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SHORT COMMUNICATIONS

Worpitzky Identity for Multipermutations

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Worpitzky's identity [1] (see also [2]) expresses x^n as a linear combination of binomial polynomials,

$$x^n = \sum_{p=1}^{n} \binom{x+p-1}{n} a_{n,p}.$$

The coefficients $a_{n,p}$ have an interesting combinatorial interpretation. For a permutation $\sigma \in S_n$, we say that i is an *index of lowering* if $i = n$ or $\sigma(i) > \sigma(i+1)$ for $i < n$. In this case, $a_{n,p}$ is equal to the number of permutations with p indices of lowering. For example, S_3 has a single permutation with three indices of lowering (namely, 321), four permutations with two indices of lowering (132, 213, 231, 312), and a single permutation with one index of lowering (123). Thus, $a_{3,1} = 1$, $a_{3,2} = 4$, $a_{3,3} = 1$, and

$$x^3 = \binom{x}{3} + 4\binom{x+1}{3} + \binom{x+2}{3}.$$

The numbers $a_{n,p}$ are referred to as the *Euler numbers*. The indices of lowering and the Euler numbers admit a natural generalization for permutations on multisets. Let $\mathbf{n} = 1^{k_1} \dots n^{k_n}$ be a multiset, i.e., a set with elements $1, 2, \dots, n$, where every element i is repeated k_i times. Let $S_{\mathbf{n}}$ be the set of permutations on a multiset $\mathbf{n}$. Note that

$$|S_{\mathbf{n}}| = \binom{k_1 + \dots + k_n}{k_1 \dots k_n} = \frac{(k_1 + \dots + k_n)!}{k_1! \dots k_n!}.$$

As in the case of ordinary permutations, for $\sigma \in S_{\mathbf{n}}$, we say that i is an *index of lowering* if $i = n$ or $\sigma(i) > \sigma(i+1)$ for $i < n$. Denote by $\operatorname{des}\sigma$ the number of the indices of lowering,

$$\operatorname{des}\sigma = |\{i \mid i = n \text{ or } \sigma(i) > \sigma(i+1),\ 1 \le i < n\}|.$$

The Euler number $a_{\mathbf{n},p}$ for a multiset $\mathbf{n}$ is defined as the number of multipermutations with p indices of lowering,

$$a_{\mathbf{n},p} = |\{\sigma \in S_{\mathbf{n}} \mid \operatorname{des}\sigma = p\}|.$$

Let $A_{\mathbf{n}} = \sum_{p>0} a_{\mathbf{n},p} x^p$ be the Euler polynomial of the multiset $\mathbf{n}$.

Example. We have

$$S_{1^2 2^3,1} = \{11222\}, \qquad S_{1^2 2^3,2} = \{12122, 12212, 12221, 21122, 22112, 22211\},$$
$$S_{1^2 2^3,3} = \{21212, 21221, 22121\},$$
$$A_{1^2 2^3} = x + 6x^2 + 3x^3.$$

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Theorem 1. *For any nonnegative integers* $k_1, \dots, k_n$*,*

$$\prod_{i=1}^{n} \binom{x + k_i - 1}{k_i} = \sum_{p>0} \binom{x + k_1 + \dots + k_n - p}{k_1 + \dots + k_n} a_{\mathbf{n},p},$$

where the symbols $a_{\mathbf{n},p}$ *stand for the Euler numbers for the permutations of the multiset* $\mathbf{n} = 1^{k_1} \cdots n^{k_n}$.

This theorem follows from the results of [3]. The objective of this note is to give a simple proof of Worpitzky's identity for multipermutations.

The proof uses the notion of barred permutations [4]. Let $\sigma \in S_{\mathbf{n}}$. A *barred permutation* with base σ is a permutation σ with bars between the components of σ such that every index of lowering has at least one bar. Let $B_{\mathbf{n}}$ be the set of barred permutations on a multiset $\mathbf{n} = 1^{k_1} \cdots n^{k_n}$. Let $b_{\mathbf{n},p}$ be the number of barred permutations with p bars. For example, $B_{1^2 2^2}$ has nine barred permutations with two bars, namely,,

$$\{1122//, 112/2/, 11/22/, 1/122/, /1122/, 12/12/, 122/1/, 2/112/, 22/11/\}.$$

Thus, $b_{1^2 2^2, 2} = 9$.

