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JORDAN ELEMENTS AND LEFT-CENTER OF A FREE LEIBNIZ ALGEBRA

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Abstract. An element of a free Leibniz algebra is called Jordan if it belongs to a free Leibniz-Jordan subalgebra. Elements of the Jordan commutant of a free Leibniz algebra are called weak Jordan. We prove that an element of a free Leibniz algebra over a field of characteristic 0 is weak Jordan if and only if it is left-central. We show that free Leibniz algebra is an extension of a free Lie algebra by left-center. We find the dimensions of the homogeneous components of the Jordan commutant and the base of its multilinear part. We find criterion for an element of free Leibniz algebra to be Jordan.

1. Introduction

Let K be a field of characteristic 0 and $\mathcal{K} = K\langle t_1, t_2, \ldots\rangle$ be a free magma, i.e., a space of non-associative non-commutative polynomials with generators $t_1, t_2, \ldots$. An ideal I of $\mathcal{K}$ is called T-*ideal* if for any $f(t_1, \ldots, t_k) \in I$ and for any endomorphism ϕ of $\mathcal{K}$,

$$f(\phi(t_1), \ldots, \phi(t_k)) \in I.$$

For non-associative, non-commutative polynomials $f_1, \ldots, f_l \in \mathcal{K}$, denote by $J(f_1, \ldots, f_l)$ the T-ideal of $\mathcal{K}$ generated by these elements.

Leibniz algebras were introduced by J.L. Loday [3]. They are defined by the identity $lei = 0$, where

$$lei = lei(t_1, t_2, t_3) = (t_1t_2)t_3 - t_1(t_2t_3) + t_2(t_1t_3).$$

Let

$$acom = acom(t_1, t_2) = t_1 \star t_2 = t_1t_2 + t_2t_1,$$

and

$$jac = jac(t_1, t_2, t_3) = t_1(t_2t_3) + t_2(t_3t_1) + t_3(t_1t_2),$$

be anti-commutative and Jacobi polynomials, respectively.

Let $(A, \circ)$ be an algebra with vector space A over a field K and multiplication $A \times A \to A, (a, b) \mapsto a \circ b$. Define the Lie and Jordan commutators (anti-commutator) by

$$[a, b] = a \circ b - b \circ a, \qquad \text{and} \qquad a \star b = a \circ b + b \circ a.$$

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Call the algebras $(A, [\ ,\ ])$ and $(A, \star)$ the *minus*- and *plus*-algebras, respectively, of A.

Let
$$[q] = \{1, 2, \ldots, q\}.$$
Here q might be infinite. Let $F(X)$ be the free Leibniz algebra defined on a set of generators $X = \{x_i | i \in [q]\}$. Let $F^+(X)$ be the subalgebra of the plus-algebra $(F(X), \star)$ generated by X. Let us introduce the following non-commutative, non-associative polynomials
$$com = t_1t_2 - t_2t_1, \qquad leibjor = (t_1t_2)(t_3t_4)$$
(commutativity and metabelian polynomials). We will see that $F^+(X)$ is a free algebra of the variety given by the polynomial identities $com = 0$ and $leibjor = 0$ (Theorem 1.2). If q is infinite we will sometimes write F and F^+ instead of $F(X)$ and $F^+(X)$.

For $a_1, \ldots, a_n \in F(X)$, denote by $a_1 \cdots a_n$ or $a_{1\cdots n}$ a right-bracketed element $a_1 \circ (\cdots (a_{n-1} \circ a_n) \cdots)$. Note that
$$a_1 \star (\cdots (a_{n-2} \star (a_{n-1} \star a_n)) \cdots) = a_1 \circ (\cdots (a_{n-2} \circ (a_{n-1} \star a_n)) \cdots),$$
so, in fact, in an expression of the form $a_1 \star (\cdots (a_{n-2} \star (a_{n-1} \star a_n)) \cdots)$ one can change all Jordan multiplications $\star$, except the last one, to the Leibniz multiplication $\circ$.

In [3] it is proved that the following set of elements
$$\mathcal{V}(X) = \cup_n \{x_{i_1 \ldots i_n} \stackrel{def}{=} x_{i_1} \cdots x_{i_n} | x_{i_1}, \ldots, x_{i_n} \in X\}$$
forms a base of the free Leibniz algebra $F(X)$. For $v = x_{i_1} \cdots x_{i_{n-1}} x_{i_n} \in \mathcal{V}(X)$, we say that v has *degree* n and that $x_{i_{n-1}}$ is the *pre-head* and x_{i_n} is the *head* of v.

Let $P_+, p_+, p_- : F(X) \to F(X)$ be linear maps defined on base elements by
$$P_+(x_{i_1} \circ (\cdots (x_{i_{n-2}} \circ (x_{i_{n-1}} \circ x_{i_n})) \cdots) = x_{i_1} \star (\cdots (x_{i_{n-2}} \star (x_{i_{n-1}} \star x_{i_n})) \cdots),$$
(changing all Leibniz mutiplications by anti-commutator)
$$p_+(x_{i_1} \circ (\cdots (x_{i_{n-2}} \circ (x_{i_{n-1}} \circ x_{i_n})) \cdots) = x_{i_1} \circ (\cdots (x_{i_{n-2}} \circ (x_{i_{n-1}} \star x_{i_n})) \cdots),$$
(changing a mutiplication between pre-head and head by anti-commutator)
$$p_-(x_{i_1} \circ (\cdots (x_{i_{n-2}} \circ (x_{i_{n-1}} \circ x_{i_n})) \cdots) = x_{i_1} \circ (\cdots (x_{i_{n-2}} \circ [x_{i_{n-1}}, x_{i_n}]) \cdots)$$
(changing a multiplication between pre-head and head by commutator).

Call an element $a \in F(X)$ *Jordan* if $a \in F^+(X)$ and *weak Jordan* if
$$a \in F(X) \star F(X).$$
It is clear that any Jordan element is weak Jordan. For example, if $q > 1$, then
$$a = x_2 \star (x_1 \circ x_2)$$
is a Jordan element, since
$$\begin{aligned} a &= x_2 \circ (x_1 \circ x_2) + (x_1 \circ x_2) \circ x_2 \\ &= x_2 \circ (x_1 \circ x_2) + x_1 \circ (x_2 \circ x_2) - x_2 \circ (x_1 \circ x_2) \\ &= x_1 \star (x_2 \star x_2)/2 \in F^+(X), \end{aligned}$$
and $b = (x_1 \circ x_2) \circ (x_1 \circ x_2)$ is weak Jordan, since
$$b = (x_1 \circ x_2) \star (x_1 \circ x_2)/2 \in F(X) \star F(X).$$

However, by Theorem 1.2 given below, the element b is not Jordan:

$$b = 2x_1 \circ (x_2 \circ (x_1 \circ x_2)) - 2x_2 \circ (x_1 \circ (x_1 \circ x_2)),$$

which implies that

$$\begin{aligned} p_+(b) &= 2x_1 \circ (x_2 \circ (x_1 \circ x_2)) - 2x_2 \circ (x_1 \circ (x_1 \circ x_2)) \\ &\quad + 2x_1 \circ (x_2 \circ (x_2 \circ x_1)) - 2x_2 \circ (x_1 \circ (x_2 \circ x_1)) \\ &\neq 2b, \end{aligned}$$

and hence, $b \notin F^+(X)$ when $q > 1$.

Note that the Jordan commutant $F(X) \star F(X)$ is an ideal of $F(X)$ with trivial right-action and left-action as a derivation,

$$a \circ (b \star c) = (a \circ b) \star c + b \star (a \circ c),$$

$$(b \star c) \circ a = 0,$$

for any $a, b, c \in F(X)$. Proofs of these facts are easy. See, for example, [1].

Call an element $z \in F(X)$ *left-central* if

$$z \circ a = 0$$

for any $a \in F(X)$. Let $Z(X)$ be the *left-center,* i.e., the set of left-central elements of $F(X)$:

$$Z(X) = \{z \in F(X) | z \circ a = 0, \forall a \in F(X)\}.$$

Let

$$Z_1(X) = \{z \in F(X) | z \circ x_1 = 0\}$$

be the *left-centralizer* of the element $x_1 \in X$ in $F(X)$.

If $z \in Z(X)$, then for any $y_1, y_2 \in F(X)$,

$$(z \circ y_1) \circ y_2 = z \circ (y_1 \circ y_2) - y_1 \circ (z \circ y_2) = 0.$$

Hence, $z \circ y_1 \in Z(X)$. Similarly, $y_1 \circ z \in Z(X)$, and so $Z(X)$ is an ideal of $F(X)$. Likewise, $Z_1(X)$ is also an ideal of $F(X)$, and

$$Z(X) \subseteq Z_1(X).$$

Since

$$(a \circ b + b \circ a) \circ c = a \circ (b \circ c) - b \circ (a \circ c) + b \circ (a \circ c) - a \circ (b \circ c) = 0,$$

we have

$$F(X) \star F(X) \subseteq Z(X).$$

For the left-center $Z(X)$ of the free Leibniz algebra $F(X)$, denote by $Z(X)_{m_1 \ldots m_q}$ the homogenous component of $Z(X)$ generated by m_1 generators x_1, m_2 generators x_2, etc, m_q generators x_q. Recall that a multinomial coefficient is defined by

$$\binom{n}{m_1 \cdots m_q} = \frac{n!}{m_1! \cdots m_q!}.$$

We write $d|m_i$ if d is a divisor of $m_1, \ldots, m_q$. Recall that the Moebius function $\mu(d)$ is defined as $(-1)^k$ if d is a product of k different prime numbers and it equals 0 if d is divisible by greater than one.

