Journal of Nonlinear Mathematical Physics Volume 15, Number 1 (2008), 87–103 Article

10-Commutators, 13-commutators and odd derivations

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Received August 24, 2007; Accepted in Revised Form October 15, 2007

Abstract

The anti-symmetric sum $s_N(X_1, \dots, X_N)$ of $N!$ compositions of N vector fields $X_1, \dots, X_N \in Vect(n)$ in all possible order is said to be a N-*commutator* if $s_N(X_1, \dots, X_N) \in Vect(n)$ for any $X_1, \dots, X_N \in Vect(n)$ and does not vanish for some vector fields $X_1, \dots, X_N$. We construct 10- and 13-commutators on $Vect(3)$ and 10-commutator on the space of divergence-free vector fields $Vect_0(3)$. We show that there are no other N-commutators on $Vect(3)$ except for 2-, 10- and 13-commutators, and no other N-commutators on the Lie algebra of divergence-free vector fields $Vect_0(3)$ except for 2-, 10-commutators. These constructions are based on calculation of powers of odd derivations.

1 Introduction

If D is any odd derivation of a superalgebra, then D^2 is also a derivation. In our paper we investigate whether D^N might also be a derivation for $N > 2$. Super-derivations, in particular, odd nilpotent derivations appear on many occasions in modern mathematics and physics. They appear in BRST cohomology theory, in deformation theory and in quantum groups. They appear also in the study of Lie commutators of vector fields. The Jacobi identity can be formulated as the statement that certain odd derivations are nilpotent. Lie commutators and identities on the space of vector fields play important role in mathematical physics.

Let $X_1, \dots, X_N \in Vect(n)$ be vector fields on a smooth manifold M of dimension n. Their N–commutator is, by definition, the expression

$$[X_1, \dots, X_N] := s_N(X_1, \dots, X_N) := \sum_{\sigma \in Sym_N} sign(\sigma)\, X_{\sigma(1)} \dots X_{\sigma(N)}.$$

Generally, the right hand side is a differential operator of order $\leq N$. But for some $N = N(n)$ it might happen that s_N induces an operation on the space of vector fields, i.e.,

$$s_N(X_1, \dots, X_N) \in Vect(n) \quad \text{for any } X_1, \dots, X_N \in Vect(n)$$

and $s_N(X_1, \dots, X_N)$ does not vanish for some vector fields $X_1, \dots, X_N$. If this is the case we say that the operation s_N is an N-*commutator*. We exclude the trivial cases where

$s_N(X_1,\ldots,X_N)=0$ is an identity in the Lie algebra $Vect(n)$, i.e., we only consider the cases where $s_N(X_1,\ldots,X_N)\neq 0$ for some $X_1,\ldots,X_N\in Vect(n)$.

A pair (n,N) is said to be *critical*, if $[X_1,\ldots,X_N]$ is always a vector field and if it does not vanish for some $X_1,\ldots,X_N$. All pairs $(n,2)$ are critical. S. Lie discovered this fact about two centuries ago. In [1], it was noticed that the pairs (n,n^2+2n-2) are also critical. We thus get interesting and generally highly non–trivial new multilinear operations on vector fields. One can expect that they will play essential role in (linear or non-linear) mathematical physics just like usual commutators. A natural problem arises: *list all critical pairs*. In [1], we have shown that for $n=2$, the complete list consists only of $(2,2)$ and $(2,6)=(n,n^2+2n-2)$.

Here we study the case $n=3$ and show that, in addition to $(3,2)$ and $(3,13)=(n,n^2+2n-2)$, there is exactly one more critical pair, namely, $(3,10)$. If we restrict ourselves to divergence-free vector fields, we prove that the only critical pairs are $(3,2)$ and $(3,10)$.

Proofs involve heavy calculations assisted by the packages `Mathematica` and `Maple`. The paper version omits some technical details and *ad hoc* examples that can be found in [3].

The general strategy of calculations is as follows. As in [1], we reduce our problem to that of calculating powers of a certain *universal odd superderivation* $D=D_n$ depending only on n. The pair (n,N), where the powers are taken in the ring of superdifferential operators, is critical if and only if D_n^N is a non–zero superderivation.

To state our results precisely, we need some definitions and notation.

All vector spaces are considered over a field K of characteristic 0. Let $(A,\circ)$ be a super-algebra with grading (parity) $A=A_0+A_1$, the parity map $\omega: A\to\{0,1\}$, and multiplication $a\circ b$. A linear map $D: A\to A$ is said to be an homogeneous *q-derivation* if

$$\begin{aligned}&D: A_i\to A_{i+q},\quad i=0,1,\\ &D(a\circ b)=D(a)\circ b+(-1)^{q\,\omega(a)}a\circ D(b)\quad\text{for all } a,b\in A.\end{aligned}$$

If $q=0$, the derivation in question is called *even,* and if $q=1$, it is called *odd.* For any super-algebra, D^2 is a derivation whenever D is odd. We let $Der\,A$ denote the super-Lie algebra of derivations of A.

Define a *super-Lagrangian algebra* $\mathcal{L}_n$ to be a super-commutative associative super-algebra generated by the odd elements (η_i,α), where $\eta_1,\ldots,\eta_n$ are just symbols, and $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbf{Z}_+^n$, where $\mathbf{Z}_+$ denotes the set of non-negative integers. For $i=1,\ldots,n$, let $\partial_i:\mathcal{L}_n\to\mathcal{L}_n$ be the even derivation of $\mathcal{L}_n$ defined on generators by

$$\partial_i(\eta_j,\alpha)=(\eta_j,\alpha+\varepsilon_i),$$

where $\varepsilon_i=(0,\ldots,0,1,0,\ldots,0)\in\mathbf{Z}_+^n$ with the 1 occupying the ith slot. Set

$$\eta_i:=(\eta_i,0)\qquad\text{and } \partial^\alpha(\eta_i)=(\eta_i,\alpha).$$

Notice that the even part $(\mathcal{L}_n)_0$ is generated by elements of the form $\partial^{\alpha^{(1)}}(\eta_{i_1})\cdots\partial^{\alpha^{(k)}}(\eta_{i_k})$ with k even, while the odd part $(\mathcal{L}_n)_1$ is generated by such elements with k odd.

Let A_k be a product of generators of the form $\partial^{\alpha}\eta_i$ such that $|\alpha| := \sum_{i=1}^{n} \alpha_i = k+1$. We say that A_k has *length* l_k (designated also as $l(A_k)$), if A_k is a product of l_k generators,

$$A_k = \partial^{\alpha^{(1)}}\eta_{i_1} \cdots \partial^{\alpha^{(l_k)}}\eta_{i_{l_k}}, \quad \text{where} \quad \alpha^{(1)}, \dots, \alpha^{(l_k)} \in \mathbf{Z}_+^n.$$

We say that a monomial $u = A_{-1}A_0 \cdots A_s$ has type $l_{-1}l_0 \cdots l_s$, if $l(A_i) = l_i$ and $i = -1, \dots, s$.

