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ANTI-COMMUTATIVE ALGEBRAS WITH SKEW-SYMMETRIC IDENTITIES

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Generalizing Lie algebras, we consider anti-commutative algebras with skew-symmetric identities of degree > 3. Given a skew-symmetric polynomial f, we call an anti-commutative algebra f-Lie if it satisfies the identity $f = 0$. If s_n is a standard skew-symmetric polynomial of degree n, then any s_4-Lie algebra is f-Lie if $\deg f \geq 4$. We describe a free anti-commutative super-algebra with one odd generator. We exhibit various constructions of generalized Lie algebras, for example: given any derivations D, F of an associative commutative algebra U, the algebras $(U, D \wedge F)$ and $(U, \mathrm{id} \wedge D^2)$ are s_4-Lie. An algebra $(U, \mathrm{id} \wedge D^3 - 2D \wedge D^2)$ is s_5'-Lie, where s_5' is a non-standard skew-symmetric polynomial of degree 5.

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1. Introduction

The functions algebras under the Poisson bracket are Lie algebras. At the same time, there are many other brackets that do not generate a Lie algebra structure. For example, so is the Jacobi bracket (the Mayer bracket, [14, 17])

$$\omega(a,b) = \begin{vmatrix} \dfrac{\partial(a)}{\partial x} & \dfrac{\partial(a)}{\partial p} & \dfrac{\partial(a)}{\partial z} \\ \dfrac{\partial(b)}{\partial x} & \dfrac{\partial(b)}{\partial p} & \dfrac{\partial(b)}{\partial z} \\ -p & 0 & 1 \end{vmatrix}.$$

It satisfies the standard skew-symmetric identity of degree four, $s_4 = 0$ [7]. We prove that not only the Jacobi bracket, but also any multiplication of the form $u_1 D_2 \wedge D_3 + u_2 D_3 \wedge D_1 + u_3 D_1 \wedge D_2$ satisfies the identity $s_4 = 0$. A particular case of this result in which the derivations D_1, D_2, D_3 commute was obtained in [7].

We consider anti-commutative algebras with a skew-symmetric identity as a natural generalization of Lie algebras. We show that the theory of generalized Lie algebras has interesting examples of algebras which are constructed by means of differentiation and integration operators. The theory of s_4-Lie algebras might be as reach as the theory of Lie algebras. The aim of the present paper is to demonstrate some fragments of such a theory by constructing some examples of simple generalized Lie algebras.

Let $K\{t_1, t_2, \dots, t_k\}$ be the space of non-commutative non-associative polynomials in variables $t_1, t_2, \dots, t_k$. To simplify presentation, we assume that the main field K has characteristic 0. In fact many of our results hold true for $char\, K \neq 2, 3$.

A polynomial $f \in K\{t_1, t_2, \dots, t_k\}$ is called *skew-symmetric* if

$$f(t_{\sigma(1)}, \dots, t_{\sigma(k)}) = \operatorname{sign} \sigma\, f(t_1, \dots, t_k),$$

for any permutation $\sigma \in \mathrm{Sym}_k$. We define the polynomial $\mathrm{acom} \in K\{t_1, t_2\}$ (the anti-commutativity polynomial) by

$$\mathrm{acom}(t_1, t_2) = t_1 t_2 + t_2 t_1.$$

Let $s_k \in K\{t_1, \dots, t_k\}$ be the standard skew-symmetric polynomial

$$s_k(t_1, \dots, t_k) = \sum_{\sigma \in \mathrm{Sym}_k} \operatorname{sign} \sigma (\cdots ((t_{\sigma(1)} t_{\sigma(2)}) t_{\sigma(3)}) \cdots) t_{\sigma(k)}.$$

In the associative case, all skew-symmetric polynomials are generated by standard skew-symmetric polynomials. This is no longer true in the non-associative case.

Let $(A, \circ)$ be an algebra with vector space A and multiplication $\circ$. Recall some definitions about polynomial identities. We say that A has an *identity* $f = 0$, where $f \in K\{t_1, t_2, \dots, t_k\}$, if

$$f(a_1, a_2, \dots, a_k) = 0$$

for any substitutions $t_1 := a_1, t_2 := a_2, \dots, t_k := a_k$ by elements of A, where multiplications are calculated in terms of the multiplication $\circ$. If any algebra with an identity $f = 0$ also satisfies an identity $g = 0$, we say that the identity $g = 0$ is *a consequence* of the identity $f = 0$ and use the notation

$$f = 0 \Rightarrow g = 0.$$

Recall that the multiplication induced by the anti-commutativity polynomial $\mathrm{acom}(a, b) = a \circ b + b \circ a$ is usually called the Jordan multiplication.

Lie algebras are defined as anti-commutative algebras with the skew-symmetric identity of degree 3, i.e., they are algebras with the identities

$$\mathrm{acom} = 0, \quad s_3 = 0.$$

In this paper, we are interested in the following generalizations of Lie algebras. Let $K^{-}\{t_1,\dots,t_k\}$ be the subspace of multi-linear skew-symmetric anti-commutative polynomials, i.e., skew-symmetric multilinear polynomials in the variety of anti-commutative algebras. For a subspace $S \subseteq K^{-}\{t_1,\dots,t_k\}$, we call $A=(A,\circ)$ *S-Lie* if it satisfies the identities $f=0$ for all $f\in S$. If S is generated by polynomials $f_1,\dots f_n$, then we call A $\{f_1,\dots,f_n\}$-Lie. In the case of $n=1$, we often write f-Lie instead of $\{f\}$-Lie.

We describe free anti-commutative super-algebras with one odd generator.

We are interested in f-Lie algebras for $f\in K^{-}\{t_1,\dots,t_k\}$. Note that non-trivial examples of f-Lie algebras appear, beginning with degrees $k=3,4,\dots$. If $k=3$,

$$K^{-}\{t_1,t_2,t_3\}=\langle s_3\rangle$$

is 1-dimensional and $\{s_3\}$-Lie algebras are nothing else than the usual Lie algebras. If $k=4$,

$$K^{-}\{t_1,t_2,t_3,t_4\}=\langle s_4\rangle,$$

and anti-commutative algebras with skew-symmetric identity of degree 4 are s_4-Lie algebras. If $k=5$, then $K^{-}\{t_1,t_2,t_3,t_4,t_5\}$ is 2-dimensional and

$$K^{-}\{t_1,t_2,t_3,t_4,t_5\}=\langle s_5,s_5'\rangle,$$

where s_5' is the skew-symmetrization of the polynomial $((t_1t_2)t_3)(t_4t_5)$.

We prove that any s_d-Lie algebra is f-Lie for any multi-linear skew-symmetric polynomial f of degree $\deg f\geq d$ if $d=3,4$. If $d\geq 5$ this result is not true.

We give non-trivial examples of simple s_4-Lie algebras and s_5-Lie, s_5'-Lie algebras.

Let U be an associative commutative algebra with multiplication $(a,b)\mapsto ab$ and derivations D and F. We can endow U with the multiplication

$$(a,b)\mapsto (D\wedge F)(a,b)=D(a)F(b)-D(b)F(a).$$

We establish that $(U,D\wedge F)$ is s_4-Lie. Moreover, if D_1,D_2,D_3 are derivations of an associative commutative algebra U and $u_1,u_2,u_3\in U$, then (U,ω) is s_4-Lie for

$$\omega=u_1D_2\wedge D_3+u_2D_3\wedge D_1+u_3D_1\wedge D_2.$$

If U is an algebra of smooth functions on a manifold and D,F are vector fields then, by the Frobenius theorem, $(U,D\wedge F)$ is s_3-Lie if and only if the vector fields D and F are in involution. We prove that $(U,\mathrm{id}\wedge D^2)$ is also s_4-Lie, where

$$\mathrm{id}\wedge D^2(a,b)=aD^2(b)-bD^2(a).$$

The algebra $(U,D\wedge D^2)$ is also s_4-Lie and $(U,\mathrm{id}\wedge D^2)$ is a homomorphic image of the algebra $(U,D\wedge D^2)$.

Let U be an associative commutative algebra with derivation D. Endow it with the new multiplication

$$a \star b = aD^3(b) - 2D(a)D^2(b) + 2D^2(a)D(b) - D^3(a)b.$$

The algebra $(U, \star)$ satisfies the identity $g = 0$ (see the definition of g in Sec. 9) and any anti-commutative algebra with the identity $g = 0$ is s_5'-Lie. Moreover, the identity $g = 0$ is a consequence of Filippov's identity $h = 0$. The identities $g = 0$ and $h = 0$ are not equivalent. In particular, the algebra $(U, \star)$ and Malcev algebras are examples of s_5'-Lie algebras.

Algebras with identities of degree 4 were also studied in [12, 15, 16, 7, 11]. The functions algebras under the Jacobi bracket are s_4-Lie [7]. This claim is a special case of our Theorem 5.4.

Closing the introduction, let us pose some problems.