The aim of this paper is to prove that the left-center of a free Leibniz algebra $F(X)$ is generated by the squares $a \circ a, a \in F(X)$.

Theorem 1.1. *Let $F(X)$ be a free Leibniz algebra over a field K of characteristic 0 generated by a set $X = \{x_i | i \in [q]\}$. Then, for any $a \in F(X)$ of degree greater than one, the following conditions are equivalent:*

- $a \in F(X) \star F(X)$
- $a \in Z(X)$
- $a \in Z_1(X)$.

In particular,

$$Z_1(X) = Z(X) = F(X) \star F(X),$$

and $a \in F(X)$ is weak Jordan if and only if $a \circ x_1 = 0$.

The set of elements of the form $x_{i_1} \star x_{i_2 \dots i_n}$, where $x_{i_1}, \dots, x_{i_n} \in X$, spans the space of weak Jordan elements $F(X) \star F(X)$. The elements of the form $x_{i_1} \star x_{i_2 \dots i_q}$, where $i_1 \dots i_q$ are permutations of the set $\{1, \dots, q\}$, such that $i_1 \neq q$, form a base of the multilinear part of $F(X) \star F(X)$.

Homogeneous components of the left-center have dimension

$$\dim\, Z(X)_{m_1 \dots m_q} = \frac{n-1}{n} \binom{n}{m_1 \, \cdots \, m_q} - \frac{1}{n} \sum_{d | m_i, d>1} \mu(d) \binom{n/d}{m_1/d \, \cdots \, m_q/d},$$

where $n = m_1 + \dots + m_q$. In particular, multilinear part of the left center $Z(X)$ of degree q has dimension $(q-1)(q-1)!$.

The dimension of degree n part of the left-center generated by q generators is equal to

$$\dim\, Z(X)_n = \frac{n-1}{n} q^n - \frac{1}{n} \sum_{d | m_i, d>1} \mu(d) q^{n/d}.$$

Theorem 1.2. *Let K be a field of characteristic 0. Then $F^+(X)$, the subalgebra of $(F(X), \star)$ generated by $X = \{x_i | i \in [q]\}$, is isomorphic to a free algebra generated by X of the variety given by the commutativity and the metabelian identities.*

For any $a \in F(X)$ of degree greater than one, the following conditions are equivalent:

- *a is Jordan element*
- $P_+(a) = 2a$
- $p_+(a) = 2a$
- $p_-(a) = 0$.

The dimension of the homogeneous part $F^+(X)_{m_1 \dots m_q}$, i.e., the dimension of a subspace of $F^+(X)$ generated by m_i elements x_i, where $i = 1, \dots, q$, is equal to

$$\dim F^+(X)_{m_1 \dots m_q} = \frac{1}{2} \left(\frac{\sum_{i=1}^q m_i^2}{n(n-1)} + \frac{n-2}{n-1} \right) \binom{n}{m_1 \, \cdots \, m_q},$$

where $n = \sum_{i=1}^q m_i$. In particular, the multilinear part of $F^+(X)$ has dimension $q!/2$.

Let us give some applications of these theorems for small q and n. If $q = 1$ and $F_2(X)$ is a space generated by elements of degree more than one, then

$$F_2(X) = F(X) \star F(X).$$

If $q = 2$ then any weak Jordan element of degree no more than 3 is Jordan.

Another application concerns tetrads. Call an element of the form

$$u = x_1 \circ (\cdots (x_{n-1} \circ x_n) \cdots) + (\cdots ((x_n \circ x_{n-1}) \circ x_{n-2}) \cdots) \circ x_1$$

reversible of degree n. Reversible elements of degree 4 are called *tetrads.*

If A is a free associative algebra, then its Jordan elements are reversible, and the space of reversible elements is generated by Jordan elements and tetrads, i.e., by elements of the form $abcd + dcba$ [2]. Tetrads are not Jordan elements. A criterion for an element of the free associative algebra to be Jordan is not known. For the case of Leibniz algebras, Theorem 1.1 gives us the following criterion for weak Jordan elements: *any element $a \in F(X)$ is weak Jordan if and only if z is left-central.* Opposite to the associative case, tetrads for Leibniz algebras are weak Jordan elements. Let $a, b, c, d \in F(X)$. Then

$$\begin{aligned} a \circ (b \circ (c \circ d)) + ((d \circ c) \circ b) \circ a &= (a \circ b) \circ (c \circ d) + b \circ (a \circ (c \circ d)) \\ &\quad + (d \circ c) \circ (b \circ a) - b \circ ((d \circ c) \circ a) \\ &= -(b \circ a) \circ (c \circ d) + b \circ (a \circ (c \circ d)) \\ &\quad - (c \circ d) \circ (b \circ a) + b \circ ((c \circ d) \circ a) \\ &= -(b \circ a) \star (c \circ d) + b \circ (a \star (c \circ d)) \\ &= -(b \circ a) \star (c \circ d) + b \star (a \star (c \circ d)) \\ &\in F(X) \star F(X). \end{aligned}$$

It is not true, however, that all reversible elements of Leibniz algebras are weak Jordan. For example, take

$$R = R(a, b, c) = a \circ (b \circ c) + (c \circ b) \circ a$$

Then

$$\begin{aligned} R \circ d &= (a \circ (b \circ c) + (c \circ b) \circ a) \circ d \\ &= (a \circ (b \circ c)) \circ d + ((c \circ b) \circ a) \circ d \\ &= (a \circ (b \circ c)) \circ d - (a \circ (c \circ b)) \circ d \\ &= 2(a \circ (b \circ c)) \circ d - (a \circ (c \star b)) \circ d \\ &= 2(a \circ (b \circ c)) \circ d - (a \star (c \star b)) \circ d \\ &= 2(a \circ (b \circ c)) \circ d \\ &= 2(a \circ (b \circ (c \circ d)) - a \circ (c \circ (b \circ d)) \\ &\quad - b \circ (c \circ (a \circ d)) + c \circ (b \circ (a \circ d))) \\ &= 2[a, b \circ c] \circ d \end{aligned}$$

Note that $[a, b \circ c] \in F(X) \star F(X)$ if b and c are linearly dependent, and $R \in Z(X)$ in this case. If b and c are linearly independent (it might happen if $q = |X| > 2$), then it is not necessary that $R \circ d = 0$. For example,

$$R(x_1, x_2, x_3) = x_1 \circ (x_2 \circ x_3) + (x_3 \circ x_2) \circ x_1 \notin F(X) \star F(X), \quad q > 2.$$

So, reversible elements of degree 3 might not be left-central. On the other hand, the Jacobian of three elements is left-central:

$$\begin{aligned} jac(a, b, c) &= a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b) \\ &= (a \circ b) \circ c + b \circ (a \circ c) + b \circ (c \circ a) + c \circ (a \circ b) \\ &= (a \circ b) \star c + b \circ (a \star c) \\ &= (a \circ b) \star c + b \star (a \star c) \\ &\in F(X) \star F(X) \subseteq Z(X). \end{aligned}$$

Since,

$$\begin{aligned}[[a,b],c]+[[b,c],a]+[[c,a],b] &= (a\circ b)\circ c+(b\circ c)\circ a+(c\circ a)\circ b\\ &= a\circ[b,c]+b\circ[c,a]+c\circ[a,b]\\ &= jac(a,b,c)-jac(a,c,b)\\ &\in Z(X),\end{aligned}$$

any Leibniz algebra satisfies the following identity of degree 4:

$$([[a,b],c]+[[b,c],a]+[[c,a],b])\circ d=0.$$

If one considers identities of Leibniz algebras under the Lie commutator, then there are no non-trivial identities for Leibniz-Lie algebras until degree 5. When the degree is 5, two identities appear for Leibniz algebras under the Lie commutator. These facts and other properties of left-central elements can be found in [3] and [1].

