 = \sum_{\sigma\in Sym_3} sign\,\sigma\,((t_4t_{\sigma(1)})t_{\sigma(2)})t_{\sigma(3)},$$

$$\begin{aligned}f_5 = {} & 2(((t_3t_1)t_2)t_4)t_5 - 2(((t_3t_1)t_4)t_2)t_5 - (((t_3t_1)t_2)t_5)t_4 + (((t_3t_1)t_4)t_5)t_2\\ & -(((t_3t_2)t_1)t_4)t_5 + (((t_3t_2)t_1)t_5)t_4 + (((t_3t_2)t_4)t_5)t_1 - (((t_3t_2)t_5)t_1)t_4\\ & +(((t_3t_4)t_1)t_2)t_5 - (((t_3t_4)t_1)t_5)t_2 - (((t_3t_4)t_2)t_5)t_1 + (((t_3t_4)t_5)t_1)t_2\\ & +(((t_3t_5)t_1)t_2)t_4 - (((t_3t_5)t_1)t_4)t_2\end{aligned}$$

be non-commutative non-associative polynomials. Then

- $f_i = 0, 1\le i\le 5,$ *are identities for* $(U,\star)$
- *Identities* $f_2=0, f_3=0, f_4=0, f_5=0$ *are independent*
- $f_2=0 \Rightarrow f_1=0$
- $f_1=0, f_4=0, f_5=0$ *are identities for* $(U,\star_u)$
- $f_2=0, f_3=0$ *are identities of the algebra* $(U,\star_u)$ *iff* $u=1$.

Here $ass(t_1,t_2,t_3) = (t_1,t_2,t_3) = t_1(t_2t_3) - (t_1t_2)t_s$ *is an associator.*

We omit proof of this result. It needs long calculations. Just note that the multiplication $(a,b) \mapsto \partial(a)\partial^2(b)$ is opposite to the multiplication $a * b = \partial^2(a)\partial(b)$. For the last multiplication Theorem 7.3 partially is proved above.

8. Simple 1-Alia algebra $(\mathbb{K}[x], \circ)$ with multiplication $a \circ b = \partial(\partial(a)b)$

Let

$$a \circ b = \partial(\partial(a)b).$$

Note that

$$2\partial(\partial(a)b) = a\partial^2(b) - \partial^2(a)b + \partial^2(ab).$$

Therefore, $(U, \circ)$ can be obtained by standard construction of 1-Alia algebras $\mathcal{A}_1(U, \cdot, f, g)$, if one sets

$$f(a) = \partial^2(a)/2, g(a) = \partial^2(a)/2.$$

Any commutative or anti-commutative algebra is 1-Alia. It will be interesting to describe simple algebras with minimal identity $alia^{(q)} = 0$ for $q = 0, \pm 1$. Minimality condition exclude from the consideration standard examples of q-Alia algebras, like Lie algebras, (anti)-commutative algebras, right-commutative algebras, left-symmetric algebras. One of such non-trivial examples of 1-Alia algebras gives us the algebra $(\mathbb{K}[x], \circ)$.

Theorem 8.1. *The algebra* $(\mathbb{K}[x], \circ)$ *is simple.*

Proof. Let

$$e_i = x^{i+2}, \quad i \geq -2.$$

Then

$$e_i \circ e_j = (i+2)(i+j+3)e_{i+j}, \qquad -2 \leq i, j.$$

For example,

$$e_{-2} \circ e_j = 0,$$
$$e_j \circ e_{-2} = (j+2)(j+1)e_{j-2},$$
$$e_{-1} \circ e_j = (j+2)e_{j-1},$$
$$e_j \circ e_{-1} = (j+2)^2 e_{j-1},$$
$$e_0 \circ e_j = 2(j+3)e_j,$$
$$e_j \circ e_0 = (j+3)(j+2)e_j.$$

Suppose that non-trivial ideal J has element $X = \sum_{i \geq i_0} \lambda_i e_i \in J$, such that $\lambda_{i_0} \neq 0$ and i_0 is minimal with this property,

$$\sum_j \mu_j e_j \in J \Rightarrow \mu_j = 0, \forall j < i_0.$$

Prove that $i_0=-2$. Suppose that it is not true.