Let U be an associative commutative algebra with derivations $D_i, F_i, i = 1, \ldots, n$, and let

$$\omega_n = \sum_{i=1}^{n} D_i \wedge F_i$$

be a new multiplication on U. If $n = 1$, the algebra (U, ω_1) satisfies the identity $s_4 = 0$. The identity $s_4 = 0$ is the best one in the following sense:

- $s_4 = 0$ is an identity of (U, ω_1), $\omega_1 = D_1 \wedge F_1$, for any associative commutative algebra U with any derivations D_1, F_1.
- There exist some associative commutative algebra U and some derivations D_1 and F_1 such that the identity $s_4 = 0$ is minimal for the algebra (U, ω_1).
- The algebra (U, ω_1) is s_3-Lie iff D_1, F_1 are in involution.

Question 1. What can be said about an analog of the s_4-identity for ω_n in the case of $n > 1$? What is the best identity for (U, ω_n)?

In particular, what should be the best identity for the Jacobi bracket

$$[a, b] = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i}\left(\frac{\partial b}{\partial x_i} + p_i \frac{\partial b}{\partial z}\right) - \frac{\partial b}{\partial p_i}\left(\frac{\partial a}{\partial x_i} + p_i \frac{\partial a}{\partial z}\right),$$

where a, b are functions in $2n + 1$ variables $(x_1, \ldots, x_n, p_1, \ldots, p_n, z)$?

More generally, assume $D_1, \ldots, D_n \in \operatorname{Der} U$ and let

$$\tau_n = D_1 \wedge \cdots \wedge D_n$$

be an n-ary multiplication. Then (U, τ_n) is n-Lie in the sense of Filippov [6] iff the derivations $D_1, \ldots, D_n$ are in involution [4], [8].

Question 2. What can be said about an analog of the s_4-identity for the n-ary multiplication τ_n?

Assume given derivations $D_1, \dots, D_m \in \operatorname{Der} U$ and elements $u_{i_1,\dots,i_n} \in U$. Let an n-ary multiplication

$$\eta = \sum_{1 \le i_1 < \cdots < i_n \le m} u_{i_1,\dots,i_n} D_{i_1} \wedge \cdots \wedge D_{i_n}$$

be defined as a linear combination of multiplications of the form τ_n.

Question 3. What is the minimal identity for (U, η) that does not follow from the anti-commutativity identity for η?

Call an f-Lie algebra A *absolutely minimal* f-Lie if $f = 0$ is a minimal identity for A. Call A minimal f-Lie if $f = 0$ is minimal in the space of skew-symmetric identities. Any Lie algebra is s_4-Lie, but not (absolutely) minimal s_4-Lie. The algebra $(K[x]/K, \mathrm{id} \wedge \partial^2)$ is absolutely minimal s_4-Lie and simple.

Problem 4. Classify all simple (absolutely) minimal s_4-Lie algebras.

2. Cup-Product in an Anti-Commutative Module

Given vector spaces A and M, for $k > 0$ we denote by $T^k(A, M)$ the space of multilinear maps

$$\psi : \underbrace{A \times \cdots \times A}_{k \text{ times}} \to M.$$

We also set

$$T^0(A, M) = M,$$
$$T^k(A, M) = 0 \quad \text{if } k < 0.$$

Let $C^k(A, M)$ be the subspace of skew-symmetric maps:

$$\psi(a_1, \dots, a_k) = \operatorname{sign} \sigma \, \psi(a_{\sigma(1)}, \dots, a_{\sigma(k)})$$

for any $\sigma \in \mathrm{Sym}_k$. Put

$$T^*(A, M) = \oplus_k T^k(A, M), \quad C^*(A, M) = \oplus_k C^k(A, M).$$

Let M, N and S be vector spaces. Suppose that we are given a bilinear map

$$M \times N \to S, \quad (m, n) \mapsto m \smile n.$$

Such a map is called a *cup-product*, and the element of S corresponding to $m \in M, n \in N$ is denoted by $m \smile n$. Prolong this cup-product to a bilinear map (call it as before cup-product)

$$T^*(A, M) \times T^*(A, N) \to T^*(A, S), \quad (\psi, \phi) \mapsto \psi \smile \phi,$$

by

$$(\psi \smile \phi)(a_1, \dots, a_k, a_{k+1}, \dots, a_{k+r}) = \psi(a_1, \dots, a_k) \smile \phi(a_{k+1}, \dots, a_{k+r})$$

if

$$\psi \in C^k(A, M), \quad \phi \in C^r(A, N).$$

Let Sym_n be the permutation group on n elements, and let $\mathrm{Sym}_{k,r} \subseteq \mathrm{Sym}_{k+r}$ be the subset consisting of shuffle permutations

$$\tau = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & k+r \\ i_1 & \cdots & i_k & j_1 & \cdots & j_r \end{pmatrix} \in \mathrm{Sym}_{k+r}, \quad i_1 < \cdots < i_k, \quad j_1 < \cdots < j_r.$$

We prolong a cup-product $M \times N \to S$ to a bilinear map

$$C^*(A, M) \times C^*(A, N) \to C^*(A, S), \quad (\psi, \phi) \mapsto \psi \wedge \phi,$$

called *wedge-product*, by the following rule. If

$$\psi \in C^k(A, M), \quad \phi \in \mathrm{Sym}^r(A, N),$$

then

$$\psi \wedge \phi \in C^{k+r}(A, S)$$

and

$$\psi \wedge \phi(a_1, \ldots, a_{k+r}) = \sum_{\tau \in \mathrm{Sym}_{k,r}} \mathrm{sign}\, \tau\, \psi(a_{\tau(1)}, \ldots, a_{\tau(k)}) \smile \phi(a_{\tau(k+1)}, \ldots, a_{\tau(k+r)}).$$

In the case when all the spaces M, N and S coincide, the cup-product endows M with an algebra structure. In this case $T^*(A, M)$ and $C^*(A, M)$ as well have algebra structures induced by the cup-product and the wedge-product. Algebraic properties of the algebra M give rise to similar algebraic properties of $T^*(A, M)$ and $C^*(A, M)$. For example, if M is an associative algebra, then $T^*(A, M)$ is an associative algebra, too.

If M satisfies the identity of degree 2, then the cup (wedge) products satisfy the corresponding super-identities on $T^*(A, M)$ $(C^*(A, M))$.

Suppose that a cup product $M \times M \to M$ is given which satisfies the anti-commutative identity

$$m \smile n = -n \smile m, \quad \forall m, n \in M.$$

In other words, M has the structure of an anti-commutative algebra. Then

$$\psi \wedge \phi(a_1, \ldots, a_{2n}) = -(-1)^n \phi \wedge \psi(a_1, \ldots, a_{2n}),$$

for all $\psi, \phi \in C^n(A, M)$. This can be easily proved by using the following property of the shuffle-product:

$$\mathrm{sgn}\, \sigma = (-1)^n$$

for any shuffle-permutation

$$\sigma = \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & 2n \\ i_1 & \cdots & i_n & j_1 & \cdots & j_n \end{pmatrix} \in \mathrm{Sym}_{2n}, \quad i_1 < \cdots < i_n, \quad j_1 < \cdots < j_n.$$

In particular,

$$\psi \wedge \psi = 0$$

if n is even, $\mathrm{char}\, K \neq 2$.

Now, let $A = K^{-}\{t_1, t_2, \ldots\}$ be the space of multilinear skew-symmetric polynomials in the anti-commutative variety. Let s_k be a standard skew-symmetric polynomial

$$s_k(t_1, \ldots, t_k) = \sum_{\sigma \in \mathrm{Sym}_k} (\cdots (t_{\sigma(1)} t_{\sigma(2)}) \cdots) t_{\sigma(k)}, \quad k > 1,$$
$$s_1(t_1) = t_1.$$

Then

$$s_k = s_{k-1} \wedge s_1, \quad k > 1.$$

3. Free Anti-Commutative Algebras with One Odd Generator

Let $Q = Q_0 + Q_1$ be a free anti-commutative super-algebra with a parity map $q : Q \to \{0, 1\}$,

$$ab = -(-1)^{q(a)q(b)} ba,$$
$$Q_0 Q_0 \subseteq Q_0, \quad Q_0 Q_1 \subseteq Q_1, \quad Q_1 Q_1 \subseteq Q_0.$$

The elements of Q_0 are called even, and the elements of Q_1 odd.

We consider super-algebras with one generator x. If x is even, then $x^2 = 0$. So, a free anti-commutative super-algebra with one even generator has no non-trivial elements of degree larger than 1.

A more interesting case appears when x is odd. In this case the kth power of x (independently of bracketing) is even (odd) if k is even (odd). Note that

$$ab = -ba$$

if one of a, b is an even element, and

$$ab = ba$$

for odd elements a, b. In particular, powers of even powers of x (independently of bracketing) vanish. So, a base of a free anti-commutative algebra with one odd generator x can be constructed by induction on length.