Corollary 1.3. *Any reversible element of even degree is weak Jordan.*

Proof. By the identity $(a\circ b)\circ c=-(b\circ a)\circ c$, we have,

$$\begin{aligned}((\cdots((x_n\circ x_{n-1})\circ x_{n-2})\cdots)\circ x_1)\circ x_1 &= -((\cdots((x_{n-1}\circ x_n)\circ x_{n-2})\cdots)\circ x_1)\circ x_1\\ &= ((\cdots(x_{n-2}\circ(x_{n-1}\circ x_n))\cdots)\circ x_1)\circ x_1\\ &\ \ \vdots\\ &= (-1)^{n-1}(x_1\circ(\cdots(x_{n-2}\circ x_n)\cdots))\circ x_1\\ &= (-1)^{n-1}(x_1\cdots x_n)\circ x_1.\end{aligned}$$

Therefore, if n is even, then

$$u=x_1\circ(\cdots(x_{n-1}\circ x_n)\cdots)+(\cdots((x_n\circ x_{n-1})\circ x_{n-2})\cdots)\circ x_1\in Z_1(X).$$

So, by Theorem 1.1 $u\in F(X)\star F(X)$ if n is even and $q\geq n$.

Corollary 1.4. *(char $K=0$) The left-center of a free Leibniz algebra is abelian. Moreover, the factor-algebra of a free Leibniz algebra $F(X)$ over the left-center $Z(X)$ is isomorphic to a free Lie algebra $L(X)$.*

In other words, the free Leibniz algebra $F(X)$ is an extension of the free Lie algebra $L(X)$ by an anti-symmetric abelian module:

$$0\to Z(X)\to F(X)\to L(X)\to 0.$$

Recall that modules of Lie algebras are usually considered as symmetric, i.e., left and right-actions are connected by the relation

$$xm+mx=0,$$

for any x of the Lie algebra L and for any element m the of L-module M. Here, we consider a Lie algebra $L(X)$ as a skew-symmetric Leibniz algebra and consider $Z(L)$ as an anti-symmetric module over a Leibniz algebra, i.e., the left action is induced by multiplication in the Leibniz algebra and the right-action is trivial:

$$zx=0,\quad xz=x\circ z,\qquad x\in L(X),z\in Z(X).$$

So, in the realization of the free Leibniz algebra as an extension of free Lie algebra

$$F(X)=L(X)+Z(X),$$

the multiplication is given by

$$(a+z_1)\circ(b+z_2)=[a,b]+a\circ z_2,\quad a,b\in L(X), z_1,z_2\in Z(X).$$

It would be interesting to find a criterion specifying when elements of free Leibniz algebras are Lie elements. Note that any weak Jordan element of degree more than 2 is weak Lie,

$$a\star(b\circ c)=[a,c\circ b]+[b,a\circ c]-[c,a\circ b].$$

There are weak Lie elements that are not weak Jordan. For example, $[a,b\circ c]$ is a such element:

$$[a,b\circ c]=a\circ(b\circ c)-b\circ(c\circ a)+c\circ(b\circ a),$$

which implies

$$p_+([a,b\circ c])-2[a,b\circ c]=-a\circ[b,c]+b\circ[c,a]-c\circ[b,a]\neq 0.$$

2. The Left-center and the Jordan commutant

Lemma 2.1. *The condition $z\in Z(X)$, where $X=\{x_1,\dots,x_q\}$, is equivalent to the condition $z\circ\ x_i=0$ for any $i=1,2,\dots q$.*

Proof. If $z\circ a=0$ for any $a\in F(X)$, then, in particular, $z\circ x_i=0$ for any generator $x_i\in X$.

Conversely, suppose that $z\circ x_i=0$ for any generator $x_i\in X$. Let v be any base element of the free Leibniz algebra $F(X)$. By induction on the degree n of $v\in\mathcal{V}(X)$, let us prove that $z\circ v=0$. The base of induction $n=1$ is trivially true by our assumption. Suppose that our statement is true for any base element of length $n-1$. Then any base element of length n can be presented in the form $v=x_i\circ u$, where u is a base element of length $n-1$. Therefore,

$$z\circ v=z\circ(x_i\circ u)=(z\circ x_i)\circ u+x_i\circ(z\circ u)=0.$$

So, our statement is true for n. □

Consider the following set of permutations

$$S(k,n)=\{\sigma\in Sym_n|\sigma(1)<\cdots<\sigma(k-1)<\sigma(k)=n$$
$$>\sigma(k+1)>\cdots>\sigma(n)\}.$$

Let $S(n)=\cup_{k=0}^n S(k,n)$. Note that $|S(n)|=2^{n-1}$.

The following multiplication rule holds in free Leibniz algebras.

Lemma 2.2. *For any $a_1,\dots,a_{n+1}\in F(X)$,*

$$a_{1\dots n}\circ a_{n+1}=\sum_{k=0}^{n}(-1)^{n-k}\sum_{\sigma\in S(k,n)}a_{\sigma\ n+1}.$$

In particular, for any $u,v\in\mathcal{V}(X)$, the product $u\circ v$ is a linear combination of elements $w\in\mathcal{V}(X)$ whose heads coincide with a head of v.

Proof. We will use induction on n. If $n=1$, then there is nothing to prove. Suppose that our statement is true for $n-1$. Then

$$\begin{aligned}a_{12\dots n}\circ a_{n+1}&=(a_1\circ a_{2\dots n})\circ a_{n+1}\\&=a_1\circ(a_{2\dots n}\circ a_{n+1})-a_{2\dots n}\circ(a_1\circ a_{n+1})\\&=a_1\circ\sum_{\beta\in S'}(-1)^{|\bar\beta|}a_{\beta\,rev(\bar\beta)\,n+1}-\sum_{\beta\in S'}(-1)^{|\bar\beta|}a_{\beta\,rev(\bar\beta)\,1\,n+1},\end{aligned}$$

where S' is a set of subsequences $\beta = \beta_1 \dots \beta_k$ of the sequence $2 \dots n$ such that $\beta_k = n$ and $\beta_1 < \dots < \beta_k$, and $\bar{\beta} = \bar{\beta}_1 \dots \bar{\beta}_{n-k-1}$ is the complement subsequence of β in $2 \dots n$, such that $\bar{\beta}_1 < \dots < \bar{\beta}_{n-k-1}$. Hence,

$$a_{12\dots n} \circ a_{n+1} = \sum_{\beta \in S'} (-1)^{|\bar{\beta}|} a_{1\, \beta\, rev(\bar{\beta})\, n+1} - \sum_{\beta \in S'} (-1)^{|\bar{\beta}|} a_{\beta\, rev(\bar{\beta})\, 1\, n+1}.$$

Note that

$$S(n) = S_1 \cup S_2,$$

where S_1 is a set of subsequences $\alpha = \alpha_1 \dots \alpha_k$ of the sequence $1 \dots n$ such that $\alpha_1 = 1$ and $\alpha_k = n$, and S_2 is a set of subsequences $\alpha = \alpha_1 \dots \alpha_k$ of the sequence $1 \dots n$ such that $\alpha_1 > 1$ and $\alpha_k = n$. Then $\alpha = \alpha_1 \dots \alpha_k \in S_1$ implies that $\alpha = 1\,\beta$, where $\beta = \alpha_2 \dots \alpha_k \in S'$. Furthermore, $\alpha = \alpha_1 \dots \alpha_k \in S_2$ implies that $\bar{\alpha} = 1\,\bar{\beta}$, where $\bar{\beta} = \bar{\alpha}_2 \dots \overline{\alpha_{n-k}}$ and $\beta = \beta_1 \dots \beta_k \in S'$. Therefore,

$$\begin{aligned} a_{12\dots n} \circ a_{n+1} &= \sum_{\alpha \in S_1} (-1)^{|\bar{\alpha}|} a_{\alpha\, rev(\bar{\alpha})\, n+1} - \sum_{\alpha \in S_2} (-1)^{|\bar{\alpha}|-1} a_{\alpha\, rev(\bar{\alpha})\, n+1} \\ &= \sum_{\alpha \in S(n)} (-1)^{|\overline{\alpha}|} a_{\alpha\ rev(\bar{\alpha})\ n+1}. \end{aligned}$$

Thus, our statement is true for n as well. □

Example 1. Lemma 2.2 allows us to construct a multiplication table for a free Leibniz algebra. For example,

$$\begin{aligned} a_1a_2a_3a_4 \circ b = a_1a_2a_3a_4b - a_1a_2a_4a_3b - a_1a_3a_4a_2b - a_2a_3a_4a_1b \\ + a_1a_4a_3a_2b + a_2a_4a_3a_1b + a_3a_4a_2a_1b - a_4a_3a_2a_1b, \end{aligned}$$

and the product of two base elements $u = x_3x_1x_3x_2$ and $v = x_2x_1$ is

$$\begin{aligned} x_3x_1x_3x_2 \circ x_2x_1 = x_3x_1x_3x_2x_2x_1 - x_3x_1x_2x_3x_2x_1 - x_3x_3x_2x_1x_2x_1 \\ - x_1x_3x_2x_3x_2x_1 + x_3x_2x_3x_1x_2x_1 + x_1x_2x_3x_3x_2x_1 \\ + x_3x_2x_1x_3x_2x_1 - x_2x_3x_1x_3x_2x_1. \end{aligned}$$

Lemma 2.3. *Let $a \in F(X)$. Then z is left-central if and only if $z \circ x_1 = 0$. In particular, $Z(X) = Z \cap F(X)$, where Z is the left-center of $F = F(x_1, x_2, \dots)$.*