The expression

$$D = \sum_i \eta_i \partial_i \quad \text{where } D(\xi) = \sum_i \eta_i \partial_i(\xi) \quad \text{for any} \quad \xi \in \mathcal{L}_n,$$

is an odd derivation of $\mathcal{L}_n$.

Let U be an associative commutative algebra with derivations $d_1, \dots, d_n$. Extend the action of ∂_i to $\mathcal{L}_n \otimes U$ by setting

$$\partial_i(\xi \otimes u) = \partial(\xi) \otimes u + \xi \otimes d_i(u).$$

We consider U as a super-algebra with the zero odd part, and make $\mathcal{L}_n \otimes U$ into a super-algebra under the natural parity:

$$(\mathcal{L}_n \otimes U)_0 = (\mathcal{L}_n)_0 \otimes U, \qquad (\mathcal{L}_n \otimes U)_1 = (\mathcal{L}_n)_1 \otimes U.$$

In particular, for the polynomial algebra $U = K[x_1, \dots, x_n]$ with partial derivatives $\frac{\partial}{\partial x_i}$ (denoted for simplicity by ∂_i), we obtain an odd derivation D of the super-algebra $\mathcal{L}_n \otimes K[x_1, \dots, x_n]$. Recall that the divergence of $D = \sum_i \eta_i$ is $Div\, D = -\sum_{i=1}^{3} \partial_i \eta_i$.

The main results of this paper are the following theorems.

Theorem 1. *Let* $D = \sum_{i=1}^{3} \eta_i \partial_i \in Der\, \mathcal{L}_3$. *Then*

$$\begin{aligned} &D^{10} \in Der\, \mathcal{L}_3 \otimes K[x_1, x_2, x_3], \\ &D^{13} \in Der\, \mathcal{L}_3 \otimes K[x_1, x_2, x_3], \quad D^i \neq 0 \quad \textit{for} \;\; i \leq 13, \\ &D^{13} = (Div\, D)D^{12}, \qquad D^{14} = 0. \end{aligned}$$

If $D^N \in Der\, \mathcal{L}_3 \otimes K[x_1, x_2, x_3]$, *then* $N = 2, 10, 13$.

Theorem 2. *Let* $Div\, D = 0$. *Then*

$$\begin{aligned} &D^{10} \in Der\, \mathcal{L}_3 \otimes K[x_1, x_2, x_3], \\ &Div\, D^{10} = 0, \quad D^i \neq 0, \; i \leq 10, \qquad D^{11} = 0. \end{aligned}$$

If $D^N \in Der\, \mathcal{L}_3 \otimes K[x_1, x_2, x_3]$ *and* $Div\, D = 0$, *then* $N = 2$ *or* 10.

Let ∂deg denote the *differential order*, i.e., the order of the differential operator. For $n = 2, 3$, the differential order of D^k is as follows:

$n=2$	k	1	2	3	4	5	6	7							
	$\partial deg\, D^k$	1	1	2	2	2	1	$-\infty$							
$n=3$	k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	$\partial deg\, D^k$	1	1	2	2	3	3	3	3	3	1	2	2	1	$-\infty$

If $D \in Der_0\mathcal{L}_n$, i.e., $Div\, D = 0$, then the growth of differential orders of D^k is given by:

$n=2$	k	1	2	3	4	5	6	7				
	$\partial deg\, D^k$	1	1	2	2	1	$-\infty$	$-\infty$				
$n=2$	k	1	2	3	4	5	6	7	8	9	10	11
	$\partial deg\, D^k$	1	1	2	2	3	3	3	3	3	1	$-\infty$

Observe a dramatic drop of $\partial deg\, D^k$ for $(n, k) = (3, 10)$ from 3 to 1. This means that D^{10} is a derivation. In fact, our calculations give more information about D^{10} and D^{13}.

Let Y be a differential monomial of a form $\partial^\alpha \eta_j$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_+$ and $\partial^\alpha = \prod_{i=1}^{n} \partial_i^{\alpha_i}$. We say that Y is of *weight* $|\alpha| - 1$, where $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The differential polynomial super-algebra $\mathcal{L}_n$ (for details, see §2) has a basis consisting of differential monomials of a form $X = X_{-1}X_0X_1 \cdots X_k$, where X_s is a product of differential monomials of weight s. For a basis element X, we say that X has *type* $(l_{-1}, l_0, l_1, \dots, l_k)$ if X_s is a product of l_s elements of weight s for any $s = -1, 0, 1, \dots, k$. For example, $\eta_1\eta_2\eta_3\partial_1\eta_1\partial_1\eta_2\partial_1^2\partial_3\eta_3$ has type $(3, 2, 0, 1)$.

In [1] was established that D^{13} is a linear combination of basis elements of type 382. In other words, D^{13} is a linear combination, with integer coefficients, of monomials of the form $A_{-1}A_0A_1\partial_i$, where $A_{-1} = \eta_1\eta_2\eta_3$ is a product of 3 generators of the form η_i, A_0 is a product of 8 generators of the form $\partial_i\eta_j$, and A_1 is a product of 2 generators of the form $\partial_i\partial_j\eta_s$. We prove that D^{10} is a linear combination of basis elements of types 271, 352, and 3601.

In terms of vector fields (first order partial differential operators), these results mean the following: For any 10 vector fields $X_1, \dots, X_{10}$ on a 3-dimensional manifold, the anti-symmetric sum of their 10-compositions $X_{\sigma(1)} \cdots X_{\sigma(10)}$ over all permutations $\sigma \in Sym_{10}$ is again a vector field. Similarly, for any 13 vector fields $X_1, \dots, X_{13} \in Vect(3)$, the anti-symmetric sum of the compositions $X_{\sigma(1)} \cdots X_{\sigma(13)}$ over all permutations $\sigma \in Sym_{13}$ is also a vector field.

So, $Vect(3)$ has three well-defined tensor operations: the Lie commutator (in our terms, 2-commutator), the 10-commutator, and the 13-commutator. In particular, s_{13} is uniquely defined by 2-jets of vector fields, and s_{10} is uniquely defined by 3-jets of vector fields. For exact relation between odd derivations and N-commutators, see Theorem 3.

2 Differential polynomial super-algebra $\mathcal{L}_n$

We consider $\mathbf{Z}_+^n$ and $\mathbf{Z}_+^n \times \{1, \dots, n\}$ as linearly ordered, where $\alpha < \beta$ if

either $|\alpha| < |\beta|$ or $(|\alpha| = |\beta|$ and $\alpha_1 = \beta_1, \dots, \alpha_{s-1} = \beta_{s-1}, \alpha_s > \beta_s$ for some $s = 1, \cdots, n)$.