Let $l = l(a)$ be the length of an element a (the number of occurrences of x in a). For instance, $l((xx)x) = 3$. Denote by $\mathrm{base}_{[l]}$ the base elements of length l of a free anti-commutative super-algebra with one odd generator. Let

$$\mathrm{base} = \bigcup_{k \geq 1} \mathrm{base}_{[k]}.$$

For $l = 1$ take

$$\mathrm{base}_{[1]} = \{x\}.$$

Suppose that $\mathrm{base}_{[k]}$ is constructed for all $k < l$. Order the elements of $\mathrm{base}_{[k]}$ somehow.

If a and b are base elements of degrees k_1 and k_2, and $k_1 + k_2 = l$, then ab can be chosen as a base element of degree l if $k_1 > k_2$. Suppose that $k_1 = k_2$. Then l should be even. If l is divisible by 4 and $k_1 = k_2 = l/2$, we have to exclude the element ab from $\mathrm{base}_{[l]}$ if $a = b$. If l is not divisable by 4, then such cases cannot appear. In the case when l is even, but $l \not\equiv 0 (\mathrm{mod}\, 4)$, we include ab into $\mathrm{base}_{[l]}$ if $a > b$.

This is a repetition of the arguments in [18], where a base was constructed for free commutative and anti-commutative algebras. Therefore, we refer to this paper for the details of proof. Thus, we have proved the following

Theorem 3.1. *Let Q be a free anti-commutative super-algebra with one generator. Then* base *constructed above forms a base of Q. The dimensions of homogeneous parts,*

$$\mu_n = |\mathrm{base}_{[n]}|,$$

satisfy the following recurrence relations:

$$\mu_n = \sum_{i<n/2} \mu_i \mu_{n-i}, \quad \text{if } n \not\equiv 0 (\mathrm{mod}\, 2),$$

$$\mu_n = \sum_{i<n/2} \mu_i \mu_{n-i} + (\mu_{n/2}^2 + \mu_{n/2})/2, \quad \text{if } n \equiv 0 (\mathrm{mod}\, 2), \quad n \not\equiv 0 (\mathrm{mod}\, 4),$$

$$\mu_n = \sum_{i<n/2} \mu_i \mu_{n-i} + (\mu_{n/2}^2 - \mu_{n/2})/2, \quad \text{if } n \equiv 0 (\mathrm{mod}\, 4),$$

for $n \geq 2$. Here we set $\mu_1 = 1$.

Corollary 3.2.

$$\mu_1 = 1, \quad \mu_2 = 1, \quad \mu_3 = 1, \quad \mu_4 = 1, \quad \mu_5 = 2, \quad \mu_6 = 4,$$

and

$$\begin{aligned}
\mathrm{base}_{[1]} &= \{x\}, \\
\mathrm{base}_{[2]} &= \{xx\}, \\
\mathrm{base}_{[3]} &= \{(xx)x\}, \\
\mathrm{base}_{[4]} &= \{((xx)x)x\}, \\
\mathrm{base}_{[5]} &= \{(((xx)x)x)x,\ ((xx)x)(xx)\}, \\
\mathrm{base}_{[6]} &= \{((((xx)x)x)x)x,\ (((xx)x)(xx))x,\ (((xx)x)x)(xx),\ ((xx)x)((xx)x)\}.
\end{aligned}$$

Let y be a base element of a free anti-commutative algebra with one odd generator of length k. Change all x in y to $t_1, \ldots, t_k$ from left to right. Consider the alternating sum of the resulting elements over the parameters $t_1, \ldots, t_k$. This construction was done in [22], and the so-obtained element is denoted by $Skew(y)$. For example,

$$\mathrm{Skew}((xx)x) = (t_1 t_2) t_3 + (t_2 t_3) t_1 + (t_3 t_1) t_2 - (t_2 t_1) t_3 - (t_3 t_2) t_1 - (t_1 t_3) t_2.$$

Note that $\mathrm{Skew}(y)$ coincides with the skew-symmetric polarization of y.

By the results of [20–23] we can give another formulation of Theorem 3.1.

Theorem 3.3. *Let $K^-\{t_1, \ldots, t_n\}$ be the space of anti-commutative multilinear skew-symmetric polynomials. Then the elements* $\mathrm{Skew}(x)$, *where x runs through the base elements of a free anti-commutative algebra with one odd generator, form a base for $K^-\{t_1, \ldots, t_n\}$.*

Whence we infer a classification of skew-symmetric anti-commutative multilinear polynomials of degree $d \leq 6$.

Corollary 3.4. *The following polynomials form a base for skew-symmetric multilinear polynomials in the variety of anti-commutative polynomials of degree d.*

$$\begin{aligned}
d &= 3, \quad \{s_3\},\\
d &= 4, \quad \{s_4\},\\
d &= 5, \quad \{s_5, s'_5 = 1/6 s_3 \wedge s_2\},\\
d &= 6, \quad \{s_6, s_6^{(2)} = s'_5 \wedge s_1, s_6^{(3)} = s_4 \wedge s_2, s_6^{(4)} = s_3 \wedge s_3\}.
\end{aligned}$$

Note that

$$s'_5(t_1, t_2, t_3, t_4, t_5) = \sum_{\sigma \in \mathrm{Sym}_5, \sigma(1)<\sigma(2), \sigma(4)<\sigma(5)} \mathrm{sign}\, \sigma((t_{\sigma(1)} t_{\sigma(2)}) t_{\sigma(3)})(t_{\sigma(4)} t_{\sigma(5)}).$$

4. The Identities $s_d = 0, d = 3, 4$, and Skew-Symmetric Identities of Degree $\geq d$

Theorem 4.1. *Let A be an s_d-Lie algebra, where $d = 3$ or $d = 4$. Let f be any skew-symmetric multi-linear anti-commutative polynomial of degree no less than d. Then A is f-Lie.*

Proof. Let

$$\begin{aligned}
r &= r(t_1, t_2, t_3, t_4, t_5)\\
&= -s_4(t_1, t_2, t_3, t_4)t_5 + s_4(t_1, t_2, t_3, t_5)t_4 - s_4(t_1, t_2, t_4, t_5)t_3\\
&\quad + s_4(t_1, t_3, t_4, t_5)t_2 - s_4(t_2, t_3, t_4, t_5)t_1 + s_4(t_1 t_2, t_3, t_4, t_5)\\
&\quad - s_4(t_1 t_3, t_2, t_4, t_5) + s_4(t_1 t_4, t_2, t_3, t_5) - s_4(t_1 t_5, t_2, t_3, t_4)\\
&\quad + s_4(t_2 t_3, t_1, t_4, t_5) - s_4(t_2 t_4, t_1, t_3, t_5) + s_4(t_2 t_5, t_1, t_3, t_4)\\
&\quad + s_4(t_3 t_4, t_1, t_2, t_5) - s_4(t_3 t_5, t_1, t_2, t_4) + s_4(t_4 t_5, t_1, t_2, t_3).
\end{aligned}$$

Prove that

$$-2s'_5 = r. \tag{1}$$

For $y \in K^-\{t_1, \ldots, t_5\}$, denote by $\mathrm{coef}_1(y)$ the coefficient of y in $(((t_1 t_2)t_3)t_4)t_5$ and denote by $\mathrm{coef}_2(y)$ the corresponding coefficient in $((t_1 t_2)t_3)(t_4 t_5)$.

Note that the polynomial r is skew-symmetric. Therefore, by Theorem 3.3,

$$r = \lambda s_5 + \mu s'_5$$

for some $\lambda, \mu \in K$. Calculate the coefficients in $(((t_1t_2)t_3)t_4)t_5$ on the left- and right-hand sides of this equality. We have

$$\mathrm{coef}_1(r) = \mathrm{coef}_1\left(-s_4(t_1,t_2,t_3,t_4)t_5 + s_4(t_1t_2,t_3,t_4,t_5)\right) = -1+1 = 0,$$
$$\mathrm{coef}_1(s_5) = 1, \quad \mathrm{coef}_1(s_5') = 0.$$

Thus,

$$\lambda = 0.$$

Further,

$$\begin{aligned}\mathrm{coef}_2(r) &= \mathrm{coef}_2(s_4(t_4t_5,t_1,t_2,t_3))\\ &= \mathrm{coef}_2(-((t_1t_2)t_3)(t_4t_5) + ((t_2t_1)t_3)(t_4t_5)) = -2.\end{aligned}$$

Therefore, (1) is proved.

By (1) any algebra with the identity $s_4 = 0$ satisfies the identity $s_5' = 0$,

$$s_4 = 0 \Rightarrow s_5' = 0.$$

Since $s_5 = s_4 \wedge s_1$, the implication

$$s_4 = 0 \Rightarrow s_5 = 0$$

is evident.