Proof. By Lemma 2.2 an element $z \circ x_1$ is a linear combination of base elements with head x_1, and an element $z \circ x_k$ can be obtained from the element $z \circ x_1$ by changing heads of base elements; that is, changing x_1 to x_k.Therefore, the conditions $z \circ x_k = 0$ and $z \circ x_1 = 0$ are equivalent for any $k = 1, \dots, q$. So, by Lemma 2.1

$$z \in Z(X) \iff z \in Z_1(X).$$

□

Lemma 2.4. *Let z be a left-central element generated by $x_1, \dots, x_q$,*

$$z = z(x_1, \dots, x_q) = \sum \lambda_{i_1 \dots i_n} x_{i_1} \cdots x_{i_n} \in Z(X),$$

where the summation runs over $i_1 \dots i_n \in [q]$. Then z is left-central as an element of the free Leibniz algebra F. Moreover, for any substitution of x_i by elements $a_i \in F, i = 1, \dots, q$, we once again obtain a left-central element

$$z' = z(a_1, \dots, a_q) = \sum_{i_1, \dots, i_n \in [q]} \lambda_{i_1 \dots i_n} a_{i_1} \cdots a_{i_n} \in Z.$$

Proof. By Lemma 2.3, $z \in Z(X)$ implies that $z \in Z$. By Lemma 2.2, for any $i_1 \dots i_n \in [q]$ and any $u \in F$, we have

$$(x_{i_1} \circ (\cdots (x_{i_{n-1}} \circ x_{i_n}) \cdots) \circ u = \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n)} x_{i_{\sigma(1)}} \circ (\cdots (x_{i_{\sigma(n)}} \circ u) \cdots).$$

Therefore, for any substitution $x_i \mapsto a_i$, we have

$$(a_{i_1} \circ (\cdots (a_{i_{n-1}} \circ a_{i_n}) \cdots) \circ u = \sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n)} a_{i_{\sigma(1)}} \circ (\cdots (a_{i_{\sigma(n)}} \circ u) \cdots).$$

So, the condition

$$z = \sum_{i_1, \dots, i_n \in [q]} \lambda_{i_1 \dots i_n} x_{i_1} \circ (\cdots (x_{i_{n-1}} \circ x_{i_n}) \cdots) \in Z(X)$$

implies the condition

$$\sum_{k=0}^{n} (-1)^{n-k} \sum_{\sigma \in S(k,n)} \sum_{i_1, \dots, i_n \in [q]} \lambda_{i_1 \dots i_n} x_{i_{\sigma(1)}} \circ (\cdots (x_{i_{\sigma(n)}} \circ u) \cdots) = 0.$$

Now take $u = x_l$ and collect all coefficients of $z \circ x_l$ at base elements $x_{j_1 \dots j_n\, l}$, where $j_1, \dots, j_n \in [q]$, and denote their sum as $\gamma_{j_1 \dots j_n\, l}$. Note that $\gamma_{j_1 \dots j_n\, l} = \gamma_{j_1 \dots j_n}$ does not depend on l. Therefore,

$$z \circ x_l = \sum_{j_1 \dots j_n \in [q]} \gamma_{j_1 \dots j_n} x_{j_1 \dots j_n\, l}.$$

By Lemma 2.2 the element $z' = z(a_1, \dots, a_q)$ constructed from $z = z(x_1, \dots, x_q)$ by replacing x_i with $a_i \in F$ has the same property:

$$z' \circ x_l = \sum_{j_1 \dots j_n} \gamma_{j_1 \dots j_n} a_{j_1} \circ (\cdots (a_{j_n} \circ x_l) \cdots).$$

Hence, the condition

$$z = \sum_{i_1, \dots, i_n \in [q]} \lambda_{i_1 \dots i_n} x_{i_1} \cdots x_{i_n} \in Z(X)$$

implies that $\gamma_{j_1 \dots j_n} = 0$ for all $j_1, \dots, j_n \in [q]$. Consequently,

$$z' \circ x_l = \sum_{j_1 \dots j_n} \gamma_{j_1 \dots j_n} a_{j_1} \circ (\cdots (a_{j_n} \circ x_l) \cdots) = 0.$$

In other words,

$$z = z(x_1, \dots, x_q) \in Z(X) \Rightarrow z' = z(a_1, \dots, a_q) \in Z.$$

□

Let $F(X)_{m_1 \dots m_q}$ be a subspace of $F(X)$ generated by m_s elements x_s, where $s = 1, 2, \dots, q$. Let $F_n(X)$ be the subspace of $F(X)$ generated by the base elements $v \in \mathcal{V}(X)$ of length n. Then

$$F_n(X) = \oplus_{k \geq 1} \oplus_{m_1 + \cdots + m_q = n} F_{m_1 \dots m_q}.$$

Lemma 2.5. *For any non-negative integers* $i_1, \dots, i_q, j_1, \dots, j_q$,

$$F(X)_{i_1 \dots i_q} \circ F(X)_{j_1,\dots,j_q} \subseteq F(X)_{i_1+j_1,\dots,i_q+j_q}.$$

In particular, for any positive integers n, m,

$$F(X)_n \circ F(X)_m \subseteq F_{n+m}(X).$$

Proof. Follows from Lemma 2.2. □

Let $\mathbf{Z}_0$ be the set of non-negative integers and

$$\mathbf{Z}_0^n = \underbrace{\mathbf{Z}_0 \times \cdots \times \mathbf{Z}_0}_{n \text{ times}}.$$

Let $\pi_{m_1,\dots,m_q} : F(X) \to F(X)_{m_1,\dots,m_q}$ be a projection map.

Lemma 2.6. *Let* $z = z(x_1, \dots, x_q) \in Z(X)$. *Then for any* $\alpha \in \mathbf{Z}_0^q$, $\pi_\alpha z \in Z(X)$.

Proof. For any $\lambda_1, \dots, \lambda_q \in K$ by Lemma 2.4

$$z' = z(\lambda_1 x_1, \dots, \lambda_q x_q) \in Z(X).$$

Present z as a sum of homogeneous components:

$$z = z(x_1, \dots, x_q) = \sum_{i_1 \dots i_q \in \mathbf{Z}_0^q} \pi_{i_1 \dots i_q} z.$$

Then,

$$z' = \sum_{i_1 \dots i_q \in \mathbf{Z}_0^q} \lambda_1^{i_1} \cdots \lambda_q^{i_q} \pi_{i_1 \dots i_q} z \in Z(X).$$

Since $\lambda_1, \dots, \lambda_q \in K$ are arbitrary elements of the infinite field K, standard reasonings based on the Vandermonde determinant shows that

$$\pi_{i_1 \dots i_q} z \in Z(X)$$

for any non-negative integers $i_1, \dots, i_q$. □

For a permutation $\sigma \in Sym_q$ written in one-line form, denote by $l(\sigma)$ and $r(\sigma)$ the parts of σ to the left and to the right of q, respectively. Denote by $rev(\sigma)$ the sequence σ written in reverse order. For example, if $\sigma = 3264751$, then $l(\sigma) = 3264$, $r(\sigma) = 51$ and $rev(\sigma) = 1574623$.

Recall that a shuffle product $\alpha \sqcup\!\sqcup \beta$ of sequences $\alpha = \alpha_1 \dots \alpha_k$ and $\beta = \beta_1 \dots \beta_l$ is defined as a sum of sequences $\gamma = \gamma_1 \dots \gamma_{k+l}$, such that $\gamma_i \in \{\alpha_1, \dots, \alpha_k, \beta_1, \dots \beta_l\}$, for any $i = 1, \dots, k+l$. Moreover, if $\gamma_{i_1} = \alpha_1, \dots \gamma_{i_k} = \alpha_k, \gamma_{j_1} = \beta_1, \dots, \gamma_{j_l} = \beta_l$, then $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$. We will write $\tau \in \alpha \sqcup\!\sqcup \beta$ if τ is one such summands. For example, if $\alpha = 14, \beta = 32$, then

$$\alpha \sqcup\!\sqcup \beta = 1432 + 1342 + 1324 + 3142 + 3124 + 3214,$$

$$1342 \in \alpha \sqcup\!\sqcup \beta, \qquad 3421 \notin \alpha \sqcup\!\sqcup \beta.$$

Lemma 2.7. *If* $b = \sum_{\mu \in Sym_n, \mu(n)=n} \lambda_\mu a_\mu$, *then*

$$b \circ a_{n+1} = (-1)^n \sum_{\sigma \in Sym_n} (-1)^{|l(\sigma)|+1} \left(\sum_{\tau \in l(\sigma) \sqcup\!\sqcup rev(r(\sigma))} \lambda_{\tau\, n} \right) a_{\sigma\, n+1}.$$

