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q-LEIBNIZ ALGEBRAS

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Abstract. An algebra $(A, \circ)$ is called Leibniz if $a\circ(b\circ c) = (a\circ b)\circ c - (a\circ c)\circ b$ for all $a, b, c \in A$. We study identities for the algebras $A^{(q)} = (A, \circ_q)$, where $a \circ_q b = a \circ b + q\, b \circ a$ is the q-commutator. Let char $K \neq 2, 3$. We show that the class of q-Leibniz algebras is defined by one identity of degree 3 if $q^2 \neq 1$, $q \neq -2$, by two identities of degree 3 if $q = -2$, and by the commutativity identity and one identity of degree 4 if $q = 1$. In the case of $q = -1$ we construct two identities of degree 5 that form a base of identities of degree 5 for -1-Leibniz algebras. Any identity of degree < 5 for -1-Leibniz algebras follows from the anti-commutativity identity.

1. Introduction. Denote by $A = (A, \circ)$ an algebra with vector space A over a field K of characteristic $\neq 2, 3$ and multiplication $(a, b) \mapsto a \circ b$. Let $(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$ be the associator and $a \circ_q b = a \circ b + q\, b \circ a$ be the q-commutator, where $q \in K$. Denote by $A^{(q)} = (A, \circ_q)$ the algebra with the q-commutator. Notice that $a \circ_{-1} b = a \circ b - b \circ a$ is a commutator (Lie bracket,

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usually denoted by $[a, b]$) and $a \circ_1 b = a \circ b + b \circ a$ is an anti-commutator (Jordan bracket, sometimes denoted by $\{a, b\}$).

Example. If A is an associative algebra, then $A^{(-1)} = (A, [\ ,\ ])$ is a Lie algebra,

$$[a, b] = -[b, a],$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0,$$

and $A^{(+1)} = (A, \{\ ,\ \})$ is a Jordan algebra,

$$\{a, b\} = \{b, a\},$$

$$\{\{a, a\}, \{b, a\}\} = \{\{\{a, a\}, b\}, a\}.$$

Usually, q-commutators are studied in the frame of quantum groups. It seems that the study of q-identities has their own interest. We try to demonstrate it in the class of Leibniz algebras. We call an algebra A *Leibniz* (more exactly *right-Leibniz*) if for all $a, b, c \in A$

$$a \circ (b \circ c) = (a \circ b) \circ c - (a \circ c) \circ b.$$

Leibniz algebras were introduced in [2], [7]. In other words, Leibniz algebras are algebras with the identity $\mathrm{lei} = 0$, where

$$\mathrm{lei} = \mathrm{lei}(t_1, t_2, t_3) = t_1(t_2t_3) - (t_1t_2)t_3 + (t_1t_3)t_2.$$

Example. Let $(L, \star)$ be a Lie algebra with multiplication $\star$ and let M be an L-module under the right action $(M, L) \to M$, $(m, a) \mapsto ma$. Make M a trivial left L-module: $am = 0$, $a \in L$, $m \in M$. Then the vector space $L \oplus M$ becomes a right-Leibniz algebra under the multiplication

$$(a + m) \circ (b + n) = a \star b + mb.$$

Indeed,

$$\begin{aligned}(a + m) \circ ((b + n) \circ (c + s)) &= (a + m) \circ (b \star c + nc)\\ &= a \star (b \star c) + m(b \star c) = (a \star b) \star c - (a \star c) \star b + (mb)c - (mc)b\\ &= ((a + m) \circ (b + n)) \circ (c + s) - ((a + m) \circ (c + s)) \circ (b + n).\end{aligned}$$

We call the so-obtained algebra $L + M$ (a semi-direct sum of Leibniz algebras) *standard* Leibniz.

Endow a standard Leibniz algebra $(L + M, \circ)$ with the commutator $[\ ,\ ]$. Then

$$\begin{aligned}[a + m, b + n] &= (a + m) \circ (b + n) - (b + n) \circ (a + m)\\ &= (a \star b) + mb - (b \star a) - na = 2[a, b] + mb - na,\end{aligned}$$

where $[a,b] = a\star b - b\star a$. The algebra $(L+M, [\ ,\ ])$ (more exactly, $L+M$ under multiplication $[a,b] + (mb - na)/2$) is called *Omni-Lie* [6], [9].

Given non-associative polynomials $f_1, \dots, f_s$, we let $\mathrm{Var}(f_1, \dots, f_s)$ denote the variety of algebras defined by identities $f_1 = 0, \dots, f_s = 0$. Let $\mathfrak{Lei}$ be the class of Leibniz algebras, i.e., the variety of algebras defined by the (right)-Leibniz identity $\mathrm{lei} = 0$.

In this paper we construct identities for q-(right)-Leibniz algebras. In particular, we describe identities for Omni-Lie algebras.

We prove that the category of q-Leibniz algebras is equivalent to the category of Leibniz algebras if $q^2 \neq 1, q \neq -2$. This means that, for $q \neq \pm 1, -2$, every algebra with identity $\mathrm{lei}^{(q)} = 0$ can be obtained as $A^{(q)}$ from some Leibniz algebra A and, conversely, if B is an algebra with identity $\mathrm{lei}^{(q)} = 0$, then $B^{(-q)}$ is right-Leibniz. In the case of $q = -2$ we should add to the identity $\mathrm{lei}^{(q)} = 0$ the identity $\mathrm{lei}_1^{(q)} = 0$ in order to obtain equivalent categories.

Theorem 1.1. *Let $q \neq -1, 1, -2$. The class of q-Leibniz algebras $\mathfrak{Lei}^{(q)}$ satisfies the identity* $\mathrm{lei}^{(q)} = 0$, *where*

$$\mathrm{lei}^{(q)} = \mathrm{lei}^{(q)}(t_1, t_2, t_3)$$

$$= (q^2-1)\,(t_1(t_2t_3) - t_2(t_1t_3)) + (q^2+q-1)\,(t_2t_1)t_3 + (t_2t_3)t_1 - t_1(t_3t_2) - q\,t_3(t_1t_2).$$

The varieties $\mathfrak{Lei}$, $\mathfrak{Lei}^{(q)}$ and $\mathrm{Var}(\mathrm{lei}^{(q)})$ *are equivalent.*

In particular, $\mathrm{Var}(\mathrm{lei}^{(q)})$ has no special identity for $\mathfrak{Lei}^{(q)}$ if $q \neq -2, q^2 \neq 1$. The identity $\mathrm{lei}_1^{(q)} = 0$ is a consequence of the identity $\mathrm{lei}^{(q)} = 0$ if $q \neq -2, q^2 \neq 1$.

Theorem 1.2. *Let $q = -2$. The class of q-Leibniz algebras $\mathfrak{Lei}^{(-2)}$ satisfies the identities* $\mathrm{lei}^{(-2)} = 0$ *and* $\mathrm{lei}_1^{(-2)} = 0$, *where* $\mathrm{lei}^{(q)}$ *is given above and*

$$\mathrm{lei}_1^{(q)} = \mathrm{lei}_1^{(q)}(t_1, t_2, t_3) = -t_1(t_2t_3 + t_3t_2) + q(t_2t_3 + t_3t_2)t_1.$$

The varieties $\mathfrak{Lei}$, $\mathfrak{Lei}^{(-2)}$ and $\mathrm{Var}(\mathrm{lei}^{(-2)}, \mathrm{lei}_1^{(-2)})$ *are equivalent.*

So the identity $\mathrm{lei}_1^{(-2)} = 0$ is a special identity for $\mathrm{Var}(\mathrm{lei}^{(-2)})$ which does not follow from the identity $\mathrm{lei}^{(-2)} = 0$. All other special identities for $\mathfrak{Lei}^{(-2)}$ follow from $\mathrm{lei}^{(-2)} = 0$.

Let acom, com and ljac be non-commutative non-associative polynomials defined by

$$\begin{aligned}\mathrm{acom} &= t_1t_2 + t_2t_1,\\ \mathrm{com} &= t_1t_2 - t_2t_1,\\ \mathrm{ljac} &= (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2.\end{aligned}$$

Define non-commutative non-associative polynomials leilie_1, leilie_2 of degree five by

$$\mathrm{leilie}_1(t_1,t_2,t_3,t_4,t_5) = 2\,\mathrm{ljac}(\mathrm{ljac}(t_1,t_2,t_3),t_4,t_5) - [\mathrm{ljac}(t_1,t_2,t_3),[t_4,t_5]],$$

$$\begin{aligned}\mathrm{leilie}_2(t_1,t_2,t_3,t_4,t_5) = -\frac{1}{2}\sum_{\sigma\in \mathrm{Sym}(2,3,4,5)} &\mathrm{sign}\,\sigma(-4(((t_{\sigma(2)}t_{\sigma(3)})t_{\sigma(4)})t_{\sigma(5)})t_1\\ &+ 2(((t_{\sigma(2)}t_{\sigma(3)})t_1)t_{\sigma(4)})t_{\sigma(5)} + 2(((t_{\sigma(2)}t_{\sigma(3)})t_{\sigma(4)})t_1)t_{\sigma(5)}\\ &+ ((t_1t_{\sigma(2)}t_{\sigma(3)})(t_{\sigma(4)}t_{\sigma(5)}) + ((t_1t_{\sigma(2)})(t_{\sigma(4)}t_{\sigma(5)}))t_{\sigma(3)}).\end{aligned}$$

For a non-commutative non-associative polynomial $f(t_1,\dots,t_k)$, denote by $\mathrm{Alt}(f)$ its skew-symmetrization

$$\mathrm{Alt}\ f(t_1,\dots,t_k) = \sum_{\sigma\in \mathrm{Sym}_k} \mathrm{sign}\,\sigma\, f(t_{\sigma(1)},\dots,t_{\sigma(k)}).$$

Let

$$\mathrm{leilie}(t_1,t_2,t_3,t_4,t_5) = \mathrm{Alt}(4(((t_1t_2)t_3)t_4)t_5 - ((t_1t_2)t_3)(t_4t_5)).$$

Theorem 1.3. *Let $q=-1$. Let A be a right-Leibniz algebra. Then $A^{(-1)}$ satisfies the identities* $\mathrm{acom}=0$, $\mathrm{leilie}_1=0$ *and* $\mathrm{leilie}_2=0$. *Any multilinear identity of $\mathfrak{Lei}^{(-1)}$ of degree no more than 4 follows from the anti-commutativity identity. Any multilinear identity of $\mathfrak{Lei}^{(-1)}$ of degree 5 follows from the identities* $\mathrm{acom}=0$, $\mathrm{leilie}_1=0$ *and* $\mathrm{leilie}_2=0$.