Let now

$$s_6^{(4)} = s_3 \wedge s_3$$

be the skew-symmetric polynomial corresponding to the bracketing type $((\bullet\bullet)\bullet)((\bullet\bullet)\bullet)$. Note that

$$\begin{aligned}&s_6^{(4)}(t_1,t_2,t_3,t_4,t_5,t_6)\\ &\quad= \sum_{\sigma\in \mathrm{Sym}_{3,3},\sigma(1)<\sigma(4)} \mathrm{sign}\,\sigma[\mathrm{jac}(t_{\sigma(1)},t_{\sigma(2)},t_{\sigma(3)}),\mathrm{jac}(t_{\sigma(4)},t_{\sigma(5)},t_{\sigma(6)})],\end{aligned}$$

where

$$\mathrm{jac}(t_1,t_2,t_3) = (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2$$

is the Jacobian and $[\ ,\ ]$ is the usual commutator, $[a,b] = ab - ba$.

Let p be a polynomial in 6 variables given by

$$\begin{aligned}&p(t_1,t_2,t_3,t_4,t_5,t_6)\\ &\quad= \sum_{\sigma\in \mathrm{Sym}_{3,3}} \mathrm{sign}\,\sigma\, s_4(\mathrm{jac}(t_{i_1},t_{i_2},t_{i_3}),t_{i_4},t_{i_5},t_{i_6})\\ &\qquad - s_6(t_1,t_2,t_3,t_4,t_5,t_6) + 2\sum_{i=1}^{6}(-1)^i s_5'(t_1,\dots,\hat{t_i},\dots,t_6)t_i.\end{aligned}$$

Prove that

$$-2s_6^{(4)} = p. \tag{2}$$

For $y \in K^{-}\{t_1, t_2, t_3, t_4, t_5, t_6\}$, denote by $\mathrm{coef}_1(y)$, $\mathrm{coef}_2(y)$, $\mathrm{coef}_3(y)$ and $\mathrm{coef}_4(y)$ the coefficients in $((((t_1t_2)t_3)t_4)t_5)t_6$, $(((t_1t_2)t_3)(t_4t_5))t_6$, $(((t_1t_2)t_3)t_4)(t_5t_6)$ and $((t_1t_2)t_3)((t_4t_5)t_6)$ respectively.

Note that the polynomial p is skew-symmetric in all 6 parameters. Therefore, by Theorem 3.3

$$p = \lambda_1 s_6 + \lambda_2 s_5' \wedge s_1 + \lambda_3 s_4 \wedge s_2 + \lambda_3 s_3 \wedge s_3.$$

We have

$$\begin{aligned}
\mathrm{coef}_1(p) = & \sum_{\sigma \in \mathrm{Sym}_{3,3}} \operatorname{sign}\sigma \, \mathrm{coef}_1(s_4(\mathrm{jac}(t_{i_1}, t_{i_2}, t_{i_3}), t_{i_4}, t_{i_5}, t_{i_6})) \\
& - \mathrm{coef}_1(s_6(t_1, t_2, t_3, t_4, t_5, t_6)) = 1 - 1 = 0, \\
\mathrm{coef}_2(p) = & \sum_{\sigma \in \mathrm{Sym}_{3,3}} \operatorname{sign}\sigma \, \mathrm{coef}_2(s_4(\mathrm{jac}(t_{i_1}, t_{i_2}, t_{i_3}), t_{i_4}, t_{i_5}, t_{i_6})) \\
& + 2\sum_{i=1}^{6} (-1)^i \, \mathrm{coef}_2(s_5'(t_1, \ldots, \hat{t_i}, \ldots, t_6)t_i) \\
= & \, \mathrm{coef}_2(((t_4t_5)\,\mathrm{jac}(t_1, t_2, t_3))t_6 - ((t_5t_4)\,\mathrm{jac}(t_1, t_2, t_3))t_6) \\
& + 2\,\mathrm{coef}_2((((t_1t_2)t_3)(t_4t_5))t_6) = 2 - 2 = 0, \\
\mathrm{coef}_3(p) = & \sum_{\sigma \in \mathrm{Sym}_{3,3}} \operatorname{sign}\sigma \, \mathrm{coef}_3(s_4(\mathrm{jac}(t_{i_1}, t_{i_2}, t_{i_3}), t_{i_4}, t_{i_5}, t_{i_6})) \\
& - \mathrm{coef}_3(s_6(t_1, t_2, t_3, t_4, t_5, t_6)) \\
& + 2\sum_{i=1}^{6} (-1)^i \, \mathrm{coef}_3(s_5'(t_1, \ldots, \hat{t_i}, \ldots, t_6)t_i) = 0 - 0 + 2 \cdot 0 = 0, \\
\mathrm{coef}_4(p) = & \sum_{\sigma \in \mathrm{Sym}_{3,3}} \operatorname{sign}\sigma \, \mathrm{coef}_4(s_4(\mathrm{jac}(t_{i_1}, t_{i_2}, t_{i_3}), t_{i_4}, t_{i_5}, t_{i_6})) \\
& - \mathrm{coef}_4(s_6(t_1, t_2, t_3, t_4, t_5, t_6)) + 2\sum_{i=1}^{6} (-1)^i \, \mathrm{coef}_4(s_5'(t_1, \ldots, \hat{t_i}, \ldots, t_6)t_i) \\
= & \, \mathrm{coef}_4(-((t_4t_5)t6)\,\mathrm{jac}(t_1, t_2, t_3) + ((t_5t_4)t6)\,\mathrm{jac}(t_1, t_2, t_3)) \\
= & -1 - 1 = -2.
\end{aligned}$$

So, we establish (2).

Since $s_6 = (s_4 \wedge s_1) \wedge s_1$, by (2) we see that

$$s_4 = 0 \Rightarrow s_6^{(4)} = 0.$$

It is easy to see that

$$\begin{gathered}
s_4 = 0 \Rightarrow s_6 = 0, \\
s_4 = 0 \Rightarrow s_5' = 0 \Rightarrow s_5' \wedge s_1 = 0, \\
s_4 = 0 \Rightarrow s_4 \wedge s_2 = 0.
\end{gathered}$$

Now, prove that

$$s_4 = 0 \Rightarrow f = 0$$

for any skew-symmetric multilinear anti-commutative polynomial f of degree ≥ 4. We proceed by induction on the degree of f, $n = \deg f$.

In the case $n = 4$ we have nothing to prove. The cases $n = 5, 6$ were examined above. Suppose that $n \geq 7$ and that our statement is true for $n-1$. By Theorem 3.3, any base skew-symmetric multilinear polynomial f can be represented as a wedge-product $f = g \wedge h$, where $\deg h \leq \deg g < n$ and $\deg g \geq 4$. By the induction hypothesis,

$$s_4 = 0 \Rightarrow g = 0.$$

Hence

$$s_4 = 0 \Rightarrow f = 0.$$

Since

$$s_3 = 0 \Rightarrow s_4 = s_3 \wedge s_1 = 0,$$

for any skew-symmetric multilinear anti-commutative polynomial f of degree no less than 3 we have

$$s_3 = 0 \Rightarrow f = 0.$$

The theorem is proved completely. $\square$

Corollary 4.2. *If $A = (A, \circ)$ is s_4-Lie, then (A, ω) is 3-Lie in the sense of Hanlon* [9], *where ω is any skew-symmetric map constructed from the multiplication $\circ$*, i.e.,

$$\omega(a, b, c) = [a, b] \circ c + [b, c] \circ a + [c, a] \circ b + q(a \circ [b, c] + b \circ [c, a \; c \circ [a, b])$$

for some $q \in K$. In particular, (A, jac) is 3-Lie in the sense of Hanlon.

Remark. For $d \geq 5$ Theorem 4.1 is not true. The non-Lie 7-dimensional Malcev algebra (octonians under commutator) satisfies the identity $s_5' = 0$ but $s_5 = 0$ and $s_6 = 0$ are not identities,

$$s_6(e_1, e_2, e_3, e_4, e_5, e_6) = 432 e_7.$$

Here we use the base for octonians which was constructed in [19, Exercise 3, p. 48].

Remark. Note that here the condition of multilinearity of identities is essential. For example,

$$f(t_1, t_2) = ((t_1 t_2) t_1) t_2 + ((t_1 t_2) t_2) t_1$$

is a skew-symmetric polynomial of type $(2, 2)$ in the variety of anti-commutative polynomials. If we consider f-Lie algebras for any skew-symmetric polynomial f, then in degree 4 appear two different generalizations of Lie algebras.

5. s_4-Lie Algebras

In this section we give non-trivial examples of s_4-Lie algebras.