Let $(\alpha, i) < (\beta, j)$ if either $i < j$ or $(i = j$ and $\alpha < \beta)$.

Let $\mathcal{L}_n$ be a super-commutative associative algebra over a field K which is generated by odd elements $e_{\alpha,i}$, where $\alpha \in \mathbf{Z}_+^n$, $i \in I$. Then, for any $\alpha, \beta, \gamma \in \mathbf{Z}_+^n$ and $i, j, s \in I$, we have

$$e_{\alpha,i}e_{\beta,j} = -e_{\beta,j}e_{\alpha,i}, \qquad e_{\alpha,i}(e_{\beta,j}e_{\gamma,s}) = (e_{\alpha,i}e_{\beta,j})e_{\gamma,s}.$$

The elements $e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}$ with $(\alpha, i) < (\beta, j) < \cdots < (\gamma, s)$ form a basis for $\mathcal{L}_n$. We fix this basis and call these elements the *base* elements of $\mathcal{L}_n$. The number of indexes $i, j, \cdots, s$ in a basis element e is called the *length* of e and denoted by $l(e)$.

Each basis element of $\mathcal{L}_n$ can be represented as $e = e^{[-1]}e^{[0]}e^{[1]}\cdots e^{[r]}$, where $e^{[s]}$ is a product of ordered generators of the form $e_{\alpha,i}$ with $|\alpha| = s+1$. We call $e^{[s]}$ the *s-component* of e; its length $l(e^{[s]})$ is denoted by $l_s(e)$ and called the *s-length* of e. Thus,

$$l(e) = \sum_{i \geq -1} l_i(e).$$

Let $\partial_i = \frac{\partial}{\partial_i}$, where $i \in I$, be partial derivatives of $U = K[x_1, \dots, x_n]$. We extend these maps to maps of $\mathcal{L}_n$ by setting

$$\partial_i e_{\beta,j} = e_{\alpha+\varepsilon_i,j}.$$

It is easy to see that ∂_i satisfies the Leibniz rule

$$\partial_i(e_{\beta,j}e_{\gamma,s}) = (\partial_i e_{\beta,j})e_{\gamma,s} + e_{\beta,j}(\partial_i e_{\gamma,s}) \text{ for all } \beta, \gamma \in \mathbf{Z}_+^n.$$

So, we have constructed the commuting even derivations $\partial_1, \dots, \partial_n \in Der(\mathcal{L}_n \otimes U)$, and

$$e_{\alpha,i} = \partial^\alpha e_{0,i} \text{ for any } \alpha \in \mathbf{Z}_+^n, i \in I.$$

Here $0 = (0, \dots, 0) \in \mathbf{Z}_+^n$.

The space $\mathcal{L}_n$ has three gradings. The first one, $\mathbf{Z}^n$-grading, is defined by

$$||e_{\alpha,i}|| = \alpha - \varepsilon_i$$

and is extended by multiplicativity to the other basis elements,

$$||e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}|| = \alpha - \varepsilon_i + \beta - \varepsilon_j + \cdots + \gamma - \varepsilon_s.$$

The second grading is induced by $\mathbf{Z}^n$-grading. It is the $\mathbf{Z}$-grading defined on a basis element $e = e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}$ by the formula

$$|e| = -l(e) + |\alpha| + |\beta| + \cdots + |\gamma|.$$

The third grading is given by the length. We let $l(\xi) = s$, if ξ is a nontrivial linear combination of homogeneous base elements of length s. The length defines a parity on $\mathcal{L}_n$.

We say that

$$wt(e) = |\alpha| + \cdots + |\beta| - l(e)$$

is the *weight* of e. We let $\mathcal{L}_n^{[l]}$ be the linear span of the basis elements u with $l(u) = l$, and let $\mathcal{L}_n^{[l,w]}$ be the linear span of the base elements u with $l(e) = l$ and $wt(e) = w$.

Example.

$$\mathcal{L}_n^{[1]} = \mathrm{Span}\,\langle e_{\alpha,i} \mid \alpha \in \mathbf{Z}^n, \text{ and } \ i = 1, \ldots, n\rangle\,,$$
$$\mathcal{L}_n^{[n]} = \mathrm{Span}\,\langle e_{0,1} \cdots e_{0,n}\rangle\,, \qquad \mathcal{L}_n^{[1,-1]} = \mathrm{Span}\,\langle e_{0,i}\rangle\,.$$

Proposition 1. $\mathcal{L}_n = \bigoplus_{l \geq 1, w \geq -n} \mathcal{L}_n^{[l,w]}$ *is an associative, super-commutative graded super-algebra:*

$$\mathcal{L}_n^{[l,w]}\mathcal{L}_n^{l_1,w_1]} \subseteq \mathcal{L}_n^{[l+l_1,w+w_1]} \quad \textit{for any } u, v, w \in \mathcal{L}_n.$$

Note that any basis element $u \in \mathcal{L}_n$ can be represented in the form $u_{-1}u_0 \cdots u_r$, where u_s for $s = -1, 0, \ldots, r$ are basis elements and where the u_s are products of generators of weight s. We say that a basis element $u \in \mathcal{L}_n$ has *type* $(l_{-1}, l_0, \ldots, l_r)$, if u is a product of l_s generators of weight s for $s = -1, 0, \ldots, r$.

Lemma 1. *Every basis element $u \in \mathcal{L}_n$ satisfies the following conditions:*

$$\sum_{i \geq -1} l_i(u) = l(u), \qquad \sum_{i \geq -1} i\, l_i(u) = wt(u), \qquad l_i(u) \leq n\tbinom{n+i}{i+1}, \quad \textit{where } i \geq -1.$$

Proof. The first two relations are reformulations of the grading property of $\mathcal{L}_n$ (Proposition 1). As for the last two relations, they follow from the fact that

$$|\{\alpha \in \mathbf{Z}_+^n \mid |\alpha| = i + 1\}| = \binom{n+i}{i+1}.$$

Example. The odd basis element $u = \eta_1\partial_1^2\eta_2\partial_1\partial_2\eta_2$ is of type $(1, 0, 2)$, $l(u) = 3$, $wt(u) = 1$.

Let $Diff_n$ be the algebra of differential operators on $\mathcal{L}_n$. It has a basis that consists of differential operators of the form $u\partial^\alpha$, where $\alpha \in \mathbf{Z}_+^n$ and u is a base element of $\mathcal{L}_n$. We endow $Diff_n$ with the multiplication $\cdot$ given by the rule

$$u\partial^\alpha \cdot v\partial^\beta = \sum_\gamma \binom{\alpha}{\gamma} u\partial^\gamma v\partial^{\alpha+\beta-\gamma}, \quad \text{where } \binom{\beta}{\gamma} = \prod_{i=1}^n \binom{\beta_i + \gamma_i}{\gamma_i}.$$

The multiplication $\cdot$ corresponds to the composition of differential operators.