Corollary 1.4. *Let A be a right-Leibniz algebra. Then $A^{(-1)}$ satisfies the identity* $\mathrm{leilie}=0$.

Corollary 1.5. *Every Omni-Lie algebra satisfies the polynomial identities* $\mathrm{acom}=0$, $\mathrm{leilie}_1=0$, $\mathrm{leilie}_2=0$ *and* $\mathrm{leilie}=0$. *The identities* $\mathrm{acom}=0$, $\mathrm{leilie}_1=0$ *and* $\mathrm{leilie}_2=0$ *form a base of the identities in the space of multilinear identities of degree no more than 5 for the class of Omni-Lie algebras.*

Note that the polynomials leilie_1, leilie_2 and leilie have 9, 60 and 90 terms, respectively.

Let

$$\mathrm{leijor}(t_1,t_2,t_3,t_4) = (t_1t_2)(t_3t_4).$$

Theorem 1.6. *Let $q=1$. Let A be a right-Leibniz algebra. Then $A^{(1)}$ satisfies the identities* $\mathrm{com}=0$ *and* $\mathrm{leijor}=0$. *Every multilinear identity which* *is true for any Leibniz-Jordan algebra follows from the identities* com $= 0$ *and* leijor $= 0$.

In other words, there are no special identities for the class of Leibniz-Jordan algebras.

The Leibniz operad has a dual operad defined by the identity

$$a(bc + cb) = (ab)c + (ac)b.$$

Such algebras are called Zinbiel [7], [8]. Identities for q-Zinbiel algebras are described in [3], [4].

2. Non-commutative non-associative polynomials. Let $K\{t_1, t_2, \ldots\}$ be the algebra of non-commutative non-associative polynomials in the variables $t_1, t_2, \ldots$ (the free magma algebra). For a polynomial $f = f(t_1, \ldots, t_k) \in K\{t_1, t_2, \ldots\}$, we say that $f = 0$ is an *identity* for the algebra $(A, \circ)$ if $f(a_1, \ldots, a_k) = 0$ for all $a_1, \ldots, a_k \in A$.

Recall that there exist $\frac{1}{k}\binom{2(k-1)}{k-1}$ types of bracketing for the string $t_1 \cdots t_k$. For example, there are 5 types of bracketing for 4 elements:

$$((t_1t_2)t_3)t_4,\ (t_1t_2)(t_3t_4),\ t_1(t_2(t_3t_4)),\ t_1((t_2t_3)t_4),\ (t_1(t_2t_3))t_4.$$

Order the types of bracketing somehow. If σ is a type of bracketing, denote by $\sigma(t_{i_1}, \ldots, t_{i_k})$ the string $t_{i_1} \cdots t_{i_k}$ with bracketing type σ. For example, if $k = 4$ and σ is the bracketing type $(t_1(t_2t_3))t_4$ then $\sigma(t_1, t_2, t_1, t_3) = (t_1(t_2t_1))t_3$.

Let α be some bracketing type of $t_1, \ldots, t_n$. We say that a monomial of the form $\alpha(t_{i_1}, \ldots, t_{i_n})$ has *multidegree* $(r_1, \ldots, r_k)$ if $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, k\}$ and $r_m = |\{s : i_s = m, s = 1, \ldots, n\}|$ is the number of indices i_s equal to m for any $m = 1, \ldots, k$. Call $f = f(x_1, \ldots, x_k)$ *homogeneous of degree* $(r_1, \ldots, r_k)$ if f is a linear combination of monomials of multidegree $(r_1, \ldots, r_k)$. Say that a homogeneous polynomial f has *degree* l if $r_1 + \cdots + r_k = l$.

A homogeneous polynomial $f = f(t_1, \ldots, t_k)$ of multidegree $(1, \ldots, 1)$ is called *multilinear.* Notice that the degree of a multilinear polynomial $f \in K\{t_1, \ldots, t_k\}$ is equal to the number of variables k. In other words a polynomial f is multilinear if f is a linear combination of monomials of the form $\alpha(t_{i_1}, \cdots, t_{i_k})$, where $\begin{pmatrix} 1 & \cdots & k \\ i_1 & \ldots & i_k \end{pmatrix} \in \mathrm{Sym}_k$ is a permutation of the set $\{1, \ldots, k\}$ and α is a bracketing.

Given polynomials $f_1, \ldots, f_s, g \in K\{t_1, \ldots, t_k\}$, we say that the identity $g = 0$ *follows from the identities* $f_1 = 0, \ldots, f_s = 0$, and write $\{f_1 = 0, \ldots, f_s =$

$0\} \Rightarrow g = 0$, if $g = 0$ is an identity for any algebra in the variety defined by the identities $f_1 = 0, \ldots, f_s = 0$.

Let $\mathfrak{L}$ be a variety of algebras and let $\mathfrak{L}^{(q)}$ be the class of algebras $A^{(q)}$ such that $A \in \mathfrak{L}$. Suppose that $(A, \circ_q) \in \mathfrak{L}^{(q)}$ has identities $f_1 = 0, \ldots, f_s = 0$. We say that these identities are $\mathfrak{L}^{(q)}$-*minimal* if

- for any $r = 1, \ldots, s$, the identity $f_r = 0$ does not follow from the identities $f_1 = 0, \ldots, f_{r-1} = 0, f_{r+1} = 0, \ldots, f_s = 0$;
- if $\{f_1 = 0, \ldots, f_{r-1} = 0, g = 0, f_{r+1} = 0, \ldots, f_s = 0\} \Rightarrow f_r = 0$ and $g = 0$ is an identity for $\mathfrak{L}^{(q)}$ then $\{f_1 = 0, \ldots, f_{r-1} = 0, f_r = 0, f_{r+1} = 0, \ldots, f_s = 0\} \Rightarrow g = 0$.

Let $(f, g) \to f \cdot g = fg$ be the multiplication of the algebra $K\{t_1, t_2 \ldots\}$. Let us endow the algebra with the multiplication $(f, g) \mapsto f \cdot_q g$ given by $f \cdot_q g = f \cdot g + q\, g \cdot f$. For example,

$$(t_1 + 3t_1t_2) \cdot ((t_2t_3)t_1) = t_1((t_2t_3)t_1) + 3(t_1t_2)((t_2t_3)t_1),$$

$$\begin{aligned}(t_1 + 3t_1t_2) \cdot_q ((t_2t_3)t_1) &= t_1((t_2t_3)t_1) + 3(t_1t_2)((t_2t_3)t_1)\\ &\quad + q((t_2t_3)t_1)t_1 + 3q((t_2t_3)t_1)(t_1t_2).\end{aligned}$$

Let

$$\tau_q : K\{t_1, t_2, \ldots\} \to K\{t_1, t_2, \ldots\}$$

be a linear map defined by

$$\tau_q(t_i) = t_i,$$
$$\tau_q(f \cdot g) = \tau_q(f) \cdot \tau_q(g) + q\, \tau(g) \cdot \tau_q(f),$$

for any $f, g \in K\{t_1, t_2, \ldots\}$. Then

$$\tau_q : (K\{t_1, t_2, \ldots\}, \cdot) \to (K\{t_1, t_2, \ldots\}, \cdot_q)$$

is the homomorphism

$$\tau_q(f \cdot g) = \tau_q(f) \cdot_q \tau_q(g).$$

Given a bracketing type σ, we set

$$\sigma_q = \tau_q \sigma.$$

In other words, $\sigma_q(t_1, \ldots, t_k)$ is the polynomial obtained from $\sigma(t_1, \ldots, t_k)$ by the multiplication $\circ_q$. For example, if σ is the bracketing type $(t_1t_2)t_3$, then

$$\sigma_q(t_3, t_1, t_2) = (t_3t_1)t_2 + q((t_1t_3)t_2 + t_2(t_3t_1)) + q^2 t_2(t_1t_3).$$

Lemma 2.1. *For any bracketing type* σ

$$\sigma_{-q}\sigma_q(t_{i_1}, \ldots, t_{i_k}) = (1 - q^2)^{k-1}\sigma_0(t_{i_1}, \ldots, t_{i_k}).$$

Proof. We use induction on k. For $k=2$ the statement is true:

$$\sigma_q(t_{i_1},t_{i_2}) = t_{i_1}t_{i_2} + q\,t_{i_2}t_{i_1},$$

and

$$\begin{aligned}\sigma_{-q}\sigma_q(t_{i_1},t_{i_2}) &= t_{i_1}t_{i_2} - q\,t_{i_2}t_{i_1} + q\,t_{i_2}\cdot t_{i_1} - q^2 t_{i_1}t_{i_2}\\ &= (1-q^2)t_{i_1}t_{i_2} = (1-q^2)\sigma_0(t_{i_1},t_{i_2}).\end{aligned}$$

Suppose that our statement is true for $k-1$. Let

$$\sigma(t_{i_1},\dots,t_{i_k}) = \sigma'(t_{i_1},\dots,t_{i_{k'}})\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$

for some $1\le k'\le k$ and for some bracketings σ', σ''. Then

$$\begin{aligned}\sigma_q(t_{i_1},\dots,t_{i_k}) &= \sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ &\quad + q\,\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\end{aligned}$$

and

$$\begin{aligned}\sigma_{-q}\sigma_q(t_{i_1},\dots,t_{i_k}) &= \sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ &\quad - q\,\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\\ &\quad + q\,\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\\ &\quad - q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ &= \sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ &\quad - q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k}).\end{aligned}$$

By the induction hypothesis

$$\begin{aligned}\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}}) &= (1-q^2)^{k'-1}\sigma'_0(t_{i_1},\dots,t_{i_{k'}}),\\ \sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k}) &= (1-q^2)^{k-k'-1}\sigma''_0(t_{i_{k'+1}},\dots,t_{i_k}).\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k}) &= (1-q^2)^{k-2}\sigma_0(t_{i_1},\dots,t_{i_k}),\\ -q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k}) &= -q^2(1-q^2)^{k-2}\sigma_0(t_{i_1},\dots,t_{i_k})\end{aligned}$$

and

$$\begin{aligned}\sigma_{-q}\sigma_q(t_{i_1},\dots,t_{i_k}) &= \sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ &\quad - q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ &= (1-q^2)^{k-1}\sigma_0(t_{i_1},\dots,t_{i_k}).\end{aligned}$$

From Lemma 2.1 we infer the following

Theorem 2.2. $(q^2 \neq 1)$ *Let* $f_1, \ldots, f_s$ *be homogeneous polynomials of degree* k*. Then the class of* q*-algebras* $\mathrm{Var}(f_1, \ldots, f_s)^{(q)}$ *forms a variety defined by the system of polynomial identities* $\sigma_{-q} f_1 = 0, \ldots, \sigma_{-q} f_s = 0$*. This variety is equivalent to* $\mathrm{Var}(f_1, \ldots, f_s)$ *and the equivalence can be given by* $A = (A, \star) \mapsto A^{(-q)} = (A, \star_{-q})$.