Proof. By Lemma 2.2

$$\begin{aligned}
b \circ a_{n+1} &= \sum_{\mu\in Sym_n, \mu(n)=n} \lambda_\mu a_\mu \circ a_{n+1} \\
&= \sum_{\substack{\mu\in Sym_n\\ \mu(n)=n}} \sum_{k=0}^{n} \sum_{\alpha\in S(k,n)} \lambda_\mu (-1)^{n-k} \big(a_{\mu(\alpha(1))}\dots a_{\mu(\alpha(k-1))} a_{\mu(\alpha(k))} \cdot \dots \\
&\qquad\qquad \cdot a_{\mu(\alpha(k+1))} \dots a_{\mu(\alpha(n))} a_{n+1}\big) \\
&= \sum_{\substack{\mu\in Sym_n\\ \mu(n)=n}} \sum_{k=0}^{n} \sum_{\alpha\in S(k,n)} \lambda_\mu (-1)^{n-k} \big(a_{\mu(\alpha(1))}\dots a_{\mu(\alpha(k-1))} a_{\mu(n)} \cdot \dots \\
&\qquad\qquad \cdot a_{\mu(\alpha(k+1))} \dots a_{\mu(\alpha(n))} a_{n+1}\big) \\
&= \sum_{\substack{\mu\in Sym_n\\ \mu(n)=n}} \sum_{k=0}^{n} \sum_{\alpha\in S(k,n)} \lambda_\mu (-1)^{n-k} \big(a_{\mu(\alpha(1))}\dots a_{\mu(\alpha(k-1))} a_{n} \cdot \dots \\
&\qquad\qquad \cdot a_{\mu(\alpha(k+1))} \dots a_{\mu(\alpha(n))} a_{n+1}\big) \\
&= (-1)^n \sum_{\sigma\in Sym_n} \sum_{\tau\in l(\sigma)\sqcup\sqcup rev(r(\sigma))} \lambda_{\tau\, n} (-1)^k \big(a_{\sigma(1)}\dots a_{\sigma(k-1)} a_n \cdot \dots \\
&\qquad\qquad \cdot a_{\sigma(k+1)} \dots a_{\sigma(n)} a_{n+1}\big) \\
&= (-1)^n \sum_{\sigma\in Sym_n} (-1)^{|l(\sigma)|+1} \left(\sum_{\tau\in l(\sigma)\sqcup\sqcup rev(r(\sigma))} \lambda_{\tau\, n} \right) a_{\sigma\, n+1}.
\end{aligned}$$

□

Example 2. If $n = 4$ and

$$\begin{aligned}
z &= \lambda_{1234} x_1x_2x_3x_4 + \lambda_{1324} x_1x_3x_2x_4 + \lambda_{2134} x_2x_1x_3x_4 \\
&\quad + \lambda_{2314} x_2x_3x_1x_4 + \lambda_{3124} x_3x_1x_2x_4 + \lambda_{3214} x_3x_2x_1x_4,
\end{aligned}$$

then

$$\begin{aligned}
z \circ x_5 = {} & \lambda_{1234} x_1x_2x_3x_4x_5 + (-\lambda_{1234} - \lambda_{1324} - \lambda_{3124}) x_1x_2x_4x_3x_5 \\
& + \lambda_{1324} x_1x_3x_2x_4x_5 + (-\lambda_{1234} - \lambda_{1324} - \lambda_{2134}) x_1x_3x_4x_2x_5 \\
& + (\lambda_{1324} + \lambda_{3124} + \lambda_{3214}) x_1x_4x_2x_3x_5 + (\lambda_{1234} + \lambda_{2134} + \lambda_{2314}) x_1x_4x_3x_2x_5 \\
& + \lambda_{2134} x_2x_1x_3x_4x_5 + (-\lambda_{2134} - \lambda_{2314} - \lambda_{3214}) x_2x_1x_4x_3x_5 \\
& + \lambda_{2314} x_2x_3x_1x_4x_5 + (-\lambda_{1234} - \lambda_{2134} - \lambda_{2314}) x_2x_3x_4x_1x_5 \\
& + (\lambda_{2314} + \lambda_{3124} + \lambda_{3214}) x_2x_4x_1x_3x_5 + (\lambda_{1234} + \lambda_{1324} + \lambda_{2134}) x_2x_4x_3x_1x_5 \\
& + \lambda_{3124} x_3x_1x_2x_4x_5 + (-\lambda_{2314} - \lambda_{3124} - \lambda_{3214}) x_3x_1x_4x_2x_5 \\
& + \lambda_{3214} x_3x_2x_1x_4x_5 + (-\lambda_{1324} - \lambda_{3124} - \lambda_{3214}) x_3x_2x_4x_1x_5 \\
& + (\lambda_{2134} + \lambda_{2314} + \lambda_{3214}) x_3x_4x_1x_2x_5 + (\lambda_{1234} + \lambda_{1324} + \lambda_{3124}) x_3x_4x_2x_1x_5 \\
& - \lambda_{3214} x_4x_1x_2x_3x_5 - \lambda_{2314} x_4x_1x_3x_2x_5 - \lambda_{3124} x_4x_2x_1x_3x_5 \\
& - \lambda_{1324} x_4x_2x_3x_1x_5 - \lambda_{2134} x_4x_3x_1x_2x_5 - \lambda_{1234} x_4x_3x_2x_1x_5.
\end{aligned}$$

3. Base for Leibniz-Jordan algebras

Let $[q] = \{1, 2, \ldots, q\}$ and $X = \{x_i | i \in [q]\}$. Let $y_{i_1 \ldots i_{n-2} i_{n-1} i_n}$, where $i_1, \ldots i_n \in [q]$, denote symbols that satisfy the conditions

$$y_{i_1 \ldots i_{n-2} i_{n-1} i_n} = y_{i_1 \ldots i_{n-2} i_n i_{n-1}}, \quad \forall i_1, \ldots, i_{n-2}, i_{n-1}, i_n \in [q].$$

Let $G(X)$ be the linear span of the elements $y_{i_1 \ldots i_n}$, where $i_1, \ldots, i_n \in [q]$. Take the set of elements $\{y_{i_1 \ldots i_n} | i_1, \ldots, i_n \in [q], \text{ and } i_{n-1} \le i_n \text{ if } n > 1\}$ as a base of $G(X)$. Define a multiplication on $G(X)$ by

$$y_{i_1 \ldots i_n} y_{j_1 \ldots j_m} = 0, \text{ if } n > 1 \text{ and } m > 1,$$
$$y_{i_1} y_{j_1 \ldots j_m} = y_{i_1 j_1 \ldots j_m}, \text{ if } n = 1,$$
$$y_{i_1 \ldots i_n} y_{j_1} = y_{j_1 i_1 \ldots i_n}, \text{ if } m = 1.$$

If $n = m = 1$, then

$$y_{i_1} y_{j_1} = y_{i_1 j_1} = y_{j_1 i_1} = y_{j_1} y_{i_1}$$

If $n = 1, m > 1$, then

$$y_{i_1} y_{j_1 \ldots j_m} = y_{i_1 j_1 \ldots j_m} = y_{j_1 \ldots j_m} y_{i_1}.$$

Similarly, if $n > 1, m = 1$, then

$$y_{i_1 \ldots i_n} y_{j_1} = y_{j_1 i_1 \ldots i_n} = y_{j_1} y_{i_1 \ldots i_n},$$

If $n > 1, m > 1$, then

$$y_{i_1 \ldots i_n} y_{j_1 \ldots j_m} = 0 = y_{j_1 \ldots j_m} y_{i_1 \ldots i_n}.$$

So, the multiplication of the algebra $G(X)$ is well-defined. It is easy to see that the algebra $G(X)$ is commutative and metabelian:

$$(ab)(cd) = 0, \quad \forall a, b, c, d \in G(X).$$

Moreover, $G(X)$ is isomorphic to a free algebra of the variety of metabelian commutative algebras generated on the set $X = \{x_i | i \in [q]\}$. An isomorphism can be given by the rule

$$y_{i_1 \ldots i_{n-1} i_n} \mapsto x_{i_1}(\cdots (x_{i_{n-1}} x_{i_n}) \cdots).$$

It is easy to check that this assignment yields an isomorphism. By the metabelian identity, any non right-bracketed and any non left-bracketed element should vanish. By the commutativity identity,

$$x_{i_1}(\cdots (x_{i_{n-2}}(x_{i_{n-1}} x_{i_n})) \cdots) = x_{i_1}(\cdots (x_{i_{n-2}}(x_{i_n} x_{i_{n-1}})) \cdots),$$

and hence, any left-bracketed element can be reduced to a right-bracketed element.

In [1] it was established that the free Leibniz algebra $F(X)$ under Jordan multiplication $a \star b = a \circ b + b \circ b$ satisfies the commutativity and metabelian identities. Set

$$x^+_{i_1 \ldots i_{n-2} i_{n-1} i_n} = \begin{cases} x_{i_1} \circ (\cdots (x_{i_{n-2}} \circ (x_{i_{n-1}} \star x_{i_n})) \cdots), i_{n-1} \le i_n, & \text{if } n > 1, \\ x_{i_1} & \text{if } n = 1. \end{cases}$$

Let us prove that set of elements

$$\mathcal{V}^+(X) = \{x^+_{i_1 \ldots i_n} | i_1, \ldots, i_n \in [q], \text{ and } i_{n-1} \le i_n \text{ if } n > 1\}$$

forms a base of $F^+(X)$.