If $i_0\ge 0$, then

$$X\in J\Rightarrow X\circ e_{-2}=\sum_{i\ge i_0}\lambda_i(i{+}2)(i{+}1)e_{i-2}\in J,\quad \lambda_{i_0}(i_0{+}2)(i_0{+}1)\neq 0.$$

This contradicts to minimality i_0. So, the case $i_0\ge 0$ is not possible.

Let $i_0=-1$. Then

$$e_{-1}\circ X=\sum_{i\ge i_0}\mu_i e_{i-1}\in J,$$

where

$$\mu_i=\lambda_i(i+2),\quad \mu_{-2}=\lambda_{-1}\neq 0.$$

This contradicts to minimality of i_0. We proved that the case $i_0=-1$ is also not possible.

So, we have proved that $i_0=-2$. We see that elements $X\circ e_j$ has a form $\sum_{i\ge j-2}\gamma_i e_i$ with $\gamma_{j-2}\neq 0$ if j runs elements $0,1,2,\dots$. This means that $J=\mathbb{K}[x]$. So, $(\mathbb{K}[x],\circ)$ is simple, where $a\circ b=\partial(\partial(a)b)$.

Remark. A map $f:A\to A,\quad f:a\mapsto\partial(a)$, induces a homomorphism of algebras

$$f:(A,*)\to(A,\circ),$$

where

$$a*b=\partial^2(a)\partial(b).$$

Check it:

$$f(a*b)=\partial(\partial^2(a)\partial(b))=\partial(a)\circ\partial(b)=f(a)\circ f(b).$$

So, we see that $(\mathbb{K}[x],*)$ is 1-Alia and there exists exact sequence of 1-Alia algebras

$$0\to\mathbb{K}\to(\mathbb{K}[x],*)\to(\mathbb{K}[x],\circ)\to 0.$$

In other words, $(\mathbb{K}[x],*)$ is a central extension of $(\mathbb{K}[x],\circ)$.

9. Standard construction of q-Alia algebras

Theorem 9.1. *Let $A=(A,\cdot)$ be associative commutative algebra with multiplication $a\cdot b$ and $f,g:A\to A$ linear maps. Define a multiplication $a\circ b$ by*

$$a\circ b=a\cdot f(b)-q\,b\cdot f(a)+g(a\cdot b).$$

Then $(A,\circ)$ is q-Alia.

Proof. Easy calculations. If $q^2\neq 1$, it follows from Theorem 3.1.

9.1. **Simple q-Alia algebras.**

Theorem 9.2. *Let $U=\mathbb{K}[x]$ and*

$$a\star b = a\,\partial^m(b) - q\,\partial^m(a)b + q\,\partial^m(ab).$$

Then $(U,\star)$ are q-Alia and simple for $q^2\neq 1$.

Proof. Calculate q-commutator of the multiplication $\star$

$$\begin{gathered} a\star_q b \\ = a\star b + q\,b\star a \\ = a\partial^m(b) - q\,\partial^m(a)b + q\partial^m(ab) + q\,\partial^m(a)b - q^2\,a\partial^m(b) + q^2\partial^m(ab) \\ = (1-q^2)a\partial^m(b) + (q+q^2)\partial^m(ab) \end{gathered}$$

This multiplication is standard. In other words, for associative commutative algebra U with usual polynomial multiplication $a\cdot b = ab$ and linear maps

$$f: U\to U, \qquad f(a) = (1-q^2)\partial^m(a),$$
$$g: U\to U, \qquad g(a) = (q^2+q)\partial^m(a),$$

the algebra $(U,\star_q)$ has a form $\mathcal{A}_0(U,\cdot,f,g)$. So, by Theorem 7.1 $(U,\star_q)$ is 0-Alia. Then by Theorem 3.1 the algebra $(U,\star)$ is q-Alia.