The equivalence of varieties means the following. There exist functors

$$F : \mathrm{Var}(f_1, \ldots, f_s) \to \mathrm{Var}(\sigma_{-q} f_1, \ldots, \sigma_{-q} f_s), \qquad (A, \circ) \to (A, \circ_q),$$

$$G : \mathrm{Var}(\sigma_{-q} f_1, \ldots, \sigma_{-q} f_s) \to \mathrm{Var}(f_1, \ldots, f_s), \qquad (A, \star) \to (A, \star'_q)$$

such that

$$GF(A, \circ) = (A, \circ), \qquad FG(A, \star) = (A, \star).$$

Here

$$a \star'_q b = \frac{1}{(1 - q^2)^{k-1}}\, a \star_q b.$$

Recall that all polynomials $f_1, \ldots, f_s$ are supposed homogeneous. Notice that, for any $(A, \circ), (B, \cdot) \in \mathrm{Var}(f_1, \ldots, f_s)$ and a morphism between them, i.e., a homomorphism $\psi : (A, \circ) \to (B, \cdot)$, there corresponds a morphism of algebras $\psi : F(A, \circ) \to F(B, \cdot)$ in the category $\mathrm{Var}(\sigma_{-q} f_1, \ldots, \sigma_{-q} f_s)$, i.e., a homomorphism $\psi : (A, \circ_q) \to (B, \cdot_q)$. Indeed,

$$\begin{aligned}
\psi(a_1 \circ_q a_2) &= \psi(a_1 \circ a_2 + q\, a_2 \circ a_1) \\
&= \psi(a_1 \circ a_2) + q\, \psi(a_2 \circ a_1) \\
&= \psi(a_1) \cdot \psi(a_2) + q\, \psi(a_2) \cdot \psi(a_1) \\
&= \psi(a_1) \cdot_q \psi(a_2).
\end{aligned}$$

If I is an ideal of $(A, \circ)$ then I is an ideal of $(A, \circ_q)$. Therefore, simplicity, nilpotency and solvability properties of algebras in the category $\mathrm{Var}(f_1, \ldots, f_s)$ remain the same for the corresponding algebras in $\mathrm{Var}(\sigma_{-q} f_1, \ldots, \sigma_{-q} f_s)$. If $(A, \circ)$ is free in the variety $\mathrm{Var}(f_1, \ldots, f_s)$, then $(A, \circ_q)$ is free in the variety $\mathrm{Var}(\sigma_{-q} f_1, \ldots, \sigma_{-q} f_s)$. We pay attention to the fact that the categories $\mathrm{Var}(f_1, \ldots, f_s)$ and $\mathrm{Var}(\sigma_{-q} f_1, \ldots, \sigma_{-q} f_s)$ are equivalent only in the case of $q^2 \neq 1$.

Let $g_1, \ldots, g_s, h$ be non-commutative non-associative polynomials. Suppose that, for a class $\mathfrak{L}$ of algebras, the corresponding class $\mathfrak{L}^{(q)}$ of q-algebras satisfies the identities $g_1 = 0, \ldots, g_s = 0$ and $h = 0$. In this case we say that $h = 0$ is a *special* identity or an s-identity for $\mathrm{Var}(g_1, \ldots, g_s)$.

We give another application of Lemma 2.1.

Theorem 2.3. *If $q \neq \pm 1$, then the map*

$$\tau_q : (K\{t_1, t_2, \ldots\}, \cdot) \to (K\{t_1, t_2, \ldots\}, \cdot_q)$$

is an isomorphism.

Let $\mathfrak{L}$ be some class of algebras. For a polynomial $f \in K\{t_1, t_2, \ldots\}$, we say that $f = 0$ is an identity for $\mathfrak{L}$ if every algebra $A \in \mathfrak{L}$ satisfies the identity $f = 0$. Recall that the class of all algebras satisfying given polynomial identities forms a variety.

Recall that $\mathfrak{Lei}$ is the class of Leibniz algebras and $\mathfrak{Lei}^{(q)}$ is the class of q-Leibniz algebras, i.e., algebras of the form $A^{(q)} = (A, \circ_q)$, where $A \in \mathfrak{Lei}$.

Define non-commutative polynomials rjac (*right-Jacobian*), lalia (*left-anti-Lie-admissible*), ralia (*right-Anti-Lie-admissible*), lia (*Lie-admissible*), s^l_k (*standard left-skew-symmetric*), s^r_k (*standard right-skew-symmetric*) and $s^{[r]}_k$ (*s_k-Lie-admissible*) by

$$\begin{aligned}
\operatorname{rjac}(t_1, t_2, t_3) &= t_1(t_2t_3) + t_2(t_3t_1) + t_3(t_1t_2),\\
\operatorname{lalia}(t_1, t_2, t_3) &= [t_1, t_2]t_3 + [t_2, t_3]t_1 + [t_3, t_1]t_2,\\
\operatorname{ralia}(t_1, t_2, t_3) &= t_1[t_2, t_3] + t_2[t_3, t_1] + t_3[t_1, t_2],\\
\operatorname{lia}(t_1, t_2, t_3) &= [[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2],\\
\operatorname{alia}^{(q)} &= \operatorname{lalia} + q \cdot \operatorname{ralia}, \qquad q \in K.
\end{aligned}$$

Recall that for a non-commutative non-associative polynomial $f(t_1, \ldots, t_k)$, we denote by $\operatorname{Alt}(f)$ its skew-symmetrization

$$\operatorname{Alt}\, f(t_1, \ldots, t_k) = \sum_{\sigma \in \mathrm{Sym}_k} \operatorname{sign}\sigma\, f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).$$

Let

$$\begin{aligned}
s^r_k(t_1, \ldots, t_k) &= \operatorname{Alt}(t_1(t_2(\cdots(t_{k-1}t_k)))),\\
s^l_k(t_1, \ldots, t_k) &= \operatorname{Alt}((\cdots(t_1t_2)\cdots t_{k-1})t_k,\\
s^{[r]}_k(t_1, \ldots, t_k) &= \operatorname{Alt}([t_1, [t_2, \cdots, [t_{l-1}, t_k]]]).
\end{aligned}$$

Notice that

$$\operatorname{com} = s_2, \quad \operatorname{lalia} = s^l_3, \quad \operatorname{ralia} = s^r_3, \quad \operatorname{lia} = s^l_3 - s^r_3 = \operatorname{lalia} - \operatorname{ralia}.$$

If algebras are anti-commutative, i.e., satisfy the identity $\operatorname{acom} = 0$, then

$$\operatorname{ljac} = -\operatorname{rjac},$$

$$\operatorname{lia} = 4\operatorname{ljac}.$$

3. Right-center and Lie elements. Let $F = F(V)$ be a free right-Leibniz algebra generated by a space V. Let $(F^{\rm lie}, [\ ,\ ])$ be the subspace of F generated by V under the commutator $[\ ,\ ]$. We say that $a \in F$ is a *Lie-element* if $a \in (F^{\rm lie}, [\ ,\ ])$. Homomorphic images of Lie elements of any Leibniz algebras are called Lie elements as well.

Let $(A, \circ)$ be a right-Leibniz algebra. An element $z \in A$ is called *right-central* if

$$a \circ z = 0$$

for all $a \in A$. Let $A^{\rm rann}$ be the set of right-central elements of A. It was noticed in [7] that $A^{\rm rann}$ is an ideal with trivial left action, $a \circ z = 0, z \in A^{\rm rann}, a \in A$, such that

$$\{a, b\} = a \circ b + b \circ a \in A^{\rm rann}$$

for all $a, b \in A$. We construct new right-central elements.

Observe that

$$s^l_{k+1}(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^i s^l_k(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}) \circ a_i. \tag{1}$$

Lemma 3.1. *Let $(A, \circ)$ be a right-Leibniz algebra. Then $A^{\rm rann}$ is an ideal such that*

$$a \circ z = 0.$$

For any $q \in K$,

$$(a \circ_q b) \circ c = (a \circ c) \circ_q b + a \circ_q (b \circ c). \tag{2}$$

In particular,

$$\{a, b\} \circ c = \{a \circ c, b\} + \{a, b \circ c\}.$$

For any $k \geq 3$

$$s^l_k(a_1, \ldots, a_k), \quad s^r_k(a_1, \ldots, a_k) \in A^{\rm rann}.$$

Moreover,

$$s^l_k(a_1, \ldots, a_k) = s^{[r]}_k(a_1, \ldots, a_k)$$

are Lie elements,

$$s^r_k(a_1, \ldots, a_k) = 0, k \geq 4,$$

and

$$s^r_3(a, b, c) = 2 s^l_3(a, b, c).$$

In other words, any right-Leibniz algebra A is $-1/2$-Alia, i.e.,

$$\mathrm{alia}^{(-1/2)}(a,b,c)=0$$

for all $a,b,c\in A$.

Proof. We have

$$\begin{aligned}(a\circ_q b)\circ c &= (a\circ b+qb\circ a)\circ c\\ &= a\circ(b\circ c)+(a\circ c)\circ b+qb\circ(a\circ c)+q(b\circ c)\circ a\\ &= (a\circ c)\circ_q b+a\circ_q(b\circ c).\end{aligned}$$

So, (2) is established. Thus, in the case $q=0$ we obtain the right-Leibniz identity

$$(a\circ b)\circ c=(a\circ c)\circ b+a\circ(b\circ c).$$

Let $k=3$. Notice that

$$s_3^r(a,b,c)=\mathrm{ralia}(a,b,c).$$

We have

$$\begin{aligned}\mathrm{ralia}(a,b,c) &= a\circ[b,c]+b\circ[c,a]+c\circ[a,b]\\ &= 2(a\circ(b\circ c)+b\circ(c\circ a)+c\circ(a\circ b))\\ &\qquad -a\circ\{b,c\}-b\circ\{c,a\}-c\circ\{a,b\}\\ &= 2\,\mathrm{rjac}(a,b,c).\end{aligned}$$

By the right-Leibniz identity

$$\mathrm{ljac}(a,b,c)=[a,b]\circ c+[b,c]\circ a+[c,a]\circ b=\mathrm{lalia}(a,b,c),$$

and

$$\begin{aligned}\mathrm{lia}(a,b,c) &= \mathrm{lalia}(a,b,c)-\mathrm{ralia}(a,b,c)=\mathrm{rjac}(a,b,c)-2\ \mathrm{rjac}(a,b,c)\\ &= -\,\mathrm{rjac}(a,b,c).\end{aligned}$$

So, $s_3^r(a,b,c) = 2\,\mathrm{rjac}(a,b,c) = -2\,\mathrm{lia}(a,b,c)$ is a Lie-element.