First, note that $\mathcal{V}^+(X)\subset F(X):$

$$x^+_{i_1\dots i_{n-2}i_{n-1}i_n}=x_{i_1}\star(\cdots(x_{i_{n-2}}\star(x_{i_{n-1}}\star x_{i_n}))\cdots)\in F^+(X).$$

Suppose that the elements $x^+_{i_1\dots i_n}$ are linearly dependent:

$$\sum_{i_1,\dots,i_n\in[q],i_{n-1}\le i_n}\lambda_{i_1\dots i_n}x^+_{i_1\dots i_n}=0,$$

for some $\lambda_{i_1\dots i_n}\in K, n>1$. Then,

$$\begin{aligned}0=&\sum_{i_1,\dots,i_n\in[q],i_{n-1}\le i_n}\lambda_{i_1\dots i_{n-2}i_{n-1}i_n}(x_{i_1\dots i_{n-2}i_{n-1}i_n}+x_{i_1\dots i_{n-2}i_ni_{n-1}})\\
=&\sum_{i_1,\dots,i_n\in[q],i_{n-1}<i_n}\lambda_{i_1\dots i_{n-2}i_{n-1}i_n}(x_{i_1\dots i_{n-2}i_{n-1}i_n}+x_{i_1\dots i_{n-2}i_ni_{n-1}})\\
&+\sum_{i_1\dots i_{n-1}\in[q]}2\lambda_{i_1\dots i_{n-1}i_{n-1}}x_{i_1\dots i_{n-2}i_{n-1}i_{n-1}}\\
=&\sum_{i_1,\dots,i_n\in[q],i_{n-1}<i_n}\lambda_{i_1\dots i_{n-2}i_{n-1}i_n}x_{i_1\dots i_{n-2}i_{n-1}i_n}\\
&+\sum_{i_1,\dots,i_n\in[q],i_{n-1}>i_n}\lambda_{i_1\dots i_{n-2}i_ni_{n-1}}x_{i_1\dots i_{n-2}i_{n-1}i_n}\\
&+\sum_{i_1\dots i_{n-1}\in[q]}2\lambda_{i_1\dots i_{n-1}i_{n-1}}x_{i_1\dots i_{n-2}i_{n-1}i_{n-1}}.\end{aligned}$$

Since elements $x_{i_1\dots i_n}$ are base elements of $F(X)$, this means that $\lambda_{i_1\dots i_n}=0$ for all $i_1,\dots i_n\in[q]$. In other words, elements $x^+_{i_1\dots i_n}$, where $i_{n-1}\le i_n$, if $n>1$, are linearly independent.

Now let us prove that any element $a\in F^+(X)$ can be presented as a linear combination of elements $v\in\mathcal{V}^+(X)$. We can assume that a is a homogeneous element. Let n be the degree of a. We proceed by induction on n. If $n=1$ our statement is evident. Suppose that for $n-1$ our statement is true and $n>1$. Since any element of degree n is a linear combination of anti-commutators of two base elements of degree $<n$, we have to prove that $x^+_{i_1\dots i_k}\star x^+_{j_1\dots j_{n-k}}$ is a linear combination of base elements of the form $x^+_{s_1\dots s_n}\in\mathcal{V}^+(X)$. This fact is easy to establish. If $k>1$, then

$$x^+_{i_1\dots i_k}\star x^+_{j_1\dots j_{n-k}}=0.$$

If $k=1$ and $n>2$ then

$$x^+_{i_1}\star x^+_{j_1\dots j_{n-1}}=x_{i_1}\circ x^+_{j_1\dots j_{n-1}}+x^+_{j_1\dots j_{n-1}}\circ x_{i_1}=x_{i_1}\circ x^+_{j_1\dots j_{n-1}}=x^+_{i_1j_1\dots j_{n-1}}.$$

If $k=1$ and $n=2$, then

$$x^+_{i_1}\star x^+_{j_1}=x_{i_1}\star x_{j_1}=x_{i_1j_1}.$$

So, we have proved that the set $\mathcal{V}^+(X)$ forms base of $F^+(X)$. Note that the map

$$G(X)\to F^+(X),\quad y_{i_1\dots i_n}\mapsto x^+_{i_1\dots i_n}$$

is a homomorphism of algebras and is one-to-one.

So, we have established the following result.

Lemma 3.1. *Let $X = \{x_i | i \in [q]\}$. Let $G(X)$ be a free algebra generated by X of the variety given by the commutativity identity $com = 0$ and the metabelian identity $leibjor = 0$, where*

$$com = t_1t_2 - t_2t_1, \quad leibjor = (t_1t_2)(t_3t_4).$$

Then $F^+(X)$, the subalgebra of $(F(X), \star)$ generated by $X = \{x_i | i \in [q]\}$, is isomorphic to $G(X)$. An isomorphism is given by

$$G(X) \to F^+(X),$$

$$y_{i_1\dots i_n} \mapsto x^+_{i_1\dots i_n} \stackrel{def}{=} x_{i_1\dots i_{n-2}i_{n-1}i_n} + x_{i_1\dots i_{n-2}i_ni_{n-1}},$$

where $i_1, \dots, i_n \in [q]$.

4. Criterion for Jordan elements

Lemma 4.1. *$a \in F^+(X)$ if and only if $a = p_+b$ for some $b \in F(X)$.*

Proof. If $a = p_+b$ and $b = \sum_{i_1,\dots,i_n \in [q]} \lambda_{i_1\dots i_n} x_{i_1\dots i_n}$, then by the rule

$$a \circ (b \star c) = a \star (b \star c),$$

we have

$$\begin{aligned} a = p_+b &= \sum_{i_1,\dots,i_n \in [q]} \lambda_{i_1\dots i_n}(x_{i_1\dots i_{n-2}i_{n-1}i_n} + x_{i_1\dots i_{n-2}i_ni_{n-1}}) \\ &= x_{i_1} \star (\cdots (x_{i_{n-2}} \star (x_{i_{n-1}} \star x_{i_n})) \cdots) \\ &\in F^+(X). \end{aligned}$$

Conversely, if $a \in F^+(X)$, then by Lemma 3.1 a is a linear combination of elements of a form

$$x^+_{i_1\dots i_n} = x_{i_1\dots i_{n-2}i_{n-1}i_n} + x_{i_1\dots i_{n-2}i_ni_{n-1}}, \quad i_1, \dots, i_n \in [q].$$

Since

$$x^+_{i_1\dots i_n} = p_+x_{i_1\dots i_n},$$

this means that a is a linear combination of elements of a form $p_+x_{i_1\dots i_n}$. So, $a = p_+b$ for some $b \in F(X)$. □

Lemma 4.2. $p_+^2 = 2p_+$.

Proof. For any base element $v = x_{i_1\dots i_n} \in \mathcal{V}(X)$ we have

$$p_+v = p_+x_{i_1\dots i_n} = x_{i_1\dots i_{n-2}i_{n-1}i_n} + x_{i_1\dots i_{n-2}i_ni_{n-1}}.$$

Thus,

$$p_+^2v = 2(x_{i_1\dots i_{n-2}i_{n-1}i_n} + x_{i_1\dots i_{n-2}i_ni_{n-1}}) = 2p_+v.$$

Therefore $p_+^2a = 2p_+a$, for any $a \in F(X)$. □

Lemma 4.3. *For any $a \in F(X)$ the following conditions are equivalent*

- $p_+a = 2a$
- $p_-a = 0$

Proof. It is evident that $p_-p_+ = 0$. Therefore, if $p_+a = 2a$, then

$$p_a = p_-((p_+a)/2) = p_-p_+(a)/2 = 0.$$

Conversely, suppose that $p_a = 0$ for $a = \sum_{i_1,\dots,i_n\in[q]} \lambda_{i_1\dots i_n} x_{i_1\dots i_n} \in F(X)$. Since

$$\begin{aligned} p_-a &= \sum_{i_1,\dots,i_n\in[q]} \lambda_{i_1\dots i_n}(x_{i_1\dots i_{n-2}i_{n-1}i_n} - x_{i_1\dots i_{n-2}i_n i_{n-1}}) \\ &= \sum_{i_1,\dots,i_n\in[q], i_{n-1}<i_n} (\lambda_{i_1\dots i_{n-2}i_{n-1}i_n} - \lambda_{i_1\dots i_{n-2}i_n i_{n-1}}) x_{i_1\dots i_{n-2}i_{n-1}i_n} \end{aligned}$$

the condition $p_-a = 0$ gives us that

$$\lambda_{i_1\dots i_{n-2}i_{n-1}i_n} = \lambda_{i_1\dots i_{n-2}i_n i_{n-1}}, \quad \forall i_1, \dots i_n \in [q].$$