Lemma 2. *The following equality holds*:

$$\sum_{p \ge 1} b_{\mathbf{n},p} x^p = \frac{\sum_{i>0} a_{\mathbf{n},i} x^i}{(1-x)^{k_1 + \dots + k_n + 1}}.$$

Proof. A barred permutation with p bars can be obtained from the permutations with $p - i$ indices of lowering by including i bars at $k_1 + \dots + k_n + 1$ places. This can be done in $\binom{k_1 + \dots + k_n + i}{i}$ ways. Therefore,

$$b_{\mathbf{n},p} = \sum_{i \ge 0} \binom{k_1 + \dots + k_n + i}{i} a_{\mathbf{n},p-i}. \qquad \square$$

Lemma 3. *The following equality holds*:

$$b_{\mathbf{n},p} = \prod_{i=1}^{n} \binom{p + k_i - 1}{k_i.}.$$

Proof. Consider a barred permutation with p bars. Let us enumerate the bars from left to right, $1, 2, \dots, p$. Let $x_{i,j}$ be the number of components equal to i between the $(j-1)$th and jth bars for $j > 1$, and let $x_{i,1}$ stand for the number of components equal to i before the first bar for $j = 1$. Note that the components of the base permutation between the $(j-1)$th and jth bars (if they exist) form a nondecreasing sequence. Therefore, every barred permutation with p bars satisfies the following system of n equations:

$$\sum_{j=1}^{p} x_{i,j} = k_i, \qquad i = 1, 2, \dots, n,$$

and, conversely, every nonnegative integer solution of this system corresponds to a barred permutation with p bars, and this correspondence is one-to-one. These equations are independent, and every equation has $\binom{k_i + p - 1}{k_i}$ nonnegative integer solutions. Therefore, the number of barred permutations with p bars is equal to $\prod_{i=1}^{n} \binom{k_i + p - 1}{k_i}$. $\square$

Proof of Theorem 1. By Lemmas 2 and 3,

$$\prod_{i=1}^{n}\binom{p+k_i-1}{k_i} = \sum_{i=0}^{p}\binom{k_1+\dots+k_n+i}{i}a_{\mathbf{n},p-i}$$
$$= \sum_{i=0}^{p}\binom{k_1+\dots+k_n+p-i}{p-i}a_{\mathbf{n},p} = \sum_{i=0}^{p}\binom{p+k_1+\dots+k_n-i}{k_1+\dots+k_n}a_{\mathbf{n},p};$$

where p is an arbitrary integer. Therefore, we may replace p by a formal parameter x. □

Corollary 4. *Let* $k_1 = \dots = k_n = k$. *Then*

$$\binom{x+k-1}{k}^n = \sum_{p>0}\binom{x+k-2+p}{kn}a_{\mathbf{n},p}.$$

Proof. This follows from Theorem 1, because the relation $a_{\mathbf{n},p} = a_{\mathbf{n},(n-1)k+2-p}$ holds for any $p = 1, 2, \dots, (n-1)k+1$.

In particular, in the case of $k_1 = \dots = k_n = 1$, we obtain Worpitzky's identity. □

REFERENCES

1. J. Worpitzky, J. Reine Angew. Math. **94**, 203 (1883).
2. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*: *A Foundation for Computer Science*, 2nd ed. (Addison-Wesley Publ. Co., Reading, MA, 1994; Vil'yams, Moscow, 2009).
3. J. F. Dillon and D. P. Roselle, SIAM J. Appl. Math. **17** (6), 1086 (1969).
4. I. Gessel and R. P. Stanley, J. Combinatorial Theory Ser. A **24** (1), 24 (1978).