By the right-Leibniz identity

$$\begin{aligned} u\circ \mathrm{rjac}(a,b,c) &= ((u\circ a)\circ(b\circ c) - ((u\circ(b\circ c))\circ a + ((u\circ b)\circ(c\circ a) \\ &\quad - ((u\circ(c\circ a))\circ b + ((u\circ c)\circ(a\circ b) - ((u\circ(a\circ b))\circ c \\ &= ((u\circ a)\circ b)\circ c - ((u\circ a)\circ c)\circ b - ((u\circ b)\circ c))\circ a \\ &\quad + ((u\circ c)\circ b)\circ a + ((u\circ b)\circ c)\circ a - ((u\circ b)\circ a)\circ c \\ &\quad - ((u\circ c)\circ a)\circ b + ((u\circ a)\circ c)\circ b + ((u\circ c)\circ a)\circ b \\ &\quad - ((u\circ c)\circ b)\circ a - ((u\circ a)\circ b)\circ c + ((u\circ b)\circ a)\circ c = 0. \end{aligned}$$

So, the element $s_3^l(a,b,c)$ is right-central.

Suppose that $s_k^l(a_1,\ldots,a_k) = s^{[r]}(a_1,\ldots,a_k)$ is a Lie element and is right-central. Prove that $s_{k+1}^l(a_1,\ldots,a_{k+1})$ is also a Lie element which is right-central. Since $s_k^l(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1}) \in A^{\mathrm{rann}}$ for every $i=1,\ldots,k+1$ and since A^{rann} is an ideal, we have

$$s_{k+1}^l(a_1,\ldots,a_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+k+1}s_k^l(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1})\circ a_i \in A^{\mathrm{rann}}.$$

Further,

$$s_{k+1}^{[r]}(a_1,\ldots,a_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}[a_i, s_k^{[r]}(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1})]$$

(by the induction hypothesis)

$$= \sum_{i=1}^{k+1}(-1)^{i+1}[a_i, s_k^l(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1})]$$

(since $s_k^l(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1}) \in A^{\mathrm{rann}}$)

$$= \sum_{i=1}^{k+1}(-1)^i s_k^l(a_1,\ldots,\hat{a}_i,\ldots,a_{k+1})\circ a_i = s_{k+1}^l(a_1,\ldots,a_{k+1}).$$

4. q-commutators of Leibniz algebras in case $q^2 \neq 1$.

Lemma 4.1. *For any Leibniz algebra A its q-algebra $A^{(q)}$ satisfies the identities* $\mathrm{lei}^{(q)} = 0$ *and* $\mathrm{lei}_1^{(q)} = 0$.

Proof. We have

$$\begin{aligned}
\mathrm{lei}^{(q)}(a,b,c) &= (q^2-1)\, a \circ_q (b \circ_q c) - a \circ_q (c \circ_q b) - (q^2-1)\, b \circ_q (a \circ_q c) \\
&\quad -qc \circ_q (a \circ_q b) + (q^2+q-1)\,(b \circ_q a) \circ_q c + (b \circ_q c) \circ_q a \\
&= (q^2-1)(a \circ (b \circ c) + (1+q) a \circ (c \circ b) - b \circ (a \circ c) - qb \circ (c \circ a) \\
&\quad +(q+q^2) c \circ (a \circ b) + qc \circ (b \circ a) + q(a \circ b) \circ c - q(a \circ c) \circ b \\
&\quad +(1-q)(b \circ a) \circ c + (q-1)(b \circ c) \circ a - q^2 (c \circ a) \circ b \\
&\quad +q^2 (c \circ b) \circ a) \\
&= (q^2-1)(a \circ (b \circ c) + (1+q) a \circ (c \circ b) - b \circ (a \circ c) + qb \circ (a \circ c) \\
&\quad +q^2 c \circ (a \circ b) + q(a \circ b) \circ c - q(a \circ c) \circ b + (1-q)(b \circ a) \circ c \\
&\quad +(q-1)(b \circ c) \circ a - q^2 (c \circ a) \circ b + q^2 (c \circ b) \circ a) \\
&= (q^2-1)(qa \circ (c \circ b) + (q-1) b \circ (a \circ c) + q^2 c \circ (a \circ b) \\
&\quad +q((a \circ b) \circ c - (a \circ c) \circ b) + (1-q)((b \circ a) \circ c - (b \circ c) \circ a) \\
&\quad -q^2((c \circ a) \circ b - (c \circ b) \circ a)) \\
&= (q^2-1)(q(a \circ (c \circ b) + (a \circ b) \circ c - (a \circ c) \circ b) \\
&\quad +(1-q)(-b \circ (a \circ c) + (b \circ a) \circ c - (b \circ c) \circ a) \\
&\quad -q^2(-c \circ (a \circ b) + (c \circ a) \circ b - (c \circ b) \circ a))
\end{aligned}$$

(by the right-Leibniz identity)

$$= 0.$$

Similarly,

$$\begin{aligned}
\mathrm{lei}_1^{(q)}(a,b,c) &= -a\circ_q(b\circ_q c) - a\circ_q(c\circ_q b) + q(b\circ_q c)\circ_q a + q(c\circ_q b)\circ_q a\\
&= -a\circ(b\circ c) - qa\circ(c\circ b) - q(b\circ c)\circ a - q^2(c\circ b)\circ a\\
&\quad -a\circ(c\circ b) - qa\circ(b\circ c) - q(c\circ b)\circ a - q^2(b\circ c)\circ a\\
&\quad +q(b\circ c)\circ a + q^2(c\circ b)\circ a + q^2 a\circ(b\circ c) + q^3 a\circ(c\circ b)\\
&\quad +q(c\circ b)\circ a + q^2(b\circ c)\circ a + q^2 a\circ(c\circ b) + q^3 a\circ(b\circ c)\\
&= -(a\circ b)\circ c + (a\circ c)\circ b - q(a\circ c)\circ b + q(a\circ b)\circ c\\
&\quad -q(b\circ c)\circ a - q^2(c\circ b)\circ a\\
&\quad -(a\circ c)\circ b + (a\circ b)\circ c - q(a\circ b)\circ c + q(a\circ c)\circ b\\
&\quad -q(c\circ b)\circ a - q^2(b\circ c)\circ a + q(b\circ c)\circ a + q^2(c\circ b)\circ a\\
&\quad +q^2(a\circ b)\circ c - q^2(a\circ c)\circ b + q^3(a\circ c)\circ b - q^3(a\circ b)\circ c\\
&\quad +q(c\circ b)\circ a + q^2(b\circ c)\circ a\\
&\quad +q^2(a\circ c)\circ b - q^2(a\circ b)\circ c + q^3(a\circ b)\circ c - q^3(a\circ c)\circ b\\
&= (-1+q+1-q+q^2-q^3-q^2+q^3)(a\circ b)\circ c\\
&\quad +(1-q-1+q-q^2+q^3+q^2-q^3)(a\circ c)\circ b\\
&\quad +(-q-q^2+q+q^2)(b\circ c)\circ a + (-q^2-q+q^2+q)(c\circ b)\circ a\\
&= 0.
\end{aligned}$$

Lemma 4.2. *If $q\neq -2$, then*

$$\mathrm{Alt}(\mathrm{lei}^{(q)}) = -(q+2)(q-1)\,\mathrm{alia}^{(\frac{-(2q+1)}{q+2})}\,.$$

If $q=-2$, then

$$\mathrm{Alt}(\mathrm{lei}^{(-2)}) = 9\,\mathrm{ralia}\,.$$

Proof. Consider the case $q\neq -2$. We have

$$\begin{aligned}
&\mathrm{lei}^{(q)}(t_1,t_2,t_3) + \mathrm{lei}^{(q)}(t_2,t_3,t_1) + \mathrm{lei}^{(q)}(t_3,t_1,t_2) - \mathrm{lei}^{(q)}(t_2,t_1,t_3) - \mathrm{lei}^{(q)}(t_3,t_2,t_1)\\
&\quad - \mathrm{lei}^{(q)}(t_1,t_3,t_2) = (q-1)\{(2q+1)(t_1[t_2,t_3] + t_2[t_3,t_1]] + t_3[t_1,t_2]])\\
&\quad - (q+2)([t_1,t_2]t_3 + [t_3,t_1]t_2 + [t_2,t_3]t_1\} = (2-q-q^2)\,\mathrm{ralia}^{\left(\frac{-(2q+1)}{q+2}\right)}\,.
\end{aligned}$$

The case $q=-2$ is considered in a similar manner. □

Lemma 4.3. *Let L be a free Leibniz algebra with 3 generators, $q \in K, q \neq 0, \pm 1$. Then any multilinear identity of $L^{(q)}$ of degree 3 follows from the identities $\mathrm{lei}^{(q)} = 0$ and $\mathrm{lei}_1^{(q)} = 0$. If $q \neq -2$ then $\mathrm{lei}_1^{(q)} = 0$ is a consequence of the identity $\mathrm{lei}^{(q)} = 0$. If $q = -2$, then $\mathrm{lei}^{(q)} = 0$ and $\mathrm{lei}_1^{(q)} = 0$ are independent identities.*

Proof. Let $L = (L, \circ)$ be a free Leibniz algebra generated by three elements a, b, c. Write the q-commutator in $L^{(q)}$ by $uv = u \circ v + qv \circ u$.