Therefore,

$$\begin{aligned} a = &\sum_{i_1,\dots,i_n\in[q], i_{n-1}<i_n} \lambda_{i_1\dots i_{n-2}i_{n-1}i_n}(x_{i_1\dots i_{n-2}i_{n-1}i_n} + x_{i_1\dots i_{n-2}i_n i_{n-1}}) \\ &+ \sum_{i_1,\dots,i_{n-1}\in[q]} \lambda_{i_1\dots i_{n-2}i_{n-1}i_{n-1}} x_{i_1\dots i_{n-2}i_{n-1}i_{n-1}} \\ = &\sum_{i_1,\dots,i_n\in[q], i_{n-1}<i_n} \lambda_{i_1\dots i_{n-2}i_{n-1}i_n} p_+(x_{i_1\dots i_{n-2}i_{n-1}i_n}) \\ &+ \sum_{i_1,\dots,i_{n-1}\in[q]} \lambda_{i_1\dots i_{n-2}i_{n-1}i_{n-1}} p_+(x_{i_1\dots i_{n-2}i_{n-1}i_{n-1}}/2). \end{aligned}$$

In other words, $a = p_+b$, for $b \in F(X)$ given by

$$\begin{aligned} b = &\sum_{i_1,\dots,i_n\in[q], i_{n-1}<i_n} \lambda_{i_1\dots i_{n-2}i_{n-1}i_n} x_{i_1\dots i_{n-2}i_{n-1}i_n} \\ &+ \sum_{i_1,\dots,i_{n-1}\in[q]} \lambda_{i_1\dots i_{n-2}i_{n-1}i_{n-1}} x_{i_1\dots i_{n-2}i_{n-1}i_{n-1}}/2. \end{aligned}$$

Therefore, by Lemma 4.2

$$p_+a = p_+^2 b = 2p_+b = 2a.$$

□

5. Dimension and base of left-center

Consider elements of $F(X) \star F(X)$. Call elements of the form $u^{(s)}_{i_1\dots i_s i_{s+1}\dots i_n} = x_{i_1\dots i_s} \star x_{i_{s+1}\dots i_n}$ as *s-type* elements. For 1-type elements, $u^{(1)}_{i_1\dots i_n} = x_{i_1} \star x_{i_2\dots i_n}$, where $n > 2$, call x_{i_1} the *leader*. If $n = 2$, call x_{i_1} the *leader* of $u^{(1)}_{i_1 i_2} = x_{i_1} \star x_{i_2}$ if $i_1 \le i_2$.

Lemma 5.1. *The degree n part of $F(X) \star F(X)$ is generated by 1-type elements of the form $x_{i_1} \star x_{i_2\dots i_n}$, where $x_{i_1}, \dots, x_{i_n} \in X$.*

Proof. Since $F(X) \star F(X)$ is generated by elements of the form $u^{(s)}_{i_1\dots i_s i_{s+1}\dots i_n} = x_{i_1\dots i_s} \star x_{i_{s+1}\dots i_n}$, it is enough to prove that any s-type element $u^{(s)}_{i_1\dots i_s i_{s+1}\dots i_n}$ can be presented as a linear combination of 1-type elements of the form $u^{(1)}_{i_1 i_2\dots i_n}$.

We will use induction on $s = 1, 2, \ldots, n-1$. If $s = 1$, there nothing is to prove. Suppose that the statement is true for $s-1$.Then

$$\begin{aligned}
u^{(s)}_{i_1\ldots i_n} &= (x_{i_1} \circ x_{i_2\ldots i_s}) \star x_{i_{s+1}\ldots i_n}\\
&= (x_{i_1} \circ x_{i_2\ldots i_s}) \circ x_{i_{s+1}\ldots i_n} + x_{i_{s+1}\ldots i_n} \circ (x_{i_1} \circ x_{i_2\ldots i_s})\\
&= x_{i_1} \circ (x_{i_2\ldots i_s} \circ x_{i_{s+1}\ldots i_n}) - x_{i_2\ldots i_s} \circ (x_{i_1} \circ x_{i_{s+1}\ldots i_n})\\
&\quad + (x_{i_{s+1}\ldots i_n} \circ x_{i_1}) \circ x_{i_2\ldots i_s} + x_{i_1} \circ (x_{i_{s+1}\ldots i_n} \circ x_{i_2\ldots i_s})\\
&= x_{i_1} \circ (x_{i_2\ldots i_s} \star x_{i_{s+1}\ldots i_n}) - x_{i_2\ldots i_s} \circ (x_{i_1} \circ x_{i_{s+1}\ldots i_n})\\
&\quad - (x_{i_1} \circ x_{i_{s+1}\ldots i_n}) \circ x_{i_2\ldots i_s}\\
&= x_{i_1} \circ (x_{i_2\ldots i_s} \star x_{i_{s+1}\ldots i_n}) - x_{i_2\ldots i_s} \star (x_{i_1\, i_{s+1}\ldots i_n}).
\end{aligned}$$

Now, we have

$$x_{i_1} \circ (x_{i_2\ldots i_s} \star x_{i_{s+1}\ldots i_n}) = x_{i_1} \star (x_{i_2\ldots i_s} \star x_{i_{s+1}\ldots i_n}).$$

Therefore, the element $x_{i_1} \circ (x_{i_2\ldots i_s} \star x_{i_{s+1}\ldots i_n})$ can be presented as a linear combination of 1-type elements. By induction, the element $x_{i_2\ldots i_s} \star x_{i_1\, i_{s+1}\ldots i_n}$ is also a linear combination of 1-type elements. Thus, the element $u^{(s)}_{i_1\ldots i_n}$ can be presented as a linear combination of elements of the form $u^{(1)}_{j_1\ldots j_n}$. Hence, our statement is true for s. □

Lemma 5.2. *Any multilinear 1-type element $u^{(1)}_{n\, i_1\ldots i_{n-1}}$ of degree n with leader x_n is a linear combination of multilinear 1-type elements with leader x_{i_1}, with $i_1 < n$.*

Proof. If $n = 2$ this statement is evident: $u_{21} = x_2 \star x_1 = x_1 \star x_2 = u^{(1)}_{12}$. Suppose that our statement is true for $n-1 > 1$. We then have

$$\begin{aligned}
u^{(1)}_{n\, i_1\ldots i_{n-1}} &= x_n \star x_{i_1\,\ldots\, i_{n-1}}\\
&= x_n \circ x_{i_1\,\ldots\, i_{n-1}} + x_{i_1\,\ldots\, i_{n-1}} \circ x_n\\
&= (x_n \circ x_{i_1}) \circ x_{i_2\ldots i_{n-1}} + x_{i_1} \circ (x_n \circ x_{i_2\ldots i_{n-1}})\\
&\quad + x_{i_1} \circ (x_{i_2\ldots i_{n-1}} \circ x_n) - x_{i_2\ldots i_{n-1}} \circ (x_{i_1} \circ x_n)\\
&= -(x_{i_1} \circ x_n) \circ x_{i_2\ldots i_{n-1}} + x_{i_1} \circ (x_n \circ x_{i_2\ldots i_{n-1}})\\
&\quad + x_{i_1} \circ (x_{i_2\ldots i_{n-1}} \circ x_n) - x_{i_2\ldots i_{n-1}} \circ (x_{i_1} \circ x_n)\\
&= -x_{i_1 n} \star x_{i_2\ldots i_{n-1}} + x_{i_1} \circ (x_n \star x_{i_2\ldots i_{n-1}})\\
&= -x_{i_1 n} \star x_{i_2\ldots i_{n-1}} + x_{i_1} \star (x_n \star x_{i_2\ldots i_{n-1}}).
\end{aligned}$$

By induction, the element $x'_{n-1} \star x_{1\ldots x'_{n-2}}$, where we set $x'_l = x_{i_{l+1}}, i = 1, \ldots, n-2$, and $x'_n = x_{i_1 n}$, is a linear combination of 1-type elements with leader x'_l, where $l \le n-2$. Since $x'_l = x_{i_{l+1}}$ and $i_{l+1} < n$, this means that the element $x_{i_1 n} \star x_{i_2\ldots i_{n-1}}$ is a linear combination of elements of 1-type of degree n with leader x_i, where $i < n$. It is evident that the element $x_{i_1} \star (x_n \star x_{i_2\ldots i_{n-1}})$ is a linear combniation of 1-type elements with leader x_{i_1}, where $i_1 < n$. Therefore, any 1-type element of the form $u^{(1)}_{n\, i_1\ldots i_{n-1}}$ is a linear combination of 1-type elements with leader x_{i_1}, with $i_1 < n$. Our statement is proved for n. □

Example 3.

$$x_3 \star x_{12} = x_1 \star (x_2 \star x_3) - x_2 \star x_{13},$$
$$x_4 \star x_{123} = x_1 \star (x_{23} \star x_4) - x_2 \star (x_{14} \star x_3) + x_3 \star x_{214},$$
$$x_5 \star x_{1234} = x_1 \star (x_5 \star x_{234}) - x_2 \star (x_{15} \star x_{34}) + x_3 \star (x_4 \star x_{215}) - x_4 \star x_{3215}.$$

6. Proof of Theorem 1.1

We know that $F(X) \star F(X) \subseteq Z(X)$, and by Lemma 2.3, $Z(X) = Z_1(X)$. So, to prove Theorem 1.1 it is enough to prove that

$$z = z(x_1, \dots, x_q) \in Z(X) \Rightarrow z \in F(X) \star F(X).$$

By Lemma 2.6 we can assume that the element $z \in Z(X)$ is homogeneous.