We also endow $Diff_n$ with two more multiplications, $\circ$ and $\bullet$. They are defined by the following rules:

$$u\partial^\alpha \circ v\partial^\beta = \sum_{\gamma \neq 0} \tbinom{\alpha}{\gamma} u\partial^\gamma v\partial^{\alpha+\beta-\gamma}, \qquad u\partial^\alpha \bullet v\partial^\beta = uv\partial^{\alpha+\beta}.$$

We see that

$$X \cdot Y = X \circ Y + X \bullet Y \quad \text{for all } X, Y \in Diff_n.$$

Given a basis element $X = u\partial^\alpha \in Diff_n$, we define its *length* $l(X)$, *weight* $wt(X)$, *parity* $q(X)$, and *differential order* $\partial deg(X)$ by setting

$$l(X) = l(u), \quad wt(X) = wt(u) + |\alpha|, \quad q(X) = l(u), \quad \partial deg(X) = |\alpha|.$$

Denote the space $Diff_n^{[1]}$ of first order differential operators by W_n. Let further

$$Diff_n^{[d]} = \langle X \mid \partial deg(X) = d\rangle\,, \quad Diff_n^{[l,w]} = \langle X \mid l(X) = l, wt(X) = w\rangle\,,$$
$$Diff_n^{[l,w,d]} = \langle X \mid l(X) = l,\ wt(X) = w,\ \partial deg(X) = d\rangle\,.$$

For a differential operator $X = \sum_{\alpha\in\mathbf{Z}_+^n} v_\alpha\partial^\alpha \in Diff_n$, define its *differential order* $deg(X)$ as the maximal $|\alpha|$ for which $v_\alpha \neq 0$.

Proposition 2. *The space of differential operators under the above multiplications has the following properties:*

The algebra $(Diff_n, \cdot)$ is an associative super-algebra. This algebra is graded,

$$Diff_n = \mathop{\oplus}_{l>0, w\geq -n} Diff_n^{[l,w]}, \qquad Diff_n^{[l,w]} \cdot Diff_n^{[l_1,w_1]} \subseteq Diff_n^{[l+l_1,w+w_1]}.$$

The algebra $(W_n, \circ)$ is super-left-symmetric, i.e.,

$$(X, Y, Z) = (-1)^{q(X)q(Y)}(Y, X, Z) \text{ for any first order differential operators } X, Y, Z,$$

where $(X, Y, Z) = X \circ (Y \circ Z) - (X \circ Y) \circ Z$ is the associator. Moreover, the super-left-symmetric rule is true for any $X, Y \in Diff_n^{[1]}, Z \in Diff_n$. This algebra is graded,

$$W_n = \mathop{\oplus}_{l>0,\ w\geq -n} W_n^{[l,w]}, \qquad W_n^{[l,w]} \circ W_n^{[l_1,w_1]} \subseteq W_n^{[l+l_1,w+w_1]}.$$

The algebra $(Diff_n, \bullet)$ is associative and super-commutative; it is graded by length, weight and differential order,

$$Diff_n = \mathop{\oplus}_{l>0,\ w\geq -n,\ d\geq 0} Diff_n^{[l,w,d]}, \qquad Diff_n^{[l,w,d]} \bullet Diff_n^{[l_1,w_1,d_1]} \subseteq Diff_n^{[l+l_1,w+w_1,d+d_1]}.$$

Any first order differential operator acts under the multiplication $\circ$ as a derivation on $(Diff_n, \bullet)$.

Proof. Notice that the natural action of W_n on $\mathcal{L}_n$ coincides with the left-symmetric product:

$$X(\eta) = X \circ \eta \text{ for any } X \in W_n, \eta \in \mathcal{L}_n.$$

Therefore, we have the following relation between the composition and left-symmetric multiplications:

$$(X \cdot Y)(\eta) \neq (X \circ Y)(\eta),$$

but

$$(X \cdot Y)(\eta) = X \circ (Y(\eta)) \text{ for any } X, Y \in Diff_n \text{ and } \eta \in \mathcal{L}_n.$$

Moreover, the composition of first order differential operators can be expressed in terms of the left-symmetric multiplication,

$$(X \cdot Y)(\eta) = X \circ (Y \circ \eta) \quad \text{for any } X, Y \in W_n, \eta \in \mathcal{L}_n.$$

Thus,

$$(X \circ Y + X \bullet Y)(\eta) = X \circ (Y \circ \eta); \qquad X \circ (Y \circ \eta) - (X \circ Y)(\eta) = (X \bullet Y)(\eta).$$

Since $X \circ Y \in W_n$, this means that

$$X \circ (Y \circ \eta) - (X \circ Y) \circ \eta = (X \bullet Y)(\eta) \quad \text{for any } X, Y \in W_n,\ \eta \in \mathcal{L}_n. \tag{1}$$

We see from these facts that

$$\begin{aligned}([X,Y] \cdot Z)(\eta) = \ & (X \cdot Y - (-1)^{q(X)q(Y)} Y \cdot X)(Z(\eta)) = \\ & (X \circ Y + X \bullet Y - (-1)^{q(X)q(Y)} Y \circ X - (-1)^{q(X)q(Y)} Y \bullet X) \circ (Z(\eta)) = \\ & (X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ (Z(\eta)).\end{aligned}$$

On the other hand,

$$\begin{aligned}([X,Y] \cdot Z)(\eta) = \ & (X \cdot (Y \cdot Z) - (-1)^{q(X)q(Y)} Y \cdot (X \cdot Z))(\eta) = \\ & X \circ (Y \cdot Z)(\eta) - (-1)^{q(X)q(Y)} Y \circ (X \cdot Z)(\eta) = \\ & X \circ (Y \circ Z(\eta)) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z(\eta)).\end{aligned}$$

Hence,

$$(X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ (Z(\eta)) = X \circ (Y \circ Z(\eta)) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z(\eta)).$$

In other words, for any $X, Y \in W_n, Z \in Diff_n$, we have

$$(X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ Z = X \circ (Y \circ Z) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z).$$

The other statements of the proposition are evident. □

For a basis element $X = u\partial^\alpha \in Diff_n$, we say that X has *type* $(l_{-1}, l_0, l_1, \ldots, l_r; d)$ if u has type $(l_{-1}, l_0, \ldots, l_r)$ and $|\alpha| = d$.