The multilinear part of the free magma algebra (the algebra of non-commutative non-associative polynomials) in degree 3 has dimension 12. It is generated by the following 12 monomials:

$$
\begin{aligned}
e_1 &= e_1(t_1, t_2, t_3) = t_1(t_2t_3), & e_2 &= e_2(t_1, t_2, t_3) = t_2(t_3t_1),\\
e_3 &= e_3(t_1, t_2, t_3) = t_3(t_1t_2), & e_4 &= e_4(t_1, t_2, t_3) = t_2(t_1t_3),\\
e_5 &= e_5(t_1, t_2, t_3) = t_3(t_2t_1), & e_6 &= e_6(t_1, t_2, t_3) = t_1(t_3t_2),\\
e_7 &= e_7(t_1, t_2, t_3) = (t_1t_2)t_3, & e_8 &= e_8(t_1, t_2, t_3) = (t_2t_3)t_1,\\
e_9 &= e_9(t_1, t_2, t_3) = (t_3t_1)t_2, & e_{10} &= e_{10}(t_1, t_2, t_3) = (t_2t_1)t_3,\\
e_{11} &= e_{11}(t_1, t_2, t_3) = (t_3t_2)t_1, & e_{12} &= e_{12}(t_1, t_2, t_3) = (t_1t_3)t_2.
\end{aligned}
$$

Let $X = X(t_1, t_2, t_3) = \sum_{i=1}^{12} \lambda_i e_i(t_1, t_2, t_3)$ be a polynomial such that $X(a, b, c) = 0$ is an identity on $L^{(q)}$.

Substitute the generator elements $a, b, c \in L$ for the parameters t_1, t_2, t_3. Write e_i instead of $e_i(a, b, c)$. We have

$$
\begin{aligned}
e_1 &= a \circ (b \circ c) + qa \circ (c \circ b) + q(b \circ c) \circ a + q^2(c \circ b) \circ a\\
&= (a \circ b) \circ c - (a \circ c) \circ b + q(a \circ c) \circ b - q(a \circ b) \circ c\\
&\quad + q(b \circ c) \circ a + q^2(c \circ b) \circ a.
\end{aligned}
$$

Similar calculations show that

$$
\begin{aligned}
e_2 &= (b \circ c) \circ a - (b \circ a) \circ c + q(b \circ a) \circ c - q(b \circ c) \circ a\\
&\quad + q(c \circ a) \circ b + q^2(a \circ c) \circ b,\\
e_3 &= (c \circ a) \circ b - (c \circ b) \circ a + q(c \circ b) \circ a - q(c \circ a) \circ b\\
&\quad + q(a \circ b) \circ c + q^2(b \circ a) \circ c,\\
e_4 &= (b \circ a) \circ c - (b \circ c) \circ a + q(b \circ c) \circ a - q(b \circ a) \circ c\\
&\quad + q(a \circ c) \circ b + q^2(c \circ a) \circ b,
\end{aligned}
$$

$$e_5 = (c \circ b) \circ a - (c \circ a) \circ b + q(c \circ a) \circ b - q(c \circ b) \circ a + q(b \circ a) \circ c + q^2 (a \circ b) \circ c,$$

$$e_6 = (a \circ c) \circ b - (a \circ b) \circ c + q(a \circ b) \circ c - q(a \circ c) \circ b + q(c \circ b) \circ a + q^2 (b \circ c) \circ a,$$

$$e_7 = (a \circ b) \circ c + q(b \circ a) \circ c + q(c \circ a) \circ b - q(c \circ b) \circ a + q^2 (c \circ b) \circ a - q^2 (c \circ a) \circ b,$$

$$e_8 = (b \circ c) \circ a + q(c \circ b) \circ a + q(a \circ b) \circ c - q(a \circ c) \circ b + q^2 (a \circ c) \circ b - q^2 (a \circ b) \circ c,$$

$$e_9 = (c \circ a) \circ b + q(a \circ c) \circ b + q(b \circ c) \circ a - q(b \circ a) \circ c + q^2 (b \circ a) \circ c - q^2 (b \circ c) \circ a,$$

$$e_{10} = (b \circ a) \circ c + q(a \circ b) \circ c + q(c \circ b) \circ a - q(c \circ a) \circ b + q^2 (c \circ a) \circ b - q^2 (c \circ b) \circ a,$$

$$e_{11} = (c \circ b) \circ a + q(b \circ c) \circ a + q(a \circ c) \circ b - q(a \circ b) \circ c + q^2 (a \circ b) \circ c - q^2 (a \circ c) \circ b,$$

$$e_{12} = (a \circ c) \circ b + q(c \circ a) \circ b + q(b \circ a) \circ c - q(b \circ c) \circ a + q^2 (b \circ c) \circ a - q^2 (b \circ a) \circ c.$$

So,

$$X =$$

$$(\lambda_1 - q\lambda_1 + q\lambda_3 + q^2 \lambda_5 - \lambda_6 + q\lambda_6 + \lambda_7 + q\lambda_8 - q^2\lambda_8 + q\lambda_{10} - q\lambda_{11} + q^2\lambda_{11})(a \circ b) \circ c$$
$$+(-\lambda_1 + q\lambda_1 + q^2\lambda_2 + q\lambda_4 + \lambda_6 - q\lambda_6 - q\lambda_8 + q^2 \lambda_8 + q\lambda_9 + q\lambda_{11} - q^2\lambda_{11} + \lambda_{12})(a \circ c) \circ b$$
$$+(-\lambda_2 + q\lambda_2 + q^2\lambda_3 + \lambda_4 - q\lambda_4 + q\lambda_5 + q\lambda_7 - q\lambda_9 + q^2\lambda_9 + \lambda_{10} + q\lambda_{12} - q^2\lambda_{12})(b \circ a) \circ c$$
$$+(q\lambda_1 + \lambda_2 - q\lambda_2 - \lambda_4 + q\lambda_4 + q^2\lambda_6 + \lambda_8 + q\lambda_9 - q^2\lambda_9 + q\lambda_{11} - q\lambda_{12} + q^2\lambda_{12})(b \circ c) \circ a$$
$$+(q\lambda_2 + \lambda_3 - q\lambda_3 + q^2\lambda_4 - \lambda_5 + q\lambda_5 + q\lambda_7 - q^2\lambda_7 + \lambda_9 - q\lambda_{10} + q^2 \lambda_{10} + q\lambda_{12})(c \circ a) \circ b$$
$$+(q^2\lambda_1 - \lambda_3 + q\lambda_3 + \lambda_5 - q\lambda_5 + q\lambda_6 - q\lambda_7 + q^2\lambda_7 + q\lambda_8 + q\lambda_{10} - q^2\lambda_{10} + \lambda_{11})(c \circ b) \circ a.$$

Thus we obtain the following system of equations

$$(1-q)\lambda_1 + q\lambda_3 + q^2\ \lambda_5 + (q-1)\lambda_6 + \lambda_7 + (q-q^2)\lambda_8 + q\lambda_{10} + (q^2-q)\lambda_{11} = 0,$$
$$(q-1)\lambda_1 + q^2\lambda_2 + q\lambda_4 + (1-q)\lambda_6 + (q^2-q)\lambda_8 + q\lambda_9 + (q-q^2)\lambda_{11} + \lambda_{12} = 0,$$
$$(q-1)\lambda_2 + q^2\lambda_3 + (1-q)\lambda_4 + q\lambda_5 + q\lambda_7 + (q^2-q)\lambda_9 + \lambda_{10} + (q-q^2)\lambda_{12} = 0,$$
$$q\lambda_1 + (1-q)\lambda_2 + (q-1)\lambda_4 + q^2\lambda_6 + \lambda_8 + (q-q^2)\lambda_9 + q\lambda_{11} + (q^2-q)\lambda_{12} = 0,$$
$$q\lambda_2 + (1-q)\lambda_3 + q^2\lambda_4 + (q-1)\lambda_5 + (q-q^2)\lambda_7 + \lambda_9 + (q^2-q)\lambda_{10} + q\lambda_{12} = 0,$$
$$q^2\lambda_1 + (q-1)\lambda_3 + (1-q)\lambda_5 + q\lambda_6 + (q^2-q)\lambda_7 + q\lambda_8 + (q-q^2)\lambda_{10} + \lambda_{11} = 0.$$

The 6×6-determinant composed of the first 6 rows is $(1-q)^5q^3(1+q)^3(q+2)$. So, this system has rank 6 if $q^2 \neq 1, q \neq 0, -2$. One can choose $\lambda_i, 7 \leq i \leq 12$, as free parameters. Now, we consider two cases.

Suppose that $q \neq -2$. In this case the system has the following solution

$$\lambda_1 = -\frac{-1+q+q^2}{(q+2)q}(\lambda_7 + \lambda_8 + \lambda_9 + (1-q-q^2)\lambda_{10} + (1+q)\lambda_{11} - \lambda_{12}),$$

$$\lambda_2 = -\frac{1}{(q+2)q}(\lambda_7 + (q^2+q-1)\lambda_8 + \lambda_9 - \lambda_{10} + (1-q-q^2)\lambda_{11} + (q+1)\lambda_{12}),$$

$$\lambda_3 = -\frac{1}{(q+2)q}(\lambda_7 + \lambda_8 + (q^2+q-1)\lambda_9 + (q+1)\lambda_{10} - \lambda_{11} - (q^2+q-1)\lambda_{12}),$$

$$\lambda_4 = -\frac{1}{(q+2)q}((1-q-q^2)\lambda_7 - \lambda_8 + (q+1)\lambda_9 + (q^2+q-1)\lambda_{10} + \lambda_{11} + \lambda_{12}),$$

$$\lambda_5 = -\frac{1}{(q+2)q}((1+q)\lambda_7 - (q^2+q-1)\ \lambda_8 - \lambda_9 + \lambda_{10} + (q^2+q-1)\lambda_{11} + \lambda_{12}),$$

$$\lambda_6 = -\frac{1}{(q+2)q}(-\lambda_7 + (q+1)\lambda_8 + (1-q-q^2)\lambda_9 + \lambda_{10} + \lambda_{11} + (q^2+q-1)\lambda_{12}).$$