Theorem 5.1. *Let $(U, \cdot)$ be an associative commutative algebra with derivation D. Then (U, ω) is s_4-Lie for $\omega = \mathrm{id} \wedge D^2$.*

Proof. Denote by s_4^ω the 4-linear function on U generated by $(a, b, c, d) \mapsto s_4(a, b, c, d)$. It is clear that $s_4(a, b, c, d)$ is a sum of elements of the form $Y^{i_1,i_2,i_3,i_4} = \pm D^{i_1}(a) \cdot D^{i_2}(b) \cdot D^{i_3}(c) \cdot D^{i_4}(d)$, where $i_1 + i_2 + i_3 + i_4 = 6$. Since $s_4^\omega(a, b, c, d)$ is skew-symmetric in a, b, c, d it is enough to collect all Y^{i_1,i_2,i_3,i_4} for $0 \le i_1 < i_2 < i_3 < i_4$. Note that 6 can be presented as a sum of four non-negative different integers only in one way

$$6 = i_1 + i_2 + i_3 + i_4, 0 \le i_1 < i_2 < i_3 < i_4 \Rightarrow i_1 = 0, i_2 = 1, i_3 = 2, i_4 = 3.$$

Therefore,

$$s_4^\omega = \lambda \, \mathrm{id} \wedge D \wedge D^2 \wedge D^3$$

for some $\lambda \in K$. Note that the coefficient λ is universal, i.e., it does not depend on the choice of the derivation D. Let O_1 be the algebra of divided power series,

$$O_1 = \left\langle x^{(i)} | x^{(i)} \cdot x^{(j)} = \binom{i+j}{i} x^{(i+j)}, 0 \le i, j \right\rangle$$

with a special derivation ∂ defined by

$$\partial(x^{(i)}) = x^{(i-1)}, i > 0, \quad \partial(x^{(0)}) = 0.$$

Take $U = O_1$ and $D = \partial$. Easy calculations show that

$$s_4(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = 0,$$
$$(\mathrm{id} \wedge D \wedge D^2 \wedge D^3)(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = x^{(0)}.$$

Thus, $\lambda = 0$, and

$$s_4 = 0$$

is an identity on (U, ω). □

Theorem 5.2. *Let U be an associative commutative algebra with derivation D, and $\omega = D \wedge D^2$. Then (U, ω) is s_4-Lie.*

Proof is similar to that of Theorem 5.1. □

Note that the derivation map $\partial : U \to U$ gives us the homomorphism of algebras

$$(U, \partial \wedge \partial^2) \to (U, \partial^0 \wedge \partial^2).$$

Indeed,

$$\partial(\partial \wedge \partial^2)(a, b)) = \partial \wedge \partial^3(a, b) = \partial^0 \wedge \partial^2(\partial(a), \partial(b))$$

for all $a, b \in U$. If $U = K[x]$, this gives us the isomorphism of algebras

$$(U/K, \partial \wedge \partial^2) \cong (U, \partial^0 \wedge \partial^2).$$

Theorem 5.3. *Let $(U, \cdot)$ be an associative commutative algebra with two derivations D and F. Then (U, ω) is s_4-Lie algebra for $\omega = D \wedge F$.*

Proof. Notice that s_k is a linear combination of degree k exterior products of the form $D_{i_{1,1}} D_{i_{1,2}} \cdots D_{i_{1,p_1}} \wedge \cdots \wedge D_{i_{k,1}} D_{i_{k,2}} \cdots D_{i_{k,p_k}}$ such that

- $D_{i_{j,r}}$ is either of D and F;
- $p_1 > 0, \ldots, p_k > 0$;
- $p_1 + \cdots + p_k = 2(k-1)$;
- $s_k(a_1, \ldots, a_k)$ is skew-symmetric in $a_1, \ldots, a_k$.

We see that

$$\begin{aligned}
&s_3(a,b,c)\\
&\quad = D(D(a)F(b) - D(b)F(a))F(c) - D(c)F(D(a)F(b) - D(b)F(a))\\
&\qquad + D(D(b)F(c) - D(c)F(b))F(a) - D(a)F(D(b)F(c) - D(c)F(b))\\
&\qquad + D(D(c)F(a) - D(a)F(c))F(b) - D(b)F(D(c)F(a) - D(a)F(c))\\
&\quad = -[D,F](a)(D(b)F(c) - D(c)F(b)) - [D,F](b)(D(c)F(a) - D(a)F(c))\\
&\qquad - [D,F](c)(D(a)F(b) - D(b)F(a)).
\end{aligned}$$

In other words,

$$s_3 = -[D,F] \wedge D \wedge F.$$

Therefore, s_4 is a linear combination of degree 4 exterior products of the form $H = D_{i_{1,1}} D_{i_{1,2}} \cdots D_{i_{1,p_1}} \wedge \cdots \wedge D_{i_{4,1}} D_{i_{4,2}} \cdots D_{i_{4,p_4}}$ such that $p_1 + p_2 + p_3 + p_4 = 6$, $p_1, p_2, p_3, p_4 > 0$, and $D_{i_{j,r}}$ is among D, F and $[D, F]$.

So, there are two possibilities for the p_i's:

- one element of the set $\{p_1, p_2, p_3, p_4\}$ is equal to 3 and the other three are equal to 1;
- two elements of the set $\{p_1, p_2, p_3, p_4\}$ are equal to 2 and the two other elements are equal to 1.

In the first case, we suppose, to simplify notation, that $p_4 = 3$ and $p_1 = p_2 = p_3 = 1$. Then H looks like $\pm D_{i_{1,1}} \wedge D_{i_{2,1}} \wedge D_{i_{3,1}} \wedge D_{i_{4,1}} D_{i_{4,2}} D_{i_{4,3}}$, where the three derivations $D_{i_{1,1}}, D_{i_{2,1}}, D_{i_{3,1}}$ run in the two element set $\{D, F\}$. Therefore, $H = 0$.

In the second case, we suppose, to simplify notation, that $p_1 = p_2 = 2$ and $p_3 = p_4 = 1$. Then H is a linear combination of exterior products of derivations of the form $[D,F] \wedge [D,F] \wedge D_{i_{3,1}} \wedge D_{i_{4,1}}$. Thus, $H = 0$.

Thus, we have proved that $s_4(a,b,c,d) = 0$ for any $a, b, c, d \in U$. □

Theorem 5.4. *Let U be an associative commutative algebra, let D_1, D_2, D_3 be derivations of U, and $u_1, u_2, u_3 \in U$. Let*

$$\omega = u_1 D_2 \wedge D_3 + u_2 D_3 \wedge D_1 + u_3 D_1 \wedge D_2.$$

Then (U, ω) is s_4-Lie.

Proof. Note that

$$\begin{aligned}\operatorname{jac}^{\omega} = \{&-u_1D_3(u_2)+u_1D_2(u_3)-u_2D_1(u_3)\\ &+u_2D_3(u_1)-u_3D_2(u_1)+u_3D_1(u_2)\}\cdot \mathrm{Jac}\\ &-(u_1[D_2,D_3]+u_2[D_3,D_1]+u_3[D_1,D_2])\wedge\omega,\end{aligned}$$

where

$$\operatorname{jac}^{\omega}(a,b,c)=\omega(\omega(a,b),c)+\omega(\omega(b,c),a)\omega(\omega(c,a),b),$$

$$\mathrm{Jac}(a,b,c)=\begin{vmatrix} D_1(a) & D_1(b) & D_1(c)\\ D_2(a) & D_2(b) & D_2(c)\\ D_3(a) & D_3(b) & D_3(c)\end{vmatrix}.$$

Using this fact one calculates that

$$\begin{aligned}s_4(a,b,c,d)=2(&\operatorname{jac}^{\omega}(a,b,c)d-\operatorname{jac}^{\omega}(a,b,d)c\\ &+\operatorname{jac}^{\omega}(a,c,d)b-\operatorname{jac}^{\omega}(b,c,d)a)=0.\end{aligned}$$ □

Remark. Recall in 3-dimensional space the no Plucker conditions, and $\omega = u_1D_2 \wedge D_3 + u_2D_3 \wedge D_1 + u_3D_1 \wedge D_2$ can be presented in the form $D \wedge F$ for some derivations D and F. Therefore, if U is an algebra of smooth functions on a manifold and at least one of the elements u_1, u_2, u_3 is invertible, then Theorem 5.4 follows from Theorem 5.3.

Corollary 5.5. [7] *Let U be an algebra of functions in 3 variables x, p, z and let $\omega : \wedge^2 U \to U$ be the Jacobi bracket* ([14, 17]),

$$\omega(a,b)=\begin{vmatrix} \dfrac{\partial(a)}{\partial x} & \dfrac{\partial(a)}{\partial p} & \dfrac{\partial(a)}{\partial z}\\ \dfrac{\partial(b)}{\partial x} & \dfrac{\partial(b)}{\partial p} & \dfrac{\partial(b)}{\partial z}\\ -p & 0 & 1\end{vmatrix}.$$

Then (U,ω) is s_4-Lie.

Proof. Note that

$$\omega=\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial p}-p\frac{\partial}{\partial p}\wedge\frac{\partial}{\partial z}.$$ □

6. Minimality of the Identity $s_4 = 0$ for the Multiplication $\mathrm{id} \wedge D^2$

In the previous section we have introduced several classes of s_4-Lie algebras. The natural question appears of whether the identity $s_4 = 0$ can be improved. Below we prove that this is not possible. We show that for some derivations D and F the identity $s_4 = 0$ is minimal for the algebras $(U, D \wedge F)$ and $(U, \mathrm{id} \wedge D^2)$.