Set

$$e_i = x^{i+m}/(i+m)!, \qquad i=-m,-m+1,\ldots.$$

Then

$$e_i\star e_j = (\binom{i+j+m}{i+m} - q\binom{i+j+m}{j+m} + q\binom{i+j+2m}{i+m})e_{i+j}.$$

So, $(U,\star)$ is graded,

$$U=\oplus_{i\geq -m}U_i, \qquad U_i = < e_i >,$$
$$U_i\star U_j\subseteq U_{i+j}.$$

Notice that

$$e_{-m}\star e_j = (q-1)e_{j-m}, \tag{4}$$

$$e_i\star e_j = q\binom{m}{-j}e_{-m}, \text{ if } -m<i,j<0, i+j=-m. \tag{5}$$

Let J is a non-trivial ideal of $(U,\star)$. Take $X=\sum_{-m\leq i\leq i_0}\lambda_i e_i\in J$, such that $\lambda_{i_0}\neq 0$ and i_0 is minimal with such property. Since $Y=e_{-m}\star X\in J$ and i_0 is minimal, by grading property $Y=0$. In particular, by (4),

$$\lambda_{i_0}(q-1)=0,$$

and

$$\lambda_{i_0}=0$$

if $i_0 \geq 0$. So, we can assume that $i_0 < 0$. Similar arguments that uses (5) shows that the case $i_0 > -m$ is not possible. So, $i_0 = -m$. In other words

$$e_{-m} \in J.$$

Then by (4)

$$e_j = (q-1)^{-1} e_{-m} \star e_{j+m} \in J.$$

This means that

$$J = U.$$

Therefore, $(U, \star)$ is simple.

10. Dual operads to Alia algebras

Theorem 10.1. *Koszul dual algebras to left-Alia algebras is defined by identities*

$$[t_1, t_2]t_3 = 0,$$
$$(t_1 t_2)t_3 = (t_1 t_3)t_2,$$
$$t_1(t_2 t_3) = 0.$$

Left-Alia operads are not Koszul. Dimensions of multilinear parts of Koszul dual to Left-Alia algebras are $d_1 = 1, d_2 = 2, d_3 = 1, d_4 = 1, \ldots$.

Koszul dual to 1*-Alia algebras is defined by identities*

$$(t_1 t_2)t_3 = -t_1(t_2 t_3),$$
$$(t_1 t_2)t_3 = (t_2 t_1)t_3,$$
$$(t_1 t_2)t_3 = (t_1 t_3)t_2.$$

Multilinear parts of degree n *of free algebra with these identities has the following dimensions* $d_1 = 1, d_2 = 2, d_3 = 1, d_i = 0, i > 3$.

Proof. According left-Alia identity in degree 3 there is only one non-trivial relation between 6 left-bracketed elements

(6) $(c \circ b) \circ a = (a \circ b) \circ c - (b \circ a) \circ c + (b \circ c) \circ a + (c \circ a) \circ b - (a \circ c) \circ b$

and no condition between 6 right-bracketed elements. Therefore we can take as a base elements of free left-Alia algebra of degree 3 all 12 elements except $(c \circ b) \circ a$.