Example. Let $X = \eta_1\eta_3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_2\partial_3\eta_3\partial_1\partial_2$. Then X is a basis element of $Diff_3$ of type $(2, 3, 0, 1; 2)$, weight 2, and differential order 2.

Lemma 2. *Any basis element $X \in Diff_n$ satisfies the following conditions:*

$$\sum_{i \geq -1} l_i(X) = l(X), \qquad \sum_{i \geq -1} i\, l_i(X) + deg(X) = wt(X), \qquad l_i(X) \leq n\binom{n+i}{i+1} \ \text{ for any } i \geq -1.$$

Proof. Follows from Proposition 2 and Lemma 1. □

Let $Diff_n^{(l_{-1}, l_0, \ldots, l_r; d)}$ be the subspace of $Diff_n$ generated by the basis elements of type $(l_{-1}, l_0, \ldots, l_r; d)$. Let

$$\tau_{(l_{-1}, l_0, \ldots, l_r; d)} : Diff_n \to Diff_n^{(l_{-1}, l_0, \ldots, l_r; d)}, \qquad \tau_d : Diff_n \to Diff_n^{[d]}$$

be the projection maps.

The polynomial space $U = K[x_1, \dots, x_n]$ has natural gradings:

$$||x^\alpha|| = \alpha, \qquad |x^\alpha| = |\alpha|.$$

Its standard basis is $\{x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i} \mid \alpha \in \mathbf{Z}_+^n\}$. The gradings on $\mathcal{L}_n$ and U induce gradings on $\mathcal{L}_n \otimes U$. In the previous section, we defined the parity q on $\mathcal{L}_n \otimes U$. Set

$$\eta_i = e_{0,i}, \qquad \partial^\alpha \eta_i := e_{\alpha,i}.$$

Then

$$l(\partial^{\alpha_1}\eta_{i_1} \cdots \partial^{\alpha_k}\eta_{i_k}) = k.$$

We identify $\mathcal{L}_n$ with $\mathcal{L}_n \otimes 1$ and consider $\mathcal{L}_n$ as a subalgebra of $\mathcal{L}_n \otimes U$.

3 First order differential operators on $\mathcal{L}_n$

Note $W_n = Diff_n^{[1]}$ has two algebraic structures. The first one, a super-Lie algebra structure with respect to the super-commutator is well known. Notice that

$$q(\xi\partial_i) = q(\xi) \text{ for every } \xi \in \mathcal{L}_n.$$

Recall that for any $D \in W_n$, the corresponding adjoint operator $ad\, D : W_n \to W_n$ is a derivation of W_n. Therefore, W_n can be interpreted as a super-Lie algebra of derivations of $\mathcal{L}_n$.

The second algebra structure on W_n is given by a left-symmetric multiplication. It is less known. Define a product $\circ$ by setting

$$(\xi\partial_i) \circ (\eta\partial_j) = \xi\partial_i(\eta)\partial_j.$$

For any $D_1, D_2, D_3 \in W_n$, let (D_1, D_2, D_3) be the associator. Then we have the left-symmetric identity

$$(D_1, D_2, D_3) = (-)^{q(D_1)q(D_2)}(D_2, D_1, D_3).$$

4 Leibniz binomial formula for the odd derivation

If D is any even derivation of some algebra $A = (A, \circ)$, then

$$D^n(a \circ b) = \sum_{i=0}^{n} \binom{n}{i} D^i(a) \circ D^{n-i}(b) \text{ for any } a, b \in A,.$$

The first known to me published super analog of this formula in due to Leites [4], although it should have been (and, no doubt, was) known earlier: For any odd derivation D, we have for any $a, b \in A$:

$$\begin{aligned} D^{2n}(a \circ b) = & \sum_{i=0}^{n} \tbinom{n}{i} D^{2i}(a) \circ D^{2n-2i}(b), \\ D^{2n+1}(a \circ b) = & \sum_{i=0}^{n} \tbinom{n}{i} D^{2i+1}(a) \circ D^{2n-2i}(b) + \\ & (-1)^{p(a)} \sum_{i=0}^{n} \tbinom{n}{i} D^{2i}(a) \circ D^{2n-2i+1}(b). \end{aligned}$$

5 Calculation of D^n

Let $\eta_1, \dots, \eta_n$ be odd elements and

$$D = \sum_{i=1}^{n} \eta_i \partial_i, \quad F = D \circ D = \sum_{i,j=1}^{n} \eta_i \partial_i \eta_j \partial_j.$$

Notice that since $D \in W_n^{[1,0]}$, F is an even element of W_n and $l_{-1}(F) = 1$, $l_0(F) = 1$, $l_s(F) = 0, s > 0$, it follows that $D^k \in Diff_n^{[k,0]}$.

Define the left-symmetric power $D^{\circ k}$ by setting $D^{\circ 1} = D$ and

$$D^{\circ k} = \quad D \circ D^{\circ(k-1)} \text{ if } k > 1.$$

Define the bullet power $D^{\bullet k}$ and associative power $D^{\cdot k}$ similarly. Since the multiplications $\cdot$ and $\bullet$ are associative, in the last cases $D^{\bullet k}$ and $D^{\cdot k}$ have the conventional properties of powers:

$$D^{\bullet k} \bullet D^{\bullet s} = D^{\bullet(k+s)}, \qquad D^{\cdot k} \bullet D^{\cdot s} = D^{\cdot(k+s)}.$$

These facts are no longer true for left-symmetric powers. For example,

$$D \circ (D \circ D^{\circ 2}) = (D \circ D) \circ D^{\circ 2},$$

but

$$D \circ (D^{\circ 2} \circ D) \neq (D \circ D^{\circ 2}) \circ D.$$

Lemma 3. $D^{\cdot 2} = F$.

Proof. It is straightforward that

$$D^{\cdot 2} = D \cdot D = \sum_{i,j=1}^{n} \eta_i \partial_i \eta_j \partial_j + \sum_{i,j=1}^{n} \eta_i \eta_j \partial_i \partial_j.$$

Since $\eta_i \eta_j = -\eta_j \eta_i$ and $\partial_i \partial_j = \partial_j \partial_i$, we have $\sum_{i,j=1}^{n} \eta_i \eta_j \partial_i \partial_j = 0$. Thus,

$$D^2 = \sum_{i,j=1}^{n} \eta_i \partial_i \eta_j \partial_j = D \circ D = F.$$

Lemma 4. $D^{\circ(2n)} = F^{\circ n}$ *for any* $n = 1, 2, 3, \cdots$

Proof follows by induction on n.

Lemma 5. $F \circ F^{\bullet k} = kF^{\bullet(k-1)} \bullet F^{\circ 2}$.