Let $U = K[x]$ and

$$a\circ b = a\,\partial^2(b) - b\,\partial^2(a).$$

As we established above $(U, \circ)$ is s_4-Lie. In this section we prove that this algebra is s_4-minimal.

Theorem 6.1. *Let $X = 0$ be a multilinear identity of degree 4 which does not follow from the anti-commutativity identity* $\mathrm{acom} = 0$. *Then*

$$s_4 = 0 \Rightarrow X = 0.$$

Proof. The following set consists of all fifteen types of multilinear anti-commutative polynomials of degree 4,

$$\{(t_1t_2)(t_3t_4), (t_1t_3)(t_2t_4), (t_2t_3)(t_1t_4), ((t_1t_2)t_3)t_4, ((t_1t_2)t_4)t_3,\\ ((t_1t_3)t_2)t_4, ((t_1t_3)t_4)t_2, ((t_1t_4)t_2)t_3, ((t_1t_4)t_3)t_2, ((t_2t_3)t_1)t_4,\\ ((t_2t_3)t_4)t_1, ((t_2t_4)t_1)t_3, ((t_2t_4)t_3)t_1, ((t_3t_4)t_1)t_2, ((t_3t_4)t_2)t_1\}.$$

Let

$$\begin{aligned} &X(t_1, t_2, t_3, t_4)\\ &\quad = \lambda_1(t_1t_2)(t_3t_4) + \lambda_2(t_1t_3)(t_2t_4) + \lambda_3(t_2t_3)(t_1t_4) + \lambda_4((t_1t_2)t_3)t_4\\ &\qquad + \lambda_{10}((t_1t_2)t_4)t_3 + \lambda_5((t_1t_3)t_2)t_4 + \lambda_{11}((t_1t_3)t_4)t_2 + \lambda_6((t_1t_4)t_2)t_3\\ &\qquad + \lambda_{12}((t_1t_4)t_3)t_2 + \lambda_7((t_2t_3)t_1)t_4 + \lambda_{13}((t_2t_3)t_4)t_1 + \lambda_8((t_2t_4)t_1)t_3\\ &\qquad + \lambda_{14}((t_2t_4)t_3)t_1 + \lambda_9((t_3t_4)t_1)t_2 + \lambda_{15}((t_3t_4)t_2)t_1 \end{aligned}$$

be their linear combination (a general anti-commutative polynomial of degree 4).

Let M be the set of 4-tuples $\{i, j, s, k\}$ such that

- $0 \le i, j, s, k \le 6$;
- $i + j + s + k = 6$ and
- at least one of the components of $\{i, j, s, k\}$ is 0.

It is easy to see that M has 74 elements. Let R be the set of differential monomials of the form

$$\partial^{i_1}(a(x))\partial^{i_2}(b(x))\partial^{i_3}(c(x))\partial^{i_4}(d(x)),$$

where (i_1, i_2, i_3, i_4) runs in M. Note that $X(a(x), b(x), c(x), d(x))$ is a linear combination of 74 elements of the set R.

Suppose that

$$X(t_1, t_2, t_3, t_4) = 0$$

is an identity on $(U, \circ)$. Make substitutions $t_1 := a(x), t_2 := b(x), t_3 := c(x), t_4 := d(x)$ by elements of U and calculate the coefficients of the elements of R. They should be 0. We obtain a system of 74 linear equations in 15 unknowns.

This system has a one-parameter solution:

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = -\lambda_{15},$$
$$\lambda_5 = \lambda_{15}, \quad \lambda_6 = -\lambda_{15}, \quad \lambda_7 = -\lambda_{15},$$
$$\lambda_8 = \lambda_{15}, \quad \lambda_9 = -\lambda_{15}, \quad \lambda_{10} = \lambda_{15}, \quad \lambda_{11} = -\lambda_{15},$$
$$\lambda_{12} = \lambda_{15}, \quad \lambda_{13} = \lambda_{15}, \quad \lambda_{14} = -\lambda_{15}.$$

Thus, the identity condition $X = 0$ implies that

$$X(t_1, t_2, t_3, t_4) = -\lambda_{15}\, s_4(t_1, t_2, t_3, t_4).$$

In other words, any multi-linear skew-symmetric identity of degree 4 of the algebra $(U, \circ)$ follows from the identity $s_4 = 0$. □

7. Minimality of $s_4 = 0$ for $D \wedge F$

We repeat the arguments of the previous section. We save notations of Sec. 6 and omit technical details. Suppose that $f(t_1, t_2, t_3, t_4) = 0$ is an identity for the algebra $(U, \circ)$, where

$$U = K[x_1, x_2, x_3], \quad D = \partial_1, \quad F = x_1\partial_2 + x_2\partial_3,$$

and

$$a \circ b = \partial_1(a)(x_1\partial_2(b) + x_2\partial_3(b)) - (x_1\partial_2(b) + x_2\partial_3(a)\partial_1(b)).$$

Substitute ordered 4 elements of the set of monomials of degree 2,

$$\{x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\},$$

for (t_1, t_2, t_3, t_4). We obtain a linear combination of polynomials of degree 5. Collect the coefficients of the homogeneous monomials of degree 5. They should be equal to 0. We obtain a system of 96 linear equations in 15 unknowns $\lambda_i, 1 \leq i \leq 15$. Solve this system. We obtain a solution as in previous section, and find that

$$f = -\lambda_{15} s_4.$$

This means that the identity $s_4 = 0$ is a minimal identity for algebras of the form $(U, D \wedge F)$, where $U = K[x_1, x_2, x_3]$ is an associative commutative algebra with two derivations $D = \partial_1$ and $F = x_1\partial_2 + x_2\partial_3$.

8. A Simple s_4-Lie Algebra

Theorem 8.1. *Let A be the subspace of $K[x]$ consisting of polynomials without a constant term and let*

$$a * b = a\partial^2 b - \partial^2(a)b$$

be a multiplication on A. Then $(A, *)$ is s_4-Lie and simple.

Proof. By Theorem 5.1 $(A, *)$ is s_4-Lie.

Let $e_i = x^{i+2}, i \geq -1$. Then

$$e_i * e_j = (j - i)(i + j + 3)e_{i+j}.$$

So, A is graded and filtered,

$$A = \oplus_i A_i, \quad A_i = \langle e_i \rangle, \quad A_i * A_j \subseteq A_{i+j},$$
$$A = \mathcal{A}_{-1} \supseteq \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \cdots, \quad \mathcal{A}_i = \oplus_{j \geq i} A_j, \quad \mathcal{A}_i * \mathcal{A}_j \subseteq \mathcal{A}_{i+j}.$$

Moreover, this algebra is transitive,

$$A_{-1} * A_i = 0 \Rightarrow i = -1.$$

Therefore, any non-trivial ideal J of A contains an element of the form

$$a = e_{-1} + a_0, \quad a \in \mathcal{A}_0.$$

Then

$$e_{i+1} * a = -(i+3)(i+2)e_i + b_i \in J, \quad b_i \in \mathcal{A}_{i+1}$$

for any $i \geq 0$. So, $J = A$, and A is simple. □

9. An s_5'-Lie Algebra

In this section we prove that the algebra $(K[x], \star)$, where

$$a \star b = a\partial^3 b - 2\partial(a)\partial^2(b) + 2\partial^2(a)\partial(b)) - \partial^3(a)b, \quad \partial = \frac{\partial}{\partial x},$$

satisfies the identity $s_5' = 0$ but not the identity $s_5 = 0$. Moreover we prove that the identity $s_5' = 0$ is not minimal.

Let

$$\begin{aligned}
&\mathrm{alt}_{2,3} f(t_1, t_2, t_3, t_4, t_5) \\
&\quad = f(t_1, t_2, t_3, t_4, t_5) - f(t_1, t_2, t_3, t_5, t_4) - f(t_1, t_2, t_4, t_3, t_5) + f(t_1, t_2, t_4, t_5, t_3) \\
&\qquad + f(t_1, t_2, t_5, t_3, t_4) - f(t_1, t_2, t_5, t_4, t_3) - f(t_2, t_1, t_3, t_4, t_5) + f(t_2, t_1, t_3, t_5, t_4) \\
&\qquad + f(t_2, t_1, t_4, t_3, t_5) - f(t_2, t_1, t_4, t_5, t_3) - f(t_2, t_1, t_5, t_3, t_4) + f(t_2, t_1, t_5, t_4, t_3)
\end{aligned}$$

be the operator that makes the parameters (t_1, t_2) and (t_3, t_4, t_5) skew-symmetric. Define a polynomial $g \in K\{t_1, \ldots, t_5\}$ by

$$\begin{aligned}
&g(t_1, t_2, t_3, t_4, t_5) \\
&\quad = \frac{1}{2}\mathrm{alt}_{2,3}\{-7((t_1t_2)(t_3t_4))t_5 + 7((t_1t_3)(t_4t_5))t_2 \\
&\qquad + 2((t_1t_3)t_2)(t_4t_5) - 2(((t_1t_2)t_3)t_4)t_5 + 2(((t_1t_3)t_4)t_5)t_2 + 2(((t_3t_4)t_1)t_2)t_5 \\
&\qquad - ((t_1t_2)t_3)(t_4t_5) + ((t_3t_4)t_5)(t_1t_2) - (((t_3t_4)t_1)t_5)t_2 + (((t_3t_4)t_5)t_1)t_2\}.
\end{aligned}$$

Theorem 9.1. *Let $U = K[x]$ or the Laurent polynomials algebra $K(x)$ and let $A = (U, \star)$. Then*

- *any identity of the algebra A of degree ≤ 4 follows from the anti-commutativity identity* acom $= 0$;
- *the algebra A satisfies the identity $g = 0$;*

- *any multilinear identity of degree* 5 *of* A *follows from the identity* $g = 0$;
- *the algebra* A *satisfies the identity* $s_5' = 0$ *but not identity* $s_5 = 0$;
- *the algebra* A *is simple.*

Proof. Direct calculations show that $(U, \star)$ satisfies the identity $g = 0$.