We have

$$[[a \otimes u, b \otimes v], c \otimes w] =$$
$$((a \cdot b) \cdot c) \otimes ((uv)w) - ((b \cdot a) \cdot c) \otimes ((vu)w) - (c \cdot (a \cdot b)) \otimes (w(uv)) + ((c \cdot (b \cdot a)) \otimes (w(vu)),$$

$$[[b \otimes v, c \otimes w], a \otimes u] =$$
$$((b \cdot c) \cdot a) \otimes ((vw)u) - ((c \cdot b) \cdot a) \otimes ((wv)u) - (a \cdot (b \cdot c)) \otimes (u(vw)) + ((a \cdot (c \cdot b)) \otimes (u(wv)) =$$

(according (6))

$((b{\cdot}c){\cdot}a)\otimes((vw)u)-(a\circ b)\circ c\otimes((wv)u)+(b\circ a)\circ c\otimes((wv)u)-(b\circ c)\circ a\otimes((wv)u)$
$-(c\circ a)\circ b\otimes((wv)u)+(a\circ c)\circ b\otimes((wv)u)-(a{\cdot}(b{\cdot}c))\otimes(u(vw))+((a{\cdot}(c{\cdot}b))\otimes(u(wv)),$

$$[[c\otimes w,a\otimes u],b\otimes v]=$$

$((c{\cdot}a){\cdot}b)\otimes((wu)v)-((a{\cdot}c){\cdot}b)\otimes((uw)v)-(b{\cdot}(c{\cdot}a))\otimes(v(wu))+((b{\cdot}(a{\cdot}c))\otimes(v(uw)).$

Thus,

$[[a\otimes u,b\otimes v],c\otimes w]+[[b\otimes v,c\otimes w],a\otimes u]+[[c\otimes w,a\otimes u],b\otimes v]=$

$$((a\cdot b)\cdot c)\otimes\{(uv)w-(wv)u\}-((b\cdot a)\cdot c)\otimes\{(vu)w-((wv)u\}$$
$$+((b\cdot c)\cdot a)\otimes\{(vw)u-(wv)u\}-(c\circ a)\circ b\otimes\{(wv)u-(wu)v\}$$
$$+(a\circ c)\circ b\otimes\{(wv)u-(uw)v\}-(a\cdot(b\cdot c))\otimes(u(vw))$$
$$-(c\cdot(a\cdot b))\otimes(w(uv))+((c\cdot(b\cdot a))\otimes(w(vu))+((a\cdot(c\cdot b))\otimes(u(wv))$$
$$-(b\cdot(c\cdot a))\otimes(v(wu))+((b\cdot(a\cdot c))\otimes(v(uw)).$$

Therefore Koszul dual operad is generated by relations that follow from identities

(7) $\qquad (t_1t_2)t_3=(t_2t_1)t_3,\quad (t_1t_2)t_3=(t_1t_3)t_2,\quad t_1(t_2t_3)=0$

It is easy to see that multilinear part of degree n of free algebra with identities (7) has the following base

$$n=1,\qquad \{a_1\},$$
$$n=2,\qquad \{a_1a_2,a_2a_1\},$$
$$n>2,\qquad \{(\cdots((a_1a_2)a_3)\cdots)a_n\}.$$

Thus, dimensions of multilinear parts are $d_2=2,d_i=1,i\neq 2$.

We omit long calculations that shows that first four dimensions of multilinear parts of free left-Alia algebras are 1,2,11,100.

So, generating functions are

$$f_{lalia}(x)=-x+x^2-11x^3/6+25x^4/6+O(x^5),$$
$$f_{dual(lalia)}(x)=-x+x^2-x^3/6+x^4/24+O(x^5).$$

We see that

$$f_{lalia}(f_{dual(lalia)}(x))=x-x^4/24+O(x^5)\neq x.$$

Therefore, necessary condition for Koszulity [1] for left-Alia algebras is not fulfilled.

The case of 1-Alia algebras is considered in a similar ways.

Remark. We do not know whether 1-Alia algebras form Koszul operad. Generating functions look like

$$f_{1-alia}(x) = -x + x^2 - 11x^3/3! + 100x^4/4! - 1270x^5/5! + O(x^6),$$

$$f_{dual(1-alia)}(x) = -x + x^2 - x^3/3!.$$

No contradiction for Koszulity condition until degree 5:

$$f_{1-alia}(f_{dual(1-alia)}(x)) = x + O(x^6).$$

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