Denote by $\nu_1(z) = z_1(x_1', x_1'', x_2, \dots, x_q) \in F(X')$, where $X' = \{x_1', x_1'', x_2, \dots, x_q\}$, the element

$$\nu_1(z) = z(x_1' + x_1'', x_2, \dots, x_q) - z(x_1', x_2, \dots, x_q) - z(x_1'', x_2, \dots, x_q).$$

For a homogeneous element $z = z(x_1, \dots, x_q) \in Z(X)_{m_1 \dots m_q} = Z \cap F(X)_{m_1 \dots m_q}$, define the degree $deg_{x_i} z = m_i$ if the entrance of x_i in each component of z is m_i. If $deg_{x_1} z = 1$, then $\nu_1 z = 0$. By Lemma 2.4

$$\nu_1(z) \in Z$$

and

$$deg_{x_1} z = m_1 > 1,$$

which implies

$$deg_{x_{1'}}(\nu_1(z)) < m_1 \quad \text{and} \quad deg_{x_{1''}}(\nu_1(z)) < m_1.$$

Conversely, if $\nu_1(z) = z_1(x_{1'}, x_{1''}, x_2, \dots, x_q) \in Z(X')$, then

$$z(x_1, \dots, x_q) = (2^{m_1} - 2)^{-1} \nu_1(z)(x_1, x_1, x_2, \dots, x_q) \in Z(X).$$

Moreover, if $\nu_1(z) = z_1(x_{1'}, x_{1''}, x_2, \dots, x_q) \in F(X') \star F(X')$, then

$$z(x_1, \dots, x_q) = (2^{m_1} - 2)^{-1} \nu_1(z)(x_1, x_1, x_2, \dots, x_q) \in F(X) \star F(X).$$

Repeat this procedure m_i times for each $i = 1, \dots, q$. We see that we can assume the element $z \in Z(X) \subset Z$ is not only homogeneous, but is also multilinear, i.e., $m_i = 1$, for any $i = 1, \dots, q$. Therefore, it is enough to prove that any multilinear left-central element is a Jordan element.

Consider the multilinear left-central element $z = z(x_1, \dots, x_q) \in Z(X)$. We must demonstrate that $z \in F(X) \star F(X)$.

Let $L(X)$ be the free Lie algebra with generators $X = \{x_1, x_2, \dots, x_q\}$. The multilinear part of the free Lie algebra of degree q has a base generated by elements of the form $[x_{\sigma(1)}, [\cdots, [x_{\sigma(q-1)}, x_q] \cdots]]$, where σ runs through the permutations of Sym_q such that $\sigma(q) = q$ (see [5]). Since

$$L(X) \cong K\langle X \rangle / J(acom, jac) \cong K\langle X \rangle / J(acom, lei) \cong F(X)/J(acom),$$

we can present $z \in F(X)$ in the form

$$z = \sum_{\substack{\sigma \in Sym_q \\ \sigma(q) = q}} \lambda_\sigma x_\sigma \text{ (modulo } F(X) \star F(X))$$

for some $\lambda_\sigma \in K$.

Then by Lemma 2.7,

$$z \circ x_{q+1} = (-1)^q \sum_{\sigma \in Sym_q} (-1)^{|l(\sigma)|+1} \left(\sum_{\tau \in l(\sigma) \sqcup\!\sqcup rev(r(\sigma))} \lambda_{\tau\, q} \right) x_{\sigma\, q+1}.$$

We see that the coefficient of $z \circ x_{q+1}$ at x_σ, where $\sigma(1) = q, \sigma(q+1) = q+1$, is equal to $\lambda_{rev(\sigma(2)\dots\sigma(q))}$. Example 2 given above demonstrates this fact in the case of $q = 4$. Therefore,

$$z \circ x_{q+1} = 0,$$

which implies that

$$\lambda_\sigma = 0,\ \forall \sigma \in Sym_q,\ \sigma(q) = q.$$

So, $z \in F(X) \star F(X)$, which proves the main part of Theorem 1.1. The part of Theorem 1.1 concerning dimensions is easy combinatorics.

As we have proved,

$$\dim Z(X)_{m_1\dots m_q} + \dim L(X)_{m_1\dots m_q} = \dim F(X)_{m_1\dots m_q}.$$

By the Witt theorem [5],

$$\dim L(X)_{m_1\dots m_q} = \frac{1}{n}\sum_{d|m_i} \mu(d) \binom{n/d}{m_1/d \cdots m_q/d},$$

where $n = m_1 + \cdots + m_q$. By the Loday Theorem [4],

$$\dim F(X)_{m_1\dots m_q} = \binom{n}{m_1 \cdots m_q}.$$

Therefore,

$$\dim Z(X)_{m_1\dots m_q} = \frac{(n-1)}{n}\binom{n}{m_1 \cdots m_q} - \sum_{d|m_i, d>1} \mu(d)\binom{n/d}{m_1/d \cdots m_q/d}.$$

In particular,

$$\dim Z(X)_{1\dots 1} = (q-1)(q-1)!.$$

It is easy to see that the number of 1-type multilinear elements is equal to $(q-1)(q-1)!$. Therefore, by Lemma 5.2, the set of multilinear 1-type elements forms a base of the multilinear part of $F(X) \star F(X)$.

In general, by Lemma 5.1, the set of 1-type elements generates the homogeneous part of $F(X) \star F(X)$.

7. Proof of Theorem 1.2

By the identity

$$a \circ (b \star c) = a \star (b \star c),$$

it is clear that

$$P_+ = p_+.$$

By Lemmas 4.1, 4.2, 4.3 and 3.1, all statements of Theorem 1.2 except the part concerning dimensions have been proven.

Let us calculate the dimension of the homogeneous part of $G(X)_{m_1\dots m_q}$. Let R be the number of sequences of length $n = m_1 + \cdots + m_n$ with components in $[q]$ such that last two components are equal. Then

$$\begin{aligned} R &= |\{i_1 \dots i_{n-1} i_{n-1} | i_1, \dots i_{n-1} \in [q]\}| \\ &= \sum_{s=1}^{q} \frac{(m_1 + \cdots + m_{s-1} + m_{s+1} + \cdots m_q + m_s - 2)!}{m_1! \cdots m_{s-1}!(m_s-2)! m_{s+1}! \cdots m_q!} \\ &= \binom{m_1 + \cdots + m_q - 2}{m_1 \cdots m_{s-1}\ m_s - 2\ m_{s+1} \cdots m_q} \frac{\sum_{s=1}^q m_s^2 - m_s}{(m_1 + \cdots + m_q)(m_1 + \cdots + m_q - 1)}. \end{aligned}$$

The number of sequences with components in $[q]$ where each $i \in [q]$ appears m_i times is

$$T = \binom{m_1 + \cdots + m_q}{m_1 \ \cdots \ m_q}.$$

Therefore, the number of base elements of $G(X)_{m_1 \dots m_q}$ of degree $n = m_1 + \cdots + m_q$ where each $x_i, i \in [q]$ appears m_i times is

$$|\{i_1 \cdots i_n | i_1, \dots, i_n \in [q], \text{ and } i_{n-1} \leq i_n \text{ if } n > 1\}| = (R + T)/2.$$

In other words,

$$\dim G(X)_{m_1 \dots m_q} = \frac{1}{2} \left(\frac{\sum_{i=1}^q m_i^2}{n(n-1)} + \frac{n-2}{n-1} \right) \binom{n}{m_1 \ \cdots \ m_q}.$$

In particular, the dimension of the multilinear part of $G(X)_{1 \dots 1}$ is $q!/2$.

References

[1] A. S. Dzhumadil'daev, *q-Leibniz algebras,* Serdica Math., **34** (2008), 415–440. MR 2424655

[2] N. Jacobson, "Structure and Representations of Jordan Algebras," AMS Colloq. Publ., **39**, Mathematical Society, Providence, RI, 1968. MR 0251099

[3] J.-L. Loday and T. Pirashvili, *Universal enveloping algebras of Liebniz algebras and (co)homology,* Math. Ann., **296** (1993), 139–158. MR 1213376

[4] J.-L. Loday, *Cup-product for Leibniz cohomology and dual Leibniz algebras,* Math. Scand., **77** (1995), 189–196. MR 1379265

[5] C. Reutenauer, "Free Lie Algebras," London Mathematical Society Monographs, New Series, 7, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. MR 1231799

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