Let $Z = Z(t_1, \dots, t_5)$ be a generic multi-linear anti-commutative polynomial of degree 5, i.e., a linear combination of 105 multilinear base elements of a free anti-commutative algebra generated by 5 elements $t_1, \dots, t_5$.

If $Z = 0$ is an identity for the anti-commutative algebra $(U, \star)$, then

$$Z(a(x), b(x), c(x), d(x), e(x)) = 0$$

for any 5 polynomials $a(x), b(x), c(x), d(x), e(x) \in U$. Note that $Z(a(x), b(x), c(x), d(x), e(x))$ is a linear combination of elements of the form $Y^{(i_1,\dots,i_5)} = \partial^{i_1}(a(x)) \cdots \partial^{i_5}(e(x))$, where $i_1 + \cdots + i_5 = 12$ and $i_1 \geq 0, \dots, i_5 \geq 0$. The coefficients of $Y^{(i_1,\dots,i_5)}$ should be 0. These conditions give us a system of linear equations in 105 unknowns. We solve this system using Mathematica. As a result we find that the polynomial $Y = Y(t_1, \dots, t_5)$ should be a linear combination of the following 5 polynomials

$$\begin{gathered}
-g(t_1, t_4, t_2, t_3, t_5) + g(t_1, t_5, t_2, t_3, t_4) - g(t_2, t_3, t_1, t_4, t_5),\\
+g(t_1, t_2, t_3, t_4, t_5),\\
g(t_1, t_4, t_2, t_3, t_5) + g(t_2, t_3, t_1, t_4, t_5),\\
g(t_1, t_3, t_2, t_4, t_5),\\
g(t_1, t_5, t_2, t_3, t_4) - g(t_2, t_3, t_1, t_4, t_5).
\end{gathered}$$

In other words, any multilinear identity of degree 5 follows from the identity $g = 0$.

Take a base of A consisting of the vectors $e_i = x^{i+3}$. Then

$$e_i \star e_j = (j - i)(i^2 + j^2 - ij - 7)e_{i+j}.$$

In particular,

$$\begin{aligned}
e_{-3} \star e_i &= (1 + i)(2 + i)(3 + i)e_{i-3},\\
e_{-2} \star e_i &= (-1 + i)(2 + i)(3 + i)e_{i-2},\\
e_{-1} \star e_i &= (-2 + i)(1 + i)(3 + i)e_{i-1}.
\end{aligned}$$

Let J_0 be a non-trivial ideal of $(K[x], \star)$. If

$$X = \sum_{i \geq i_0} \lambda_i e_i, \quad \lambda_{i_0} \neq 0,$$

write $\deg X = i_0$ and say that X has degree i_0. Prove that J_0 has an element X of degree -3.

Suppose that this is not true, and there exits $X \in J_0$ such that $i_0 > -3$, but there are no elements of J with degree -3. If $i_0 = -2$, then

$$X' = e_{-1} \star X = 4\lambda_{-2}e_{-3} + \sum_{i \geq -1} (i - 2)(i + 3)(i + 1)\lambda_i e_{i-1} \in J_0,$$

so we obtain an element $0 \neq X' \in J_0$ with degree -3. If $i_0 = -1$, then

$$X'' = e_{-2} \star X = -2\lambda_{-1}e_{-3} + \sum_{i \geq 0}(i+3)(i+2)(i-1)\lambda_i e_{i-2} \in J_0$$

is an element of degree -3. If $i_0 \geq 0$, then $e_{-3} \star X \in J_0$ is also a non-trivial element of J_0 with degree $i_0 - 3$. In all cases we come to a contradiction with the minimality of i_0, if $i_9 > -3$. This means that J_0 contains an element whose degree is -3.

Let $0 \neq X = \sum_{i \geq -3} \lambda_i e_i \in J_0$ be an element with $\lambda_{-3} \neq 0$. Then for any $j \geq -2$

$$X \star e_{j+3} = \sum_{i \geq j} \mu_i e_i \in J_0$$

for some $\mu_i \in K$. Notice that

$$\mu_j = \lambda_{-3}(j+5)(j+4)(j+3) \neq 0.$$

So, for any $j \geq -3$ the ideal J_0 contains non-trivial elements of the form $\sum_{i \geq j} \mu_i e_i$, with $\mu_j \neq 0$. This means that $J_0 = K[x]$. So, $(K[x], \star)$ is simple.

The simplicity of $(U, \star)$ is true for the Laurent polynomials algebra too.

Let U be the algebra of Laurent polynomials $K(x)$ and J be a non-trivial ideal of $(K(x), \star)$. Then $J_0 = K[x] \cap J$ is an ideal of $(K[x], \star)$. Prove that

$$J_0 \neq 0.$$

Let $X = \sum_{i \geq i_0} \lambda_i e_i \in J$. Suppose that $i_0 < -3$. Then

$$X \star e_{-i_0} = \sum_{j \geq 0} \gamma_j e_j \in J_0$$

for some $\gamma_j \in K$ such that

$$\gamma_0 = -2(3i_0^2 - 7)i_0 \neq 0.$$

Therefore, $J_0 \neq 0$, and, as we have proved above, $J_0 = K[x]$. In particular,

$$e_0 \in J_0.$$

Thus

$$e_0 \star e_i = (i^2 - 7)ie_i \in J.$$

If $i < -3$, then $(i^2 - 7)i \neq 0$. Thus,

$$e_i \in J$$

for all $i \in \mathbf{Z}$. So, $(K(x), \star)$ is simple.

Note that

$$\begin{aligned} & s_5'(t_1, t_2, t_3, t_4, t_5) \\ & \quad = g(t_1, t_2, t_3, t_4, t_5)/2 - g(t_1, t_3, t_2, t_4, t_5)/2 - g(t_1, t_4, t_2, t_3, t_5) \\ & \qquad + g(t_1, t_5, t_2, t_3, t_4) - 3/2g(t_2, t_3, t_1, t_4, t_5). \end{aligned}$$

Thus, the skew-symmetric identity $s_5' = 0$ is a consequence of the identity $g = 0$. Therefore the algebra $(A, \star)$ satisfies the identity $s_5' = 0$.

We have

$$s_5(1, x, x^2, x^4, x^5) = -414720 \neq 0.$$

So, $(A, \star)$ does not satisfy the identity $s_5 = 0$. $\square$

Remark. In the class of multi-linear skew-symmetric identities, the identity $s'_5 = 0$ for the algebra $(U, \mathrm{id} \wedge \partial^3 - 2\partial \wedge \partial^2)$ is a minimal identity.

10. Malcev Algebras as s'_5-Lie Algebras

Let

$$\begin{aligned}
&\mathrm{malc}(t_1, t_2, t_3, t_4) \\
&\quad = (t_1t_3)(t_2t_4) - ((t_1t_2)t_3)t_4 - ((t_2t_3)t_4)t_1 - ((t_3t_4)t_1)t_2 - ((t_4t_1)t_2)t_3.
\end{aligned}$$

An anti-commutative algebra with the identity $\mathrm{malc} = 0$ is called *Malcev.*

Theorem 10.1. *Any Malcev algebra is s'_5-Lie under the commutator. Moreover, any Malcev algebra is g-Lie, where g is the polynomial defined in Sec.* 9