Proof. Since $F \in W_n$ is an even derivation, every left-symmetric multiplication operator acts on $(Diff_n, \bullet)$ as a super-derivation (Proposition 2) and we have

$$F \circ (F \bullet F) = (F \circ F) \bullet F + F \bullet (F \circ F).$$

By commutativity of the bullet-multiplication, this means that

$$F \circ F^{\bullet 2} = 2F \bullet F^{\circ 2}.$$

An easy induction on k, based on such arguments, shows that our lemma is true for any k.

Lemma 6. $D^4 = F^{\circ 2} + F^{\bullet 2}$.

Proof. By Lemma 3 and by associativity of $\circ$,

$$D^4 = D^2 \cdot D^2 = F \cdot F = F \circ F + F \bullet F.$$

Lemma 7. $D^6 = F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}$.

Proof. By Lemma 3 and Lemma 6,

$$D^6 = D^2 \cdot D^4 = D^2 \circ D^4 + D^2 \bullet D^4 = F \circ (F^{\circ 2} + F^{\bullet 2}) + F \bullet (F^{\circ 2} + F^{\bullet 2})$$

By Lemma 5, $F \circ F^{\bullet 2} = 2F \bullet F^{\circ 2}$, so

$$D^6 = F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}.$$

Lemma 8. $D^8 = F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}$.

Proof. By Lemma 5,

$$F \circ F^{\bullet 3} = 3F^{\circ 2} \bullet F^{\bullet 2}.$$

Therefore, by Lemma 3 and Lemma 7

$$\begin{aligned} D^8 &= D^2 \cdot D^6 = D^2 \circ D^6 + D^2 \bullet D^6 = \\ &F \circ (F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}) + F \bullet (F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}) = \\ &F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 3F \bullet F^{\circ 3} + 3F^{\circ 2} \bullet F^{\bullet 2} + F \bullet F^{\circ 3} + 3F^{\bullet 2} \bullet F^{\circ 2} + F^{\bullet 4} = \\ &F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}. \end{aligned}$$

Lemma 9.

$$\begin{aligned} D^{10} &= F^{\circ 5} + 5(F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3})) + \\ &5(2F^{\circ 3} \bullet F^{\bullet 2} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2}) + 4F^{\circ 2} \bullet F^{\bullet 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5}. \end{aligned}$$

Proof is straightforward.

Lemma 10. *For every $G \in Diff_n$, we have*

$$F \circ (\prod_{r=1}^{n} \eta_r \, G) = 0.$$

Proof. We have

$$\eta_i \partial_i(\eta_j) \partial_j(\eta_1 \cdots \eta_n) = \sum_{s=1}^{n} \xi_s,$$

where

$$\xi_s = \eta_i \partial_i \eta_j \eta_1 \cdots \eta_{s-1} \partial_j(\eta_s) \eta_{s+1} \cdots \eta_n.$$

If $s \neq i$, then

$$\xi_s = \pm \eta_i \eta_i \xi_{i,s}, \quad \text{where } \xi_{i,s} = \partial_i \eta_j \partial_j \eta_s \prod_{r \neq i,s} \eta_r.$$

Since $\eta_i\eta_i = 0$, this means that
$$\xi_s = 0 \ \text{ if } s \neq i.$$
If $s = i$, then
$$\xi_s = \pm\eta_i\partial_i\eta_j\partial_j\eta_i \prod_{r\neq i}\eta_r = \partial_i\eta_j\partial_j\eta_i(\prod_r \eta_r).$$
We have
$$\sum_{i,j=1}^{n} \partial_i\eta_j\partial_j\eta_i = \theta_1 + \theta_2 + \theta_3,$$
where
$$\theta_1 = \sum_{i<j}\partial_i\eta_j\partial_j\eta_i, \quad \theta_2 = \sum_{i}\partial_i\eta_i\partial_i\eta_i, \quad \theta_3 = \sum_{i>j}\partial_i\eta_j\partial_j\eta_i.$$
Since the elements $\partial_i\eta_j$ and $\partial_j\eta_i$ are odd, $\theta_1 + \theta_3 = 0, \qquad \theta_2 = 0$. Thus,
$$F \circ (\prod_{r=1}^{n}\eta_r G) = (\sum_{i,j=1}^{n}\partial_i\eta_j\partial_j\eta_i)\prod_r \eta_r G = 0.$$

Let
$$Diff_n^{[s]} =< u\partial^\alpha \mid u \in \mathcal{L}_n, \alpha \in \Gamma_n, |\alpha| = s >$$
be the space of differential operators of order s, and let $\tau_s : Diff_n \to Diff_n^{[s]}$ be the projection.

Lemma 11. *If* $n = 3$, $D = \sum_{i=1}^{n} u_i\partial_i$, *and* u_i *are odd, then*
$$\begin{aligned}
\tau_1 D &= F^{\circ 5},\\
\tau_2 D^{10} &= 5(F^{\circ 2}\bullet F^{\circ 3} + F\circ(F\bullet F^{\circ 3})),\\
\tau_3 D^{10} &= 5(2F^{\circ 3}\bullet F^{\bullet 2} + 3F\bullet F^{\circ 2}\bullet F^{\circ 2}),\\
\tau_s D^{10} &= 0 \quad \textit{for } s > 3.
\end{aligned}$$

Proof. Follows from Lemma 9 and from the fact that $F^{\bullet s} = 0$ if $s > n$.

Conclusion. To find D^{10}, we need to calculate $F^{\circ s}$ for $s = 1, 2, 3$ and $F^{\bullet 2}$. Let us give some examples of such calculations. Below we denote by ξ the product of monomials of the form $\partial^\alpha(\eta_i)$, where $\alpha \neq 0$.

Example. Collect the terms of the form $\eta_1\eta_2\xi\partial_1^2$ for second bullet-powers of F. We have
$$F^{\bullet 2}_{\eta_1\eta_2;\partial_1^2} = -2\eta_1\eta_2\partial_1\eta_1\partial_2\eta_1\partial_1^2, \quad F^{\bullet 2}_{\eta_1\eta_3;\partial_1^2} = -2\eta_1\eta_3\partial_1\eta_1\partial_3\eta_1\partial_1^2, \quad F^{\bullet 2}_{\eta_2\eta_3;\partial_1^2} = -2\eta_2\eta_3\partial_2\eta_1\partial_3\eta_1\partial_1^2.$$

Example. Collect terms of the form $\eta_1\xi\partial_1$ for the second left-symmetric power of F. We get
$$\begin{aligned}
F^{\circ 2}_{\eta_1;\partial_1} = \ & \eta_1(-2\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3 + \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2 -\\
& \partial_2\eta_1\partial_3\eta_2\partial_1\eta_3 + \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3)\partial_1.
\end{aligned}$$

Example. Third left-symmetric powers of F. We have $F^{\circ 3} = F \circ (F \circ F)$. Collect terms of the form $\eta_1 \xi \partial_1$ for the third left-symmetric power. We see that

$$\begin{aligned} F^{\circ 3}_{\eta_1;\partial_1} = \ & \eta_1(2\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3 + 2\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3 - 6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3 \\ & -2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3 \\ & +2\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3 - 2\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3 + \partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3 \\ & -\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3 + 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_3\eta_3 - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3 + \\ & \partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3 - \partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3 - 2\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3)\partial_1. \end{aligned}$$

We similarly establish that $F^{\circ 3}_{\eta_2;\partial_1}$ has 8 terms, $F^{\circ 3}_{\eta_3;\partial_1}$ has 8 terms, $F^{\circ 3}_{\eta_1\eta_2;\partial_1}$ has 76 terms, $F^{\circ 3}_{\eta_1\eta_3;\partial_1}$ has 76 terms, and the element $F^{\circ 3}_{\eta_2\eta_3;\partial_1}$ has 71 terms.