Substitute these expressions for λ_i, $1 \leq i \leq 6$, in $X(t_1, t_2, t_3)$ and collect the coefficients of λ_j, $7 \leq j \leq 12$. We obtain a presentation of the polynomial $X(t_1, t_2, t_3)$ as a linear combination of the following 6 polynomials

$$\begin{aligned} f_1 = {} & (q-1)t_1(t_2t_3) - (q^3-q+1)t_1(t_3t_2) - (q-1)t_2(t_1t_3) \\ & - (q^2+q-1)t_2(t_3t_1) + (q^3-q)t_3(t_1t_2) + (q^3+q^2-q)(t_1t_3)t_2 + q(t_2t_3)t_1, \end{aligned}$$

$$f_2 = (-1+q^2)t_1(t_2t_3) - t_1(t_3t_2) - (q^2-1)t_2(t_1t_3) - qt_3(t_1t_2) + (q^2+q-1)(t_2t_1)t_3 + (t_2t_3)t_1,$$

$$f_3 = (-q^3+q-1)t_1(t_2t_3) - t_1(t_3t_2) + (q^3-q+1)t_2(t_1t_3) - (q^2+q-1)t_2(t_3t_1 - qt_3(t_1t_2) + (q^3+q^2-q)(t_1t_2)t_3 + (q^2+q)(t_2t_3)t_1,$$

$$f_4 = -t_1(t_2t_3) - (1+q)t_1(t_3t_2) + t_2(t_1t_3) - (q^2+q-1)t_2(t_3t_1) - t_3(t_1t_2) + (q^2+q-1)t_3(t_2t_1) + (q^2+2q)(t_2t_3)t_1,$$

$$f_5 = (1-q)t_1(t_2t_3) + (q^3-q+1)t_1(t_3t_2) - q^2t_2(t_1t_3) - (q^3-q)t_3(t_1t_2) - q(t_2t_3)t_1 + (q^3+q^2-q)(t_3t_1)t_2,$$

$$f_6 = -t_1(t_2t_3) - t_1(t_3t_2) + q(t_2t_3)t_1 + q((t_3t_2)t_1).$$

We see that if $q^2 \neq 1, q \neq -2$, then

$$f_1 = \frac{1}{(q-1)(q+1)(q+2)}(-\operatorname{lei}^{(q)}(t_1,t_2,t_3) - (-1+q+q^2)\operatorname{lei}^{(q)}(t_2,t_1,t_3) + (-1+q+q^2)^2\operatorname{lei}^{(q)}(t_3,t_1,t_2) + (-1+q+q^2)\operatorname{lei}^{(q)}(t_3,t_2,t_1)),$$

$$f_2 = \operatorname{lei}^{(q)},$$

$$f_3 = \frac{1}{(q-1)(q+1)(q+2)}(-(1+q)\operatorname{lei}^{(q)}(t_1,t_2,t_3) + (-1+q+q^2)^2\operatorname{lei}^{(q)}(t_2,t_1,t_3) - (-1+q+q^2)\operatorname{lei}^{(q)}(t_3,t_1,t_2) + (1+q)(-1+q+q^2)\operatorname{lei}^{(q)}(t_3,t_2,t_1)),$$

$$f_4 = \frac{1}{(q+1)(q-1)}(-\operatorname{lei}^{(q)}(t_1,t_2,t_3) + (-1+q+q^2)\operatorname{lei}^{(q)}(t_3,t_2,t_1)),$$

$$f_5 = \frac{1}{(q-1)(q+1)(q+2)}(\operatorname{lei}^{(q)}(t_1,t_2,t_3) + (-1+q+q^2)^2\ \operatorname{lei}^{(q)}(t_1,t_3,t_2) - (-1+q+q^2)\operatorname{lei}^{(q)}(t_2,t_3,t_1) - (-1+q+q^2)\operatorname{lei}^{(q)}(t_3,t_2,t_1)),$$

$$f_6 = \frac{1}{(q-1)(q+1)(q+2)}(-\operatorname{lei}^{(q)}(t_1,t_2,t_3) - \operatorname{lei}^{(q)}(t_1,t_3,t_2) + (-1+q+q^2)\operatorname{lei}^{(q)}(t_2,t_3,t_1) + (-1+q+q^2)\operatorname{lei}^{(q)}(t_3,t_2,t_1)).$$

Now, we consider the case $q = -2$. In this case, similar arguments show that X is a linear combination of the following polynomials

$$\begin{aligned} g_1 &= t_3(t_2t_1) + 2/3(t_1t_2)t_3 + 4/3(t_1t_3)t_2 + 4/3(t_2t_1)t_3 - 4/3(t_2t_3)t_1 \\ &\quad +5/3(t_3t_1)t_2 - 5/3(t_3t_2)t_1, \\ g_2 &= t_2(t_3t_1) + 4/3(t_1t_2)t_3 + 2/3(t_1t_3)t_2 + 5/3(t_2t_1)t_3 - 5/3(t_2t_3)t_1 \\ &\quad +4/3(t_3t_1)t_2 - 4/3(t_3t_2)t_1, \\ g_3 &= t_1(t_2t_3) - 5/3(t_1t_2)t_3 + 5/3(t_1t_3)t_2 - 4/3(t_2t_1)t_3 + 4/3(t_2t_3)t_1 \\ &\quad +4/3(t_3t_1)t_2 + 2/3(t_3t_2)t_1, \\ g_4 &= t_1(t_3t_2) + 5/3(t_1t_2)t_3 - 5/3(t_1t_3)t_2 + 4/3(t_2t_1)t_3 + 2/3(t_2t_3)t_1 \\ &\quad -4/3(t_3t_1)t_2 + 4/3(t_3t_2)t_1, \\ g_5 &= t_2(t_1t_3) - 4/3(t_1t_2)t_3 + 4/3(t_1t_3)t_2 - 5/3(t_2t_1)t_3 + 5/3(t_2t_3)t_1 \\ &\quad +2/3(t_3t_1)t_2 + 4/3(t_3t_2)t_1, \\ g_6 &= t_3(t_1t_2 + 4/3(t_1t_2)t_3 - 4/3(t_1t_3)t_2 + 2/3(t_2t_1)t_3 + 4/3(t_2t_3)t_1 \\ &\quad -5/3(t_3t_1)t_2 + 5/3(t_3t_2)t_1. \end{aligned}$$

We have

$$\begin{aligned} &g_1 - 1/3(4\,\mathrm{lei}^{(-2)}(t_1,t_2,t_3) + 3\,\mathrm{lei}^{(-2)}(t_1,t_3,t_2) + 2\,\mathrm{lei}^{(-2)}(t_2,t_1,t_3) \\ &\quad + 4\,\mathrm{lei}_1^{(-2)}(t_1,t_2,t_3) - \mathrm{lei}_1^{(-2)}(t_2,t_1,t_3)) = 1/3\,\mathrm{ralia}(t_1,t_2,t_3), \\ &g_2 - 1/3(5\,\mathrm{lei}^{(-2)}(t_1,t_2,t_3) + 6\,\mathrm{lei}^{(-2)}(t_1,t_3,t_2) + 4\,\mathrm{lei}^{(-2)}(t_2,t_1,t_3) \\ &\quad + 5\,\mathrm{lei}_1^{(-2)}(t_1,t_2,t_3) + \mathrm{lei}_1^{(-2)}(t_2,t_1,t_3)) = 8/3\ \mathrm{ralia}(t_1,t_2,t_3), \\ &g_3 - 1/3(-4\,\mathrm{lei}^{(-2)}(t_1,t_2,t_3) - 6\,\mathrm{lei}^{(-2)}(t_1,t_3,t_2) - 5\,\mathrm{lei}^{(-2)}(t_2,t_1,t_3) \\ &\quad - 4\,\mathrm{lei}_1^{(-2)}(t_1,t_2,t_3) - 5\,\mathrm{lei}_1^{(-2)}(t_2,t_1,t_3)) = -10/3\ \mathrm{ralia}(t_1,t_2,t_3), \\ &g_4 - 1/3(4\,\mathrm{lei}^{(-2)}(t_1,t_2,t_3) + 6\,\mathrm{lei}^{(-2)}(t_1,t_3,t_2) + 5\,\mathrm{lei}^{(-2)}(t_2,t_1,t_3) \\ &\quad + \mathrm{lei}_1^{(-2)}(t_1,t_2,t_3) + 5\,\mathrm{lei}_1^{(-2)}(t_2,t_1,t_3)) = 10/3\ \mathrm{ralia}(t_1,t_2,t_3), \\ &g_5 - 1/3(-5\,\mathrm{lei}^{(-2)}(t_1,t_2,t_3) - 6\,\mathrm{lei}^{(-2)}(t_1,t_3,t_2) - 4\,\mathrm{lei}^{(-2)}(t_2,t_1,t_3) \\ &\quad - 5\,\mathrm{lei}_1^{(-2)}(t_1,t_2,t_3) - 4\,\mathrm{lei}_1^{(-2)}(t_2,t_1,t_3)) = -8/3\ \mathrm{ralia}(t_1,t_2,t_3), \end{aligned}$$

$$g_6 - 1/3(2\,\mathrm{lei}^{(-2)}(t_1,t_2,t_3) + 3\,\mathrm{lei}^{(-2)}(t_1,t_3,t_2) + 4\,\mathrm{lei}^{(-2)}(t_2,t_1,t_3) \\ - \mathrm{lei}_1^{(-2)}(t_1,t_2,t_3) + 4\,\mathrm{lei}_1^{(-2)}(t_2,t_1,t_3)) = 8/3\ \mathrm{ralia}(t_1,t_2,t_3).$$

By Lemma 4.2 ralia $= 0$ is a consequence of the identity $\mathrm{lei}^{(q)} = 0$. Therefore, all the identities $g_i = 0$ are consequences of the identities $\mathrm{lei}^{(q)} = 0$ and $lei_1^{(q)} = 0$.

We have proved that any identity of degree 3 of $L^{(q)}$ for $q = -2$ follows from the identities $\mathrm{lei}^{(q)} = 0$ and $\mathrm{lei}_1^{(q)} = 0$. Notice that the equation

$$\mathrm{lei}_1^{(q)}(a,b,c) = \mu_1\,\mathrm{lei}^{(q)}(a,b,c) + \mu_2\,\mathrm{lei}^{(q)}(b,c,a) + \mu_3\,\mathrm{lei}^{(q)}(c,a,b) \\ + \mu_4\,\mathrm{lei}^{(q)}(b,a,c) + \mu_5\,\mathrm{lei}^{(q)}(c,b,a) + \mu_6\,\mathrm{lei}^{(q)}(a,c,b)$$

in $L^{(q)}$ with unknowns $\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6$ is not solvable. Therefore, this system of identities $\mathrm{lei}^{(q)} = 0, \mathrm{lei}_1^{(q)} = 0$ is $\mathfrak{Lei}^{(q)}$-minimal if $q = -2$.