Proof. Direct calculations show that

$$\begin{aligned}
&-s'_5(t_1, t_2, t_3, t_4, t_5) \\
&\quad = 3\{-\mathrm{malc}(t_1, t_2, t_3, t_4)t_5 + \mathrm{malc}(t_2, (t_1t_4), t_3, t_5) \\
&\qquad - \mathrm{malc}(t_3, (t_2t_4), t_5, t_1) + \mathrm{malc}((t_3t_4), t_1, t_2, t_5)\} \\
&\qquad + 2\{\mathrm{malc}(t_1, t_2, t_4, t_5)t_3 + \mathrm{malc}(t_1, t_2, t_5, t_4)t_3 - \mathrm{malc}(t_1, t_4, t_3, t_5)t_2 \\
&\qquad + \mathrm{malc}(t_2, t_3, t_5, t_4)t_1 - \mathrm{malc}(t_2, t_1, (t_3t_5)t_4 - \mathrm{malc}(t_3, t_2, (t_1t_5), t_4) \\
&\qquad - \mathrm{malc}(t_3, (t_1t_2), t_4, t_5) - \mathrm{malc}(t_4, t_1, (t_2t_3), t_5) - \mathrm{malc}(t_4, t_3, (t_1t_2), t_5) \\
&\qquad - \mathrm{malc}(t_4, (t_2t_3), t_1, t_5) + \mathrm{malc}((t_2t_5), t_1, t_3, t_4)\} \\
&\qquad - \mathrm{malc}(t_1, t_2, t_3, t_5)t_4 - \mathrm{malc}(t_1, t_3, t_2, t_4)t_5 + \mathrm{malc}(t_1, t_3, t_4, t_5)t_2 \\
&\qquad + \mathrm{malc}(t_1, t_3, t_5, t_4)t_2 - \mathrm{malc}(t_2, t_1, t_3, t_4)t_5 + \mathrm{malc}(t_2, t_1, t_3, t_5)t_4 \\
&\qquad + \mathrm{malc}(t_2, t_3, t_4, t_5)t_1 + \mathrm{malc}(t_3, t_1, t_4, t_5)t_2 \\
&\qquad - \mathrm{malc}(t_1, t_2t_3, t_4, t_5) - \mathrm{malc}(t_1, t_2t_4, t_3, t_5) + \mathrm{malc}(t_1, t_2t_5, t_3, t_4) \\
&\qquad + \mathrm{malc}(t_1, t_4t_5, t_2, t_3) - \mathrm{malc}(t_2, t_1, t_3t_4, t_5) - \mathrm{malc}(t_2, t_1t_3, t_4, t_5) \\
&\qquad + \mathrm{malc}(t_2, t_1t_5, t_3, t_4) - \mathrm{malc}(t_2, t_3t_4, t_1, t_5) \\
&\qquad + \mathrm{malc}(t_2, t_3t_5, t_1, t_4) - \mathrm{malc}(t_3, t_2, t_1t_4, t_5) + \mathrm{malc}(t_3, t_1t_4, t_2, t_5) \\
&\qquad - \mathrm{malc}(t_4, t_2, t_1t_3, t_5) - \mathrm{malc}(t_4, t_1t_3, t_2, t_5) - \mathrm{malc}(t_4, t_1t_3, t_5, t_2) \\
&\qquad - \mathrm{malc}(t_2t_3, t_1, t_4, t_5) - \mathrm{malc}(t_2t_4, t_1, t_3, t_5) \\
&\qquad + \mathrm{malc}(t_3t_5, t_1, t_2, t_4) + \mathrm{malc}(t_4t_5, t_1, t_2, t_3).
\end{aligned}$$

Therefore,

$$\mathrm{malc} = 0 \Rightarrow s'_5 = 0.$$

Similar calculations show that $g = 0$ is an identity for Malcev algebras. $\square$

Remark. This fact also can be deduced from Filippov's results. Let

$$
\begin{aligned}
&h(t_1,t_2,t_3,t_4,t_5)\\
&\quad = -t_1(t_2(t_5(t_3t_4))) + t_1(t_5(t_2(t_3t_4))) + t_2(t_5(t_1(t_3t_4))) - t_2(t_1(t_5(t_3t_4)))\\
&\qquad + t_5(t_1(t_4(t_2t_3))) + t_5(t_2(t_4(t_1t_3))) - t_5(t_4(t_1(t_2t_3))) - t_5(t_4(t_2(t_1t_3)))\\
&\qquad - 2t_1((t_2t_5)(t_3t_4)) - 2t_2((t_1t_5)(t_3t_4)) - 2t_5((t_1t_3)(t_2t_4)) + 2t_5((t_1t_4)(t_2t_3))
\end{aligned}
$$

be the so-called h-polynomial [5]. The polynomial h plays an important role in the theory of Malcev algebras. More detailed calculations show that

$$\mathrm{acom} = 0, h = 0 \Rightarrow g = 0 \Rightarrow s_5' = 0.$$

11. Anti-Commutative Algebras with the Identity $s_5 + 2s_5' = 0$

Recall that an algebra with the identity zinbiel $= 0$,

$$\mathrm{zinbiel}(t_1,t_2,t_3) = (t_1t_2 + t_2t_1)t_3 - t_1(t_2t_3),$$

is called Zinbiel (sometimes *chronological*) [1, 2, 13, 10].

Example of a Zinbiel algebra. Let $U = K[x]$ and let $\int a(x) = \int_0^x a(x)dx$ be the integration operator. Then $(U,\star)$, with

$$a(x) \star b(x) = b(x) \int a(x),$$

is Zinbiel.

It is proved in [3] that Zinbiel algebras under the commutator satisfy the identity tortkara $= 0$, with

$$\mathrm{tortkara} = (t_1t)(t_2t) + \mathrm{jac}(t_1,t_2,t)t.$$

Moreover, the algebra $(U,\diamond)$, where

$$a(x) \diamond b(x) = a(x) \iint b(x),$$

under the commutator also satisfies the identity tortkara $= 0$.

Calculations similar to those in previous sections show that the following theorem is true.

Theorem 11.1. *The identity $s_5 + 2s_5' = 0$ is a consequence of the identity* tortkara $= 0$.

Corollary 11.2. *Algebras of integrable functions in a single variable under the multiplication*

$$(a(x), b(x)) \mapsto a(x) \int b(x) - b(x) \int a(x)$$

or

$$(a(x), b(x)) \mapsto a(x) \iint b(x) - b(x) \iint a(x)$$

are $(s_5 + 2s_5')$-Lie algebras.

12. Independence of the Identities $s_5 = 0$ and $s_5' = 0$.

Below we construct a 5-dimensional anti-commutative algebra $A = \{e_1, e_2, e_3, e_4, e_5\}$. We give the product $e_i \circ e_j$ only for $i < j$ and set

$$e_j \circ e_i = -e_i \circ e_j \quad \text{if } i < j \text{ and } e_i \circ e_i = 0.$$

In each case we check whether the algebra A satisfies the identities $s_4 = 0$, $s_5 = 0$ and $s_5' = 0$.

If the multiplication table of A is given by

$$e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_4 = e_5, \quad e_1 \circ e_5 = 0,$$
$$e_2 \circ e_3 = 0, \quad e_2 \circ e_4 = e_1, \quad e_2 \circ e_5 = e_2,$$
$$e_3 \circ e_4 = e_3, \quad e_3 \circ e_5 = e_3, \quad e_4 \circ e_5 = 0,$$

then A satisfies the identity $s_5 = 0$, but

$$s_4(e_1, e_2, e_3, e_4) = -2e_1 \neq 0,$$
$$s_5'(e_1, e_2, e_3, e_4, e_5) = e_1 \neq 0.$$

So, $s_5 = 0$ is a minimal identity among skew-symmetric multilinear identities, and

$$s_5 = 0 \not\Rightarrow s_5' = 0.$$

If A is an algebra with the multiplication

$$e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_4 = 0, \quad e_1 \circ e_5 = 0,$$
$$e_2 \circ e_3 = e_1, \quad e_2 \circ e_4 = e_2, \quad e_2 \circ e_5 = e_3,$$
$$e_3 \circ e_4 = e_5, \quad e_3 \circ e_5 = e_5, \quad e_4 \circ e_5 = 0,$$

then $s_5' = 0$ is an identity, but $s_4 = 0$ and $s_5 = 0$ are not identities,

$$s_4(e_1, \ldots, e_4) = -2e_5 \neq 0,$$
$$s_5(e_1, \ldots, e_5) = 2e_3 - 2e_5 \neq 0.$$

So, $s_5' = 0$ is a minimal identity among skew-symmetric multi-linear identities, and $s_5 = 0$ is not a consequence of the identity $s_5' = 0$. Hence,

$$s_5' = 0 \not\Rightarrow s_5 = 0.$$

If the multiplication table of A is given by

$$e_1 \circ e_2 = e_3, \; e_1 \circ e_3 = e_4, \; e_1 \circ e_4 = e_5, \; e_1 \circ e_5 = 0,$$
$$e_2 \circ e_3 = e_5, \; e_2 \circ e_4 = e_3, \; e_2 \circ e_5 = e_1, \; e_3 \circ e_4 = e_1, \; e_3 \circ e_5 = e_5, \; e_4 \circ e_5 = 0,$$

then $s_5' = 0$ and $s_5 = 0$ are identities, but $s_4 = 0$ is not,

$$s_4(e_1, e_2, e_3, e_4) = 2e_1 + 2e_4 \neq 0.$$

Hence, the following inclusion of classes of generalized Lie algebras is strict:

$$s_4 - \text{Lie} \subset \{s_5, s_5'\} - \text{Lie}.$$

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