Lemma 12. 1) $F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3}) = 0$. 2) $F^{\circ 3} \bullet F^{\bullet 2} = 0$. 3) $F \bullet F^{\circ 2} \bullet F^{\circ 2} = 0$.

Proof of Theorem 1. By Lemmas 11 and 12, we have $\tau_2(D^{10}) = 0$ and $\tau_3(D^{10}) = 0$. Therefore, $D^{10} = \tau_1(D^{10})$. Other statements of Theorem 1 and Theorem 2 can be proved by similar calculations of left-symmetric and bullet powers of F.

6 N-commutators and super-derivations

In this section we explain how the escort forms [1] appear in calculating the powers of odd derivations. Let $I = \{1, \ldots, n\}$ and let $D = \sum_{i=1}^{n} \eta_i \partial_i \in Der\, \mathcal{L}_n$ be an odd super-derivation. For any $\alpha \in \mathbf{Z}_+^n$, set $x^{(\alpha)} = \frac{x^\alpha}{\alpha!}$ and recall that $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbf{Z}_+^n$ (the i-th coordinate is 1). Set

$Supp(s_k) := \{k$-tuples $\{(\alpha^{(1)}, i_1), \cdots, (\alpha^{(k)}, i_k)\} \mid i_1, \ldots, i_k \in I$ and $\alpha^{(1)}, \ldots, \alpha^{(k)} \in \mathbf{Z}_+^n$, such that $\sum_{p=1}^{k} \alpha^{(p)} - \epsilon_{i_p}$ has the form $-\mu$ for some $0 \neq \mu \in \mathbf{Z}_+^n$.$\}$

Let

$$L =< x^{(\alpha)}\partial_i \mid \alpha \in \mathbf{Z}_+^n, i = 1, \ldots, n >$$

be Lie algebra of derivations of $U = K[x_1, \ldots, x_n]$ (Witt algebra) and

$$L_s =< x^{(\alpha)}\partial_i \mid |\alpha| = \sum_{i=1}^{n} \alpha_i = s + 1 >$$

be homogeneous components. Then

$$L = \mathop{\oplus}_{s \geq -1} L_s, \quad [L_p, L_s] \subseteq L_{p+s}.$$

Let $\psi : \wedge^k L \to L$ be anti-symmetric multilinear graded form. This means that

$$\psi(X_1, \ldots, X_k) \in L_{|X_1| + \cdots + |X_k|},$$

for any homogeneous elements $X_1, \ldots, X_k$. Then its escort form $esc(\psi)$ is defined as a multilinear anti-symmetric form

$$esc(f) : \wedge^k L \to L_{-1}$$

such that

$$\begin{array}{lcl} esc(\psi)(X_1, \ldots, X_k) & = \psi(X_1, \ldots, X_k), & \text{if } |X_1| + \cdots + |X_k| = -1, \\ esc(\psi)(X_1, \ldots, X_k) & = 0, & \text{if } |X_1| + \cdots + |X_k| \neq -1. \end{array}$$

In particular,

$$\{(\alpha^{(1)}, i_1), \ldots, (\alpha^{(k)}, i_k)\} \notin Supp(s_k) \Rightarrow esc(s_k)(x^{(\alpha^{(1)})}\partial_{i_1}, \ldots, x^{(\alpha^{(k)})}\partial_{i_k}) = 0.$$

The main result of this section is the following

Theorem 3.

$$D^k = \sum_{(\alpha^{(1)}, i_1) < \cdots < (\alpha^{(k)}, i_k)} \partial^{\alpha^{(1)}}(u_{i_1}) \cdots \partial^{\alpha^{(k)}}(u_{i_k}) esc(s_k)(x^{(\alpha^{(1)})}\partial_{i_1}, \ldots, x^{(\alpha^{(k)})}\partial_{i_k}).$$

where the summation is taken over ordered pairs $\{(\alpha, i_1), (\beta, i_2), \ldots, (\gamma, i_k)\} \in Supp(s_k)$.

Proof. Let $U = K[x_1, \ldots, x_n]$ and let the ∂_i be partial derivatives. Let Gr_k be the Grassmann super-algebra with generators $\xi_1, \ldots, \xi_k$. For $U = K[x_1, \ldots, x_n]$, set $\mathcal{U} = U \otimes Gr_k$. Extend each derivation $\partial_i \in Der\, U$ to a derivation of $\mathcal{U}$ by setting

$$\partial_i(v \otimes \omega) = \partial_i(v) \otimes \omega.$$

We obtain a commuting system $\mathcal{D} = \{\partial_1, \ldots, \partial_n\}$ of even derivations of $\mathcal{U}$. Thus, we can consider the $\mathcal{D}$-differential super-algebra $\mathcal{U}$, its super-derivation algebra $\mathcal{L} =< f\partial_i \mid f \in \mathcal{U} >$, and its algebra of super-differential operators

$$\mathcal{D}iff =< f\partial^\alpha \mid \alpha \in \mathbf{Z}_+^n, \quad f \in \mathcal{U} > .$$

We can endow $\mathcal{D}iff$ with the composition operation, left-symmetric multiplication, and bullet multiplication. In particular, we can consider $\mathcal{L}$ to be a left-symmetric algebra as well as a super-Lie algebra. Thus, $\mathcal{L} \cong W_n \otimes Gr_k$, i.e., $\mathcal{L}$ is isomorphic to the current algebra of W_n-valued functions on the $0|k$-dimensional superspace.