Lemma 4.4. *Suppose that $q \neq 0, \pm 1$ and an algebra $(A,\star)$ satisfies the identities $\mathrm{lei}^{(q)} = 0$ and $\mathrm{lei}_1^{(q)} = 0$. Then the algebra $(A,\circ)$, where $a \circ b = (1-q^2)^{-1}(a \star b - q\, b \star a)$, is a (right)-Leibniz algebra, and the algebras $(A,\star)$ and $(A,\circ_q)$ are isomorphic.*

Proof. One checks that

$$\mathrm{lei}(t_1,t_2,t_3) = -2\ \mathrm{lei}^{(q)}(t_1,t_2,t_3) - 2/3\ \mathrm{lei}^{(q)}(t_1,t_3,t_2) - \mathrm{lei}^{(q)}(t_2,t_1,t_3) \\ +2/3\,(\mathrm{lei}^{(q)}(t_2,t_3,t_1) - 2\,\mathrm{lei}_1^{(q)}(t_1,t_2,t_3) - \mathrm{lei}_1^{(q)}(t_2,t_1,t_3)$$

for $q^2 \neq 1, q = -2$, and

$$\mathrm{lei}(t_1,t_2,t_3) = \frac{1}{(q^2-1)(q+2)}(q(q+1)\,\mathrm{lei}^{(q)}(t_1,t_2,t_3) \\ -(-1+2q+q^2)\,\mathrm{lei}^{(q)}(t_1,t_3,t_2) - (q+1)\ \mathrm{lei}^{(q)}(t_2,t_1,t_3) \\ +(1-q+q^2+q^3)\,\mathrm{lei}^{(q)}(t_2,t_3,t_1) + (q+1)\,\mathrm{lei}^{(q)}(t_3,t_1,t_2) \\ -(q+q^2)\,\mathrm{lei}^{(q)}(t_3,t_2,t_1))$$

for $q^2 \neq 1, q \neq -2$.

Therefore, for any algebra $(A,\star)$ with identities $\mathrm{lei}^{(q)} = 0$ and $\mathrm{lei}_1^{(q)} = 0$ the algebra $(A,\circ)$, where $a \circ b = (1-q^2)^{-1}(a \star b - q\, b \star a)$, satisfies the identity lei $= 0$. It is evident that

$$a \circ_q b = (1-q^2)^{-1}(a \circ b + q\, b \circ a) \\ = (1-q^2)^{-1}(a \star b - q\, b \star a + q\, b \star a - q^2\, a \star b) \\ = a \star b.$$

Proof of Theorems 1.1 and 1.2. By Lemmas 4.1, 4.3, 4.4 our theorems are true. □

5. Leibniz-Lie algebras. In this section we study identities for Leibniz-Lie algebras, i.e., algebras $(A, [\ ,\ ])$ under -1-commutator for Leibniz algebras $(A, \circ)$.

Note that $\mathrm{leilie}_1(t_1, t_2, t_3, t_4, t_5)$ has type $(3, 2)$, i.e., it is skew-symmetric in t_1, t_2, t_3 and in t_4, t_5, and $\mathrm{leilie}_2(t_1, t_2, t_3, t_4, t_5)$ has type $(1, 4)$, is skew-symmetric in t_2, t_3, t_4, t_5.

Let

$$
\begin{aligned}
&\mathrm{lei}_3^{(-1)}(t_1, t_2, t_3, t_4, t_5) = \\
&((t_1t_2)t_3)(t_4t_5) + ((t_1t_2)t_5)(t_3t_4) + ((t_1t_2)(t_3t_4))t_5 - 2((t_1t_2)(t_3t_5))t_4 \\
&+ ((t_1t_2)(t_4t_5))t_3 - ((t_1t_3)t_5)(t_2t_4) + ((t_1t_3)(t_2t_4))t_5 - ((t_1t_4)t_3)(t_2t_5) \\
&+ ((t_1t_4)t_5)(t_2t_3) + ((t_1t_4)(t_2t_3))t_5 - ((t_1t_4)(t_2t_5))t_3 + 2((t_1t_4)(t_3t_5))t_2 \\
&+ ((t_1t_5)t_3)(t_2t_4) - ((t_1t_5)(t_2t_4))t_3 + ((t_2t_3)t_5)(t_1t_4) + ((t_2t_4)t_3)(t_1t_5) \\
&- ((t_2t_4)t_5)(t_1t_3) - 2((t_2t_4)(t_3t_5))t_1 - ((t_2t_5)t_3)(t_1t_4) + ((t_3t_5)t_4)(t_1t_2) \\
&+ 2(((t_1t_2)t_3)t_5)t_4 - 6(((t_1t_2)t_4)t_3)t_5 + 6(((t_1t_2)t_4)t_5)t_3 - 2(((t_1t_2)t_5)t_3)t_4 \\
&- 2(((t_1t_3)t_2)t_4)t_5 + 2(((t_1t_3)t_4)t_2)t_5 + 6(((t_1t_3)t_5)t_2)t_4 - 6(((t_1t_3)t_5)t_4)t_2 \\
&+ 6(((t_1t_4)t_2)t_3)t_5 - 6(((t_1t_4)t_2)t_5)t_3 - 2(((t_1t_4)t_3)t_5)t_2 + 2(((t_1t_4)t_5)t_3)t_2 \\
&+ 2(((t_1t_5)t_2)t_4)t_3 - 6(((t_1t_5)t_3)t_2)t_4 + 6(((t_1t_5)t_3)t_4)t_2 - 2(((t_1t_5)t_4)t_2)t_3 \\
&+ 2(((t_2t_3)t_1)t_4)t_5 - 2(((t_2t_3)t_4)t_1)t_5 - 6(((t_2t_3)t_5)t_1)t_4 + 6(((t_2t_3)t_5)t_4)t_1 \\
&- 6(((t_2t_4)t_1)t_3)t_5 + 6(((t_2t_4)t_1)t_5)t_3 + 2(((t_2t_4)t_3)t_5)t_1 - 2(((t_2t_4)t_5)t_3)t_1 \\
&- 2(((t_2t_5)t_1)t_4)t_3 + 6(((t_2t_5)t_3)t_1)t_4 - 6(((t_2t_5)t_3)t_4)t_1 + 2(((t_2t_5)t_4)t_1)t_3 \\
&+ 2(((t_3t_4)t_1)t_2)t_5 - 2(((t_3t_4)t_2)t_1)t_5 - 4(((t_3t_4)t_5)t_1)t_2 + 4(((t_3t_4)t_5)t_2)t_1 \\
&+ 4(((t_3t_5)t_1)t_2)t_4 - 4(((t_3t_5)t_1)t_4)t_2 - 4(((t_3t_5)t_2)t_1)t_4 + 4(((t_3t_5)t_2)t_4)t_1 \\
&+ 2(((t_3t_5)t_4)t_1)t_2 - 2(((t_3t_5)t_4)t_2)t_1 + 2(((t_4t_5)t_1)t_2)t_3 - 2(((t_4t_5)t_2)t_1)t_3 \\
&- 4(((t_4t_5)t_3)t_1)t_2 + 4(((t_4t_5)t_3)t_2)t_1.
\end{aligned}
$$

Statements below need long calculations. We omit them. Details one can find in our preprint [5].

Lemma 5.1. *The identity* $\mathrm{lei}_3^{(-1)} = 0$ *is a consequence of the identities* $\mathrm{leilie}_1 = 0$, $\mathrm{leilie}_2 = 0$ *and the anti-commutativity identity.*

Lemma 5.2. *Let* $(A, \circ)$ *be a Leibniz algebra. Then the Leibniz-Lie algebra* $(A, [\ ,\ ])$ *satisfies the identities* $\mathrm{lei}_1^{(-1)} = 0$, $\mathrm{leilie}_2 = 0$.

Lemma 5.3. *Any identity of degree* 4 *for* $\mathfrak{Lei}^{(-1)}$ *follows from the identity* $\mathrm{acom} = 0$.

Proof. Let, working modulo the identity of anti-commutativity

$$X_4(t_1, t_2, t_3, t_4) =$$
$$\lambda_1(t_1t_2)(t_3t_4) + \lambda_2(t_1t_3)(t_2t_4) + \lambda_3(t_2t_3)(t_1t_4) + \lambda_4((t_1t_2)t_3)t_4$$
$$+ \lambda_{10}((t_1t_2)t_4)t_3 + \lambda_5((t_1t_3)t_2)t_4 + \lambda_{11}((t_1t_3)t_4)t_2 + \lambda_6((t_1t_4)t_2)t_3$$
$$+ \lambda_{12}((t_1t_4)t_3)t_2 + \lambda_7((t_2t_3)t_1)t_4 + \lambda_{13}((t_2t_3)t_4)t_1 + \lambda_8((t_2t_4)t_1)t_3$$
$$+ \lambda_{14}((t_2t_4)t_3)t_1 + \lambda_9((t_3t_4)t_1)t_2 + \lambda_{15}((t_3t_4)t_2)t_1$$

be a generic multilinear polynomial of degree 4. For t_1, t_2, t_3, t_4, we substitute the elements a, b, c, d of the free Leibniz algebra, and calculate $X_4(a, b, c, d)$ under the commutator $[u, v] = u \circ v - v \circ u$. We obtain

$$X_4(a, b, c, d) =$$
$$(2\lambda_1 + \lambda_4 - 2\lambda_7 - 4\lambda_{13} + 4\lambda_{15})(((a \circ b) \circ c) \circ d)$$
$$+ (-2\lambda_1 - 2\lambda_8 + \lambda_{10} - 4\lambda_{14} - 4\lambda_{15})(((a \circ b) \circ d) \circ c)$$
$$+ (2\lambda_2 + \lambda_5 + 2\lambda_7 + 4\lambda_{13} + 4\lambda_{14})(((a \circ c) \circ b) \circ d)$$
$$+ (-2\lambda_2 - 2\lambda_9 + \lambda_{11} - 4\lambda_{14} - 4\lambda_{15})(((a \circ c) \circ d) \circ b)$$
$$+ (-2\lambda_3 + \lambda_6 + 2\lambda_8 + 4\lambda_{13} + 4\lambda_{14})(((a \circ d) \circ b) \circ c)$$
$$+ (2\lambda_3 + 2\lambda_9 + \lambda_{12} - 4\lambda_{13} + 4\lambda_{15})(((a \circ d) \circ c) \circ b)$$
$$+ (-2\lambda_1 - \lambda_4 - 2\lambda_5 + 4\lambda_9 - 4\lambda_{11})(((b \circ a) \circ c) \circ d)$$
$$+ (2\lambda_1 - 2\lambda_6 - 4\lambda_9 - \lambda_{10} - 4\lambda_{12})(((b \circ a) \circ d) \circ c)$$
$$+ (2\lambda_3 + 2\lambda_5 + \lambda_7 + 4\lambda_{11} + 4\lambda_{12})(((b \circ c) \circ a) \circ d)$$
$$+ (-2\lambda_3 - 4\lambda_9 - 4\lambda_{12} + \lambda_{13} - 2\lambda_{15})(((b \circ c) \circ d) \circ a)$$
$$+ (-2\lambda_2 + 2\lambda_6 + \lambda_8 + 4\lambda_{11} + 4\lambda_{12})(((b \circ d) \circ a) \circ c)$$