We see that, for any $f_1, \ldots, f_n \in \mathcal{U}$, we can consider a homomorphism

$$\mathcal{L}_n \to \mathcal{U}, \qquad \eta_i \mapsto f_i, \quad \text{where } i = 1, \ldots, n.$$

This homomorphism can be extended to a homomorphism of left-symmetric or Lie algebras $Der\,\mathcal{L}_n \to \mathcal{L}$ as well as to a homomorphism of associative (left-symmetric) algebras $Diff_n \to \mathcal{D}iff$. We can use this homomorphism in calculating F^k for $F = \sum_{i=1}^n f_i\partial_i \in \mathcal{L}$. In other words, in the formula for D^k we can make substitutions $\eta_i \mapsto f_i$ and calculate in $\mathcal{U}$ the expressions obtained.

We use this method for calculating the coefficients $\lambda_{\{(\alpha,i_1),(\beta,i_2),\dots,(\gamma,i_k);\mu\}}$, where

$$D^k = \sum \lambda_{\{(\alpha,i_1),(\beta,i_2),\dots,(\gamma,i_k);\mu\}}\partial^\alpha(u_{i_1})\partial^\beta(u_{i_2})\cdots\partial^\gamma(u_{i_k})\partial^\mu.$$

Since the number of η_i-indices is the same as the number of ∂_i-indices, we take the summation here over $\alpha,\beta,\dots,\gamma,\mu \in \mathbf{Z}^n_+$ and $i_1,\dots,i_k \in \{1,\dots,n\}$ such that

$$\alpha+\beta+\cdots+\gamma+\mu = \sum_{s=1}^k \epsilon_{i_s}.$$

Take

$$F = X_1\otimes\xi_1+\cdots+X_k\otimes\xi_k \in \mathcal{L},$$

where $X_i \in W_n, i=1,\dots,k$ are even elements. It is evident that

$$F^k = s_k(X_1,\dots,X_k)\otimes(\xi_1\cdots\xi_k).$$

On the other hand, if $X_1 = x^{\alpha^{(1)}}\partial_{i_1}, X_2 = x^{\alpha^{(2)}}\partial_{i_2},\dots,X_k = x^{\alpha^{(k)}}\partial_{i_k}$, then F can be represented in the form

$$F = \sum_{i=1}^k f_i\partial_i \in \mathcal{L}, \quad \text{where } f_i = \sum_{s|i_s=i} x^{\alpha^{(s)}}\otimes\xi_s \in \mathcal{U}.$$

So, substituting

$$\eta_i \mapsto \sum_{s|i_s=i} x^{\alpha^{(s)}}\otimes\xi_s \in \mathcal{U}$$

in D^k and performing calculations in $\mathcal{U}$ gives us, on the one hand,

$$\lambda_{\{\alpha^{(1)},i_1),\dots,(\alpha^{(k)},i_k);\mu\}}\alpha^{(1)}!\cdots\alpha^{(k)}!\partial^\mu\otimes\xi_1\cdots\xi_k+Y,$$

where

$$Y \in < x^\alpha\partial^\beta\otimes Gr_k \mid |\alpha|>0, \alpha,\beta\in\mathbf{Z}^n_+ >,$$

and, one the other hand,

$$s_k(x^{\alpha^{(1)}}\partial_{i_1},\dots,x^{\alpha^{(k)}}\partial_{i_k})\otimes\xi_1\cdots\xi_k.$$

Take the projections $\mathcal{D}iff \to < 1 > \otimes\xi_1\dots\xi_k$ of both sides. We have

$$\lambda_{\{\alpha^{(1)},i_1),\dots,(\alpha^{(k)},i_k);\mu\}}\alpha^{(1)}!\cdots\alpha^{(k)}!\partial^\mu = esc(s_k)(x^{\alpha^{(1)}}\partial_{i_1},\dots,x^{\alpha^{(k)}}\partial_{i_k}).$$

Thus,

$$esc(s_k)(x^{(\alpha^{(1)})}\partial_{i_1},\dots,x^{(\alpha^{(k)})}\partial_{i_k}) = \lambda_{\{\alpha^{(1)},i_1),\dots,(\alpha^{(k)},i_k);\mu\}}\partial^\mu,$$

which is what we need to prove. □

Denote by s_k° the map $\wedge^k W_n \longrightarrow W_n$ given by

$$s_k^\circ(X_1,\dots,X_k) = \sum_{\sigma\in Sym_k} sign\,\sigma\, X_{\sigma(1)}\circ(X_{\sigma(2)}\circ(\cdots(X_{\sigma(k-1)}\circ X_{\sigma(k)}))),$$

where W_n is considered as a left-symmetric algebra under the multiplication $f\partial_i\circ g\partial_j = f\partial_i(g)\partial_j$.

Corollary 1. *The following statements are equivalent:* (1) $D^k \in Der\,\mathcal{L}$; (2) $D^k = D^{\circ k}$; (3) s_k *is a* k*-commutator on* W_n*;* (4) $s_k = s_k^\circ$.

Theorem 3 has two-fold applications. We use it in constructing D^k by means of s_k and, vice versa, one can use D^k in calculating k-commutators.

7 How to calculate the 10-commutator ?

Let

$$s_5^+(t_1,t_2,t_3,t_4,t_5) = \sum_{\sigma\in Sym_5} t_{\sigma(1)}\cdots t_{\sigma(5)}$$

be the associative symmetric polynomial in 5 variables.

Lemma 13.

$$s_{10}(t_1,\ldots,t_{10}) = \sum_{\sigma\in Sym'_{10}} sign\,\sigma\; s_5^+([t_{\sigma(1)},t_{\sigma(2)}],\ldots,[t_{\sigma(9)},t_{\sigma(10)}]),$$

where

$$Sym'_{10} = \{\sigma\in Sym_{10} \mid \sigma(1)<\sigma(2),\ldots,\sigma(9)<\sigma(10), \sigma(1)<\sigma(3)<\sigma(5)<\sigma(7)<\sigma(9)\}.$$

Notice that Sym'_{10} contains 945 permutations. The best way to calculate $s_N(X_1,\ldots,X_N)$ in `Mathematica` is as follows. First step: Calculate all wedge-products

$$[X_{\sigma(1)},X_{\sigma(2)}]\wedge\cdots\wedge[X_{\sigma(9)},X_{\sigma(10)}]$$

for all 945 shuffle-permutations $\sigma\in Sym'_{10}$. Usually, there are not too many non-zero wedge-products. Second step: for these wedge-products, calculate

$$s_5^+([X_{\sigma(1)},X_{\sigma(2)}],\ldots,[X_{\sigma(9)},X_{\sigma(10)}])$$

and take their anti-symmetric sum. Direct calculation of $s_{10}(X_1,\ldots,X_{10})$ needs calculating $10! = 3628800$ summands; this is very hard. Practically, direct calculation of $s_N(X_1,\ldots,X_N)$ are not possible on the modern PC if $N>7$.

Acknowledgements

I am grateful to Yu.I. Manin, M. Jibladze and D. Leites for their interest in these results and for their essential remarks.

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