$$+ (2\lambda_2 + 4\lambda_9 - 4\lambda_{11} + \lambda_{14} + 2\lambda_{15})(((b \circ d) \circ c) \circ a)$$
$$+ (-2\lambda_2 - 2\lambda_4 - \lambda_5 + 4\lambda_8 - 4\lambda_{10})(((c \circ a) \circ b) \circ d)$$
$$+ (2\lambda_2 - 4\lambda_6 - 4\lambda_8 - \lambda_{11} - 2\lambda_{12})(((c \circ a) \circ d) \circ b)$$
$$+ (-2\lambda_3 + 2\lambda_4 + 4\lambda_6 - \lambda_7 + 4\lambda_{10})(((c \circ b) \circ a) \circ d)$$
$$+ (2\lambda_3 - 4\lambda_6 - 4\lambda_8 - \lambda_{13} - 2\lambda_{14})(((c \circ b) \circ d) \circ a)$$
$$+ (-2\lambda_1 + 4\lambda_6 + \lambda_9 + 4\lambda_{10} + 2\lambda_{12})(((c \circ d) \circ a) \circ b)$$
$$+ (2\lambda_1 + 4\lambda_8 - 4\lambda_{10} + 2\lambda_{14} + \lambda_{15})(((c \circ d) \circ b) \circ a)$$
$$+ (2\lambda_3 - 4\lambda_4 - \lambda_6 + 4\lambda_7 - 2\lambda_{10})(((d \circ a) \circ b) \circ c)$$
$$+ (-2\lambda_3 - 4\lambda_5 - 4\lambda_7 - 2\lambda_{11} - \lambda_{12})(((d \circ a) \circ c) \circ b)$$
$$+ (2\lambda_2 + 4\lambda_4 + 4\lambda_5 - \lambda_8 + 2\lambda_{10})(((d \circ b) \circ a) \circ c)$$
$$+ (-2\lambda_2 - 4\lambda_5 - 4\lambda_7 - 2\lambda_{13} - \lambda_{14})(((d \circ b) \circ c) \circ a)$$
$$+ (2\lambda_1 + 4\lambda_4 + 4\lambda_5 - \lambda_9 + 2\lambda_{11})(((d \circ c) \circ a) \circ b)$$
$$+ (-2\lambda_1 - 4\lambda_4 + 4\lambda_7 + 2\lambda_{13} - \lambda_{15})(((d \circ c) \circ b) \circ a).$$

Since all 24 left-bracketed elements like $((a \circ b) \circ c) \circ d$ are linear independent elements, the condition $X_4(a, b, c, d) = 0$ gives us the system of 24 linear equations in 15 unknowns $\lambda_i, i = 1, \ldots, 15$. We see that the rank of this system is 15 and our system has the trivial solution only: $\lambda_i = 0$ for all $i = 1, 2, \ldots, 15$. In other words, any multilinear identity of degree 4 for $\mathfrak{Lei}^{(-1)}$ follows from the identity acom $= 0$.

Lemma 5.4. *Any identity of degree* 5 *for the free Leibniz algebra follows from the identities* $\mathrm{leilie}_1 = 0$*,* $\mathrm{leilie}_2 = 0$*,* $\mathrm{lei}_3^{(-1)} = 0$.

P r o o f. Let $f = f(t_1, \ldots, t_5)$ be a non-commutative non-associative polynomial such that $f = 0$ is an identity for any right-Leibniz algebra. Notice that there exist 105 anti-commutative non-associative polynomials. Present f as a linear combination of these 105 elements.

Insert in f the elements of the free Leibniz algebra generated by 5 elements u_1, u_2, u_3, u_4, u_5 and calculate the polynomial f under the commutator $[u, v] = u \circ v - v \circ u$, where $(u, v) \mapsto u \circ v$ is the multiplication in a free (right)-Leibniz algebra. Expand this expression in terms of the multiplication $\circ$ using the Leibniz rule

$$u \circ (v \circ w) = (u \circ v) \circ w - (u \circ w) \circ v.$$

We obtain an element which is a linear combination of 120 elements of the form $(((u_{\sigma(1)} \circ u_{\sigma(2)}) \circ u_{\sigma(3)}) \circ u_{\sigma(4)}) \circ u_{\sigma(5)}$, where $\sigma \in \mathrm{Sym}_5$. The identity condition $f = 0$ on $L^{(-1)}$ gives us 120 linear equations in 105 unknowns λ_i. Solve this system of equations. We do this using the computer system `Mathematica`. We find out that the system has 14 free parameters. It shows that f can be presented as a linear combination of the 14 polynomials given below

$$
\begin{aligned}
f_1 &= \mathrm{leilie}_1,\\
f_2 &= \mathrm{leilie}_1(t_1,t_2,t_4,t_3,t_5),\\
f_3 &= \mathrm{leilie}_1(t_1,t_2,t_5,t_3,t_4),\\
f_4 &= \mathrm{leilie}_1(t_1,t_3,t_4,t_2,t_5),\\
f_5 &= \mathrm{leilie}_1(t_2,t_3,t_4,t_1,t_5),\\
f_6 &= \mathrm{leilie}_1(t_1,t_3,t_5,t_2,t_4),\\
f_7 &= \mathrm{leilie}_1(t_2,t_3,t_5,t_1,t_4),\\
f_9 &= \mathrm{leilie}_1(t_1,t_4,t_5,t_2,t_3) + \mathrm{leilie}_2(t_1,t_2,t_3,t_4,t_5),\\
f_{10} &= (\mathrm{leilie}_1(t_3,t_4,t_5,t_1,t_2) + 2\,\mathrm{leilie}_2(t_1,t_2,t_3,t_4,t_5) + \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)\\
&\quad - \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_5,t_4) - \mathrm{lei}_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2,\\
f_{11} &= (2\,\mathrm{leilie}_1(t_2,t_4,t_5,t_1,t_3) + \mathrm{leilie}_1(t_3,t_4,t_5,t_1,t_2) + 2\,\mathrm{leilie}_2(t_1,t_2,t_3,t_4,t_5)\\
&\quad + \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_4,t_5) - \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_5,t_4)\\
&\quad - \mathrm{lei}_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2,\\
f_{12} &= (\mathrm{leilie}_1(t_3,t_4,t_5,t_1,t_2) + \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)\\
&\quad + \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_5,t_4) - \mathrm{lei}_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2,\\
f_{14} &= (\mathrm{leilie}_1(t_3,t_4,t_5,t_1,t_2) + \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)\\
&\quad - \mathrm{lei}_3^{(-1)}(t_1,t_2,t_3,t_5,t_4) + \mathrm{lei}_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2.
\end{aligned}
$$

So, by Lemma 5.1 the 9-term polynomial leilie_1 and the 60-term polynomial leilie_2 form a base of multilinear identities of degree 5.

P r o o f   o f   T h e o r e m 1.3. Follows from Lemmas 5.1, 5.2, 5.3 and 5.4. □

6. Leibniz-Jordan algebras.

P r o o f   o f   T h e o r e m 1.6. It is easy to check that leijor = 0 is an identity for any algebra of the form $A^{(1)}$, where A is a Leibniz algebra.

Let A be an associative algebra and let M be a right module over A. Then $A^{(-1)}$ is a Lie algebra and M can be made into an antisymmetric $A^{(-1)}$-module. Let $L = A + M$ be the standard Lebniz algebra corresponding to these Lie and antisymmetric module structures. If we denote by $\star$ the multiplication in the Leibniz algebra L, then

$$(a+m)\star(b+n) = [a,b] + mb,$$

and

$$\{a+m, b+n\} = [a,b] + mb + [b,a] + na = na + mb.$$

In particular,

$$\{a,m\} = ma, \qquad \{a,b\} = 0, \qquad \{m,n\} = 0 \tag{3}$$

for all $a, b \in A, m, n \in M$. Recall that

$$\{t_1, t_2\} = t_1t_2 + t_2t_1$$

is the Jordan commutator.

Suppose that $f = 0$ is a minimal identity for the Leibniz-Jodan algebra $(L, \{\ ,\ \})$ which does not follow from the identity leijor $= 0$. We can assume that f is multilinear and $f = f(t_1, \dots, t_k)$ is a linear combination of left-bracketed monomials of the form $((t_{i_1}t_{i_2})\cdots)t_{i_k}$. So,

$$f(t_1, \dots, t_k) = \sum_{\sigma\in \mathrm{Sym}_k} \lambda_\sigma((t_{\sigma(1)}t_{\sigma(2)})\cdots)t_{\sigma(k)}$$

for some $\lambda_\sigma \in K$. Write the condition $f(a_1, \dots, a_{k-1}, m) = 0$ by using the multiplication rules (3) for Leibniz-Jordan algebras. We have

$$f(a_1, \dots, a_{k-1}, m) = \sum_{\sigma\in \mathrm{Sym}_{k-1}} \lambda_\sigma((ma_{\sigma(1)})\cdots)a_{\sigma(k-1)} \tag{4}$$

for any $a_1, \dots, a_{k-1} \in A, m \in M$.

Take $A = \mathrm{Mat}_n$ to be the matrix algebra and $M = K^n$ the n-dimensional natural module. Then conditions (4) imply that

$$\sum_{\sigma\in \mathrm{Sym}_{k-1}} \lambda_\sigma((a_{\sigma(1)}a_{\sigma(2)})\cdots)a_{\sigma(k-1)} = 0$$

is an identity for Mat_n. By the Amitsur-Levitsky Theorem [1], matrix algebras have no identity of degree $k-1$ if $k < 2n+1$. So, $f = 0$ is not an identity for Leibniz-Jordan algebras of the form $\mathrm{Mat}_n + K^n$ if $n > (k-1)/2$. In other words, any s-identity for Leibniz-Jordan algebras follows from the identities leijor $= 0$, com $= 